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A NOTE ON SPECTRAL CONTINUITY

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Abstract. In the present note, provided $T \in \mathscr{L}(\mathscr{H})$ is biquasitriangular and Browder's theorem hold for T, we show that the spectrum σ is continuous at T if and only if the essential spectrum σ_e is continuous at T.

1. Introduction

Let $\mathscr{L}(\mathscr{H})$ denote the algebra of bounded linear operators on a separable infinite dimensional complex Hilbert space $\mathscr{H}$. Let $\mathcal{K}$ denote the set of all compact subsets of the complex plane $\mathbb{C}$. Equipping $\mathcal{K}$ with the Hausdorff metric, one may consider the spectrum σ as a function $\sigma : \mathscr{L}(\mathscr{H}) \to \mathcal{K}$ mapping operators $T \in \mathscr{L}(\mathscr{H})$ into their spectrum $\sigma(T)$. Newburgh [12] may be the first to have systematically investigated the continuity of the spectrum. He showed that the spectrum of an element of a Banach algebra is upper semicontinuous and that the spectrum is continuous at any element with totally disconnected spectrum. In addition, he showed that the spectrum is continuous on an abelian Banach algebra and on the class of operators satisfying G_1-condition. Studies identifying sets $\mathcal{C}$ of operators for which σ becomes continuous when restricted to $\mathcal{C}$ has been carried out by a number authors (see, for example, [5, 7, 8, 10]). On the other hand, Conway and Morrel [2, 3]

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have undertaken a detailed study on the continuity of various spectra in $\mathscr{L}(\mathscr{H})$.

Given an operator $T \in \mathscr{L}(\mathscr{H})$, let $\alpha(T) = \dim\ker(T)$ and $\beta(T) = \dim\ker(T^*)$. T is upper semi-Fredholm if $T\mathscr{H}$ is closed and $\alpha(T) < \infty$ and T is lower semi-Fredholm if $T\mathscr{H}$ is closed and $\beta(T) < \infty$. If T is semi-Fredholm, then the index of T, $\mathrm{ind}(T)$, is defined by $\mathrm{ind}(T) = \alpha(T) - \beta(T)$. T is said to be Fredholm if $T\mathscr{H}$ is closed and the deficiency indices $\alpha(T)$ and $\beta(T)$ are (both) finite. Let $\mathscr{F}$ denote the set of Fredholm operators and $\mathscr{S}\mathscr{F}$ the set of semi-Fredholm operators. Let

$$P_n(T) = \{\lambda \in \sigma(T) : T - \lambda \in \mathscr{S}\mathscr{F} \text{ and } \mathrm{ind}(T - \lambda) = n\}$$

for $n \in \mathbb{Z} \cup \{\pm\infty\}$ and let

$$P_{\pm}(T) = \bigcup\{P_n(T) : n \neq 0\} \text{ and } P_{-}(T) = \bigcup\{P_n(T) : -\infty \leq n \leq -1\}.$$

Recall that if $\mathscr{C}$ is the ideal of compact operators on $\mathscr{H}$ and

$$\pi : \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H})/\mathscr{C} \tag{1.1}$$

is the canonical map, then the essential spectrum of T is defined by

$$\sigma_e(T) = \sigma(\pi(T)).$$

In the following, we shall denote the set of accumulation points (resp. isolated points) of $\sigma(T)$ by $\mathrm{acc}\sigma(T)$(resp. $\mathrm{iso}\sigma(T)$) and write $\sigma_p^0(T)$ for the isolated eigenvalues of finite multiplicity. Recall that the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of T are the sets

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \in \mathscr{S}\mathscr{F} \text{ and } \sigma_b(T) = \sigma_e(T) \cup \mathrm{acc}\sigma(\mathrm{T}), \text{ respectively.}$$

Let $\pi_0(T) := \sigma(T) \setminus \sigma_b(T)$ denote the set of *Riesz points* of T. We say that *Browder's theorem holds for* T if

$$\sigma(T) \setminus \sigma_w(T) = \pi_0(T).$$

It is well known [9, Theorem 2; 9] that each of the following conditions is equivalent to Browder's theorem for T:

$$\sigma(T) = \sigma_w(T) \cup \sigma_p^0(T); \tag{1.2}$$

$$\sigma_w(T) = \sigma_b(T); \tag{1.3}$$

$$\mathrm{acc}\sigma(T) \subseteq \sigma_w(T). \tag{1.4}$$

In [6] Djordjevic and Han proved that the following;

Proposition 1.1. *If Browder's theorem holds for $T \in \mathscr{L}(\mathscr{H})$, then the followings are equivalent:*
(1) σ *is continuous at* T*;*
(2) σ_w *is continuous at* T*;*
(3) σ_b *is continuous at* T.

In the present note, provided $T \in \mathscr{L}(\mathscr{H})$ is biquasitriangular and Browder's theorem hold for T, we show that σ is continuous at T if and only if σ_e is continuous at T.

2. Main results

An operator $T \in \mathscr{L}(\mathscr{H})$ is called *quasitriangular* if there exists a sequence $\{F_n\}$ of projections of finite rank that

$$F_n \to 1 \text{ weakly and } ||F_n T F_n - T F_n|| \to 0.$$

Also, an operator $T \in \mathscr{L}(\mathscr{H})$ is called *biquasitriangular* if T and T^* are quasitriangular. It is well known [1] that an operator is *quasitriangular* if and only if for each $\lambda \in \mathbb{C}$ such that $T-\lambda$ semi-Fredholm, $\text{ind}(T-\lambda) \geq 0$. It follows that an operator T is *biquasitriangular* if and only if for each $\lambda \in \mathbb{C}$ such that $T-\lambda$ semi-Fredholm, $\text{ind}(T-\lambda) = 0$. Consequently,

$$T \text{ is biquasitriangular if and only if } P_{\pm}(T) = \emptyset. \tag{2.1}$$

We begin with the result of Conway and Morrel;

Lemma 2.1. [3, Corollary 4.3] *If $T \in \mathscr{L}(\mathscr{H})$ is biquasitriangular, then σ_e is continuous at T if and only if for each $\lambda \in \sigma_e(T)$ and $\epsilon > 0$, the ϵ-neighborhood of λ contains a component of σ_e.*

Theorem 2.2. *Let $T \in \mathscr{L}(\mathscr{H})$ be biquasitriangular. If σ is continuous at T, then σ_e is continuous at T.*

Proof. Assume that the spectrum σ is continuous at T. Since T is biquasitriangular, [11, Theorem 3] imply that $\sigma(T) = \sigma_e(T) \cup \sigma_p^0$ and $\sigma(T)$ is the closure of its isolated points. Also, $\sigma_e(T)$ is the closure of its trivial components. Thus Lemma 2.1 implies that σ_e is continuous at T. □

Now we consider the inverse of Theorem 2.2.

THEOREM 2.3. *If Browder's theorem holds for $T \in \mathscr{L}(\mathscr{H})$ and σ_e is continuous at T, then σ is continuous at T.*

Proof. To prove the theorem it would suffice to prove that if $\{T_n\} \subset \mathscr{L}(\mathscr{H})$ is a sequence of operators such that

$$\lim_{n\to\infty} ||T_n - T|| = 0$$

for some operator $T \in \mathscr{L}(\mathscr{H})$, then

$$\text{acc}\sigma(T) \subseteq \liminf_n \sigma(T_n). \tag{2.2}$$

because it is well known [12] that the function σ is upper semi-continuous and

$$\text{iso}\sigma(T) \subseteq \liminf_n \sigma(T_n). \tag{2.3}$$

Asuume that $\lambda \in \text{acc}\sigma(T)$. First, let $\lambda \in \text{acc}\sigma(T) \cap \sigma_e(T)$. Since the function σ_e is continuous at T, then

$$\lambda \in \liminf_n \sigma_e(T_n) \subseteq \liminf_n \sigma(T_n).$$

Second, let $\lambda \in \text{acc}\sigma(T) \setminus \sigma_e(T)$. Then $T - \lambda$ is Fredholm. Assume to the contrary that $\lambda \notin \liminf \sigma(T_n)$. Then there exists a $\delta > 0$, a neighbourhood $\mathcal{N}_\delta(\lambda)$ of λ and a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ such that

$$\sigma(T_{n_k}) \cap \mathcal{N}_\delta(\lambda) = \emptyset \text{ for every } k \geq 1.$$

Evidently, $T_{n_k} - \lambda$ is Fredholm, with $\text{ind}(T_{n_k} - \lambda) = 0$, and

$$\lim_{n\to\infty} ||(T_{n_k} - \lambda) - (T - \lambda)|| = 0.$$

It follows from the continuity of the index that $\text{ind}(T - \lambda) = 0$, and so $T - \lambda$ is Weyl. This is a contradiction to (1.4) because Browder's theorem holds for T. □

COROLLARY 2.4. *Let $T \in \mathscr{L}(\mathscr{H})$ be biquasitriangular and Browder's theorem holds for T. Then the essential spectrum σ_e is continuous at T if and only if the spectrum σ is continuous at T.*

Proof. It is immediately follows from combining Theorem 2.2 and Theorem 2.3. □

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