Korean J. Math. **20** (2012), No. 4, pp. 441–449
http://dx.doi.org/10.11568/kjm.2012.20.4.441

ON THE QUASI-(θ, s)-CONTINUITY

SEUNGWOOK KIM

ABSTRACT. The quasi-(θ, s)-continuity is a weakened form of the weak (θ, s)-continuity and equivalent to the weak quasi-continuity. The basic properties of those functions are investigated in concern with the other weakened continuous functions. It turns out that the open property of a function and the extremall disconnectedness of the spaces are crucial tools for the survey of these functions.

1. Introduction

The concept of (θ, s)-continuous function is introduced by Joseph and Kwack ([2]) to investigate S-closed spaces due to Thompson ([7]). T. Noiri and Saeid Jafari ([1],[6]) obtained several properties of this function and the relationships between (θ, s)-continuity, contra-continuity, regular set-connectedness and other related functions.

The weak (θ, s)-continuity is a weakened form of (θ, s)-continuity. The properties of the weak (θ, s)-continuity and the relationships with the other generalized continuity are studied in ([4]). In the present paper we introduce the quasi-(θ, s)-continuity which is a weakened form of the weak (θ, s)-continuity and study here the properties of this function.

Received October 11, 2012. Revised December 6, 2012. Accepted December 10, 2012.

2010 Mathematics Subject Classification: 54C08.

Key words and phrases: weakly (θ, s)-continuous functions, quasi continuous functions, weakly quasi continuous functions, precontinuous functions, s-,S-closed space.

This work was supported by Hankook University of Foreign Studies Research Fund 2012.

In section 4 the relationships between the quasi-(θ, s)-continuity and some weakened continuous functions such as weakly quasi-, and precontinuous functions are investigated. In section 5 a weak s-quasi-continuity which implies weak quasi-continuity is defined and we will see that under certain conditions the weakly (θ, s)-, weakly quasi-, weakly s-quasi-continuous and precontinuous functions are equivalent. In sections 4,5 we will see that the open property of the functions and the extremall disconnectedness of the spaces play the essential role for a study of the equivalence of these functions. At the end of this paper, as a remark, a table clarifying the whole relationships of the functions appeared in this paper is established.

2. Preliminaries

Throughout the present paper, X and Y are always topological spaces. For a subset S of X we denote the interior and the closure of S by $\text{Int}(S)$ and $\text{Cl}(S)$, respectively. S is said to be *semi-open* ([3]) if $S \subset \text{Cl}(\text{Int}(S))$. The family of all semi-open sets of X is denoted by $SO(X)$. We set $SO(X, x) = \{S | x \in S, S \in SO(X)\}$. S is said to be *semi-closed* if $X \setminus S$ is semi-open. S is said to be *regular open* (resp. *regular closed*) if $S = \text{Int}(\text{Cl}(S))$(resp. $S = \text{Cl}(\text{Int}(S))$).

X is called *extremally disconnected* if a closure of an open set is open.

A function $f : X \to Y$ is said to be *weakly quasi-continuous at* $x \in X$ if to each open set V in Y containing $f(x)$ and to each open set U in X containing x there is a non empty open set G contained in U such that $G \subseteq f^{-1}(Cl(V)))$. $f : X \to Y$ is said to be *weakly quasi-continuous* if it is weakly quasi-continuous at each point $x \in X$.

3. The quasi-(θ, s)-continuous functions

In this section we introduce the concept of the quasi- (θ, s)-continuity and show that this is equivalent to the weakly quasi-continuity.

DEFINITION 3.1. *A function $f : X \to Y$ is quasi- (θ, s)-continuous at $x \in X$ if to each open $V \subset Y$ containing $f(x)$ there exists a semi-open set $R \subset X$ containing x such that $f(R) \subset Cl(V)$.*

$f : X \to Y$ is said to be quasi- (θ, s)-continuous if it is quasi- (θ, s)-continuous at each $x \in X$.

THEOREM 3.2. *A function $f : X \to Y$ is quasi- (θ, s)-continuous iff f is weakly quasi-continuous.*

Proof. " $\longleftarrow$ " : Let $x \in X$ and V be open in Y containing $f(x)$ and define the set $\mathcal{G} := \bigcup\{W | W \subseteq X$, open and $f(W) \subseteq Cl(V)\}$. Then $\mathcal{G} \cup \{x\}$ is the desired semi-open set.

" $\longrightarrow$ " : Let $x \in X$ and V be open in Y containing $f(x)$. Let W be open subset in X containing x. There exists a semi-open R in X containing x such that $f(R) \subseteq Cl(V)$. Then $\emptyset \neq R \cap W \subseteq f^{-1}(Cl(V))$ where $R \cap W$ is semi-open. Hence there exists a non empty open set $T \subseteq R \cap W$ such that $f(T) \subseteq Cl(V)$. □

THEOREM 3.3. *Let $f : X \to Y$ be open. The following are equivalent.*

(a) *f is quasi- (θ, s)-continuous.*
(b) *$f^{-1}(Cl(S))$ is semi-open for each semi-open $S \subset Y$.*
(c) *$f^{-1}(Cl(V))$ is semi-open for each open $V \subset Y$.*

Proof. "$(a) \longrightarrow (b)$" : Let $S \subseteq Y$ be semi-open. We show

$$f^{-1}(Cl(S)) \subseteq Cl(Int(f^{-1}(Cl(S)))).$$

Let $y \in f^{-1}(Cl(S))$ and W be an open subset of X containing y. Then $f(y) \in Cl(S) = Cl(Q)$ where Q is an open set in Y. Thus $f(W) \cap Q \neq \emptyset$. There exists $z \in W$ such that $f(z) \in Q$. By the quasi- (θ, s)-continuity of f there exists a semi-open R such that $x \in R$ and $R \subseteq f^{-1}(Cl(Q))$. Since $W \cap R$ is semi-open set containing x, there exists a non empty open $G \subseteq W \cap R$ such that

$$G \subseteq f^{-1}(Cl(Q)) = f^{-1}(Cl(S)).$$

Therefore $G \subseteq Int(f^{-1}(Cl(S)))$ that means $W \cap Int(f^{-1}(Cl(S))) \neq \emptyset$.

The proof for "$(b) \longrightarrow (c)$" is obvious and "$(c) \longrightarrow (a)$" follows easily from the fact that the intersection of semi-open set and open set is semi-open. □

4. The relationships between functions

Recall that a function $f : X \to Y$ is said to be (θ, s)-continuous if for each $x \in X$ and each $S \in SO(Y, f(x))$, there exists an open set U in X containing x such that $f(U) \subset Cl(S)$. Next we introduce a weakened form of (θ, s)-continuity and show partial results about this function ([4]).

Definition 4.1. *A function $f : X \to Y$ is said to be weakly (θ, s)-continuous if for each $x \in X$ and each $S \in SO(Y, f(x))$, there exists $R \in SO(X, x)$ such that $f(R) \subset Cl(S)$.*

The following example shows that the weak (θ, s)-continuity does not imply the (θ, s)-continuity.

Example 4.2. *A function*

$$f : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} 1 \text{ if } x \in [0, \infty) \\ 0 \text{ if } x \in (-\infty, 0) \end{cases}$$

where on both $\mathbb{R}$ the natural topologies are given is weakly (θ, s)-continuous, but not (θ, s)-continuous at $x = 0$.

Through the following counterexample and corollary we can see that the weak quasi-continuity is generaller than the weak (θ, s)-continuity ([4]).

Example 4.3. *Consider a function*

$$f : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} x + 1 \text{ if } x \in [0, \infty) \\ x \text{ if } x \in (-\infty, 0) \end{cases}$$

where on both $\mathbb{R}$ the natural topologies are given. Then f is weakly quasi-continuous, but, considering the semi-open set $(0, 1]$ containing $f(0)$, we see that f is not weakly (θ, s)-continuous at $x = 0$.

Corollary 4.4. *Every weakly (θ, s)-continuous function is weakly quasi-continuous and also quasi-(θ, s)-continuous by Theorem 3.2.*

Remark 4.5. *The next implications are valid.*
The (θ, s)-continuity $\Rightarrow$ the weak (θ, s)-continuity $\Rightarrow$ the weak quasi-continuity $\Leftrightarrow$ the quasi-(θ, s)-continuity.

In [4] the following statements are shown:

Theorem 4.6. *The following are equivalent for a function $f : X \to Y$.*

(a) *f is weakly-(θ, s)-continuous.*
(b) *$f^{-1}(Cl(S))$ is semi-open for a semi-open set S in Y.*
(c) *$f^{-1}(R)$ is semi-open for a regular closed set R in Y.*
(d) *$f^{-1}(T)$ is semi-closed for a regular open set T in Y.*
(e) *$f^{-1}(S)$ is semi-open for a θ-semi-open set S in Y.*

With Theorems 3.2, 3.3 and 4.6 we have:

Theorem 4.7. *Let $f : X \to Y$ be open. Then the following are equivalent.*

(a) *f is weakly (θ, s)-continuous.*
(b) *f is weakly quasi-continuous.*
(c) *f is quasi- (θ, s)-continuous*

Recall that a function $f : X \to Y$ is said to be *precontinuous at* $x \in X$ if for each neighborhood V of $f(x)$ in Y, $Cl(f^{-1}(V))$ is a neighborhood of x. $f : X \to Y$ is called *precontinuous* if it is precontinuous at each $x \in X$. To obtain relations between the weak (θ, s)-, weak quasi-and precontinuity we need the following.

In [4] we obtained the following result concerned with the precontinuity.

DEFINITION 4.8. *If $Cl(A \cap B) = Cl(A) \cap Cl(B)$ for every $A, B \subset X$, we say that X satisfies the closure equality.*

THEOREM 4.9. *Let X satisfy the closure equality. If $f : X \to Y$ precontinuous and open, then $f^{-1}(Cl(S))$ $=Cl(f^{-1}(S))$for each semi-open $S \subset Y$.*

THEOREM 4.10. *Let X satisfy the closure equality and be extremally disconnected. If $f : X \to Y$ is open, the following are equivalent. ([4])*
(a) *f is weakly (θ, s)-continuous.*
(b) *f is weakly quasi-continuous.*
(c) *f is precontinuous.*

According to theorem 4.7 we can add one equivalent condition to theorem 4.10 as follows.

THEOREM 4.11. *Let X satisfy the closure equality and be extremally disconnected. If $f : X \to Y$ is open, the following are equivalent.*
(a) *f is weakly (θ, s)-continuous.*
(b) *f is weakly quasi-continuous.*
(c) *f is quasi- (θ, s)-continuous.*
(d) *f is precontinuous.*

THEOREM 4.12. *Let $f : X \to Y$ be precontinuous and open. If Y is extremally disconnected, $Cl(f^{-1}(S))$ is semi-open for every semi-open set $S \subset Y$.*

Proof. Let $S \subseteq Y$ be semi-open. We show $\text{Cl}(f^{-1}(S)) \subseteq \text{Cl}(\text{Int}(\text{Cl}(f^{-1}(S))))$. Let $x \in \text{Cl}(f^{-1}(S))$ and $x \in W, W \subseteq X$ be open . Then $W \cap f^{-1}(S) \neq \emptyset$, thus there exists $y \in W$ such that $f(y) \in S$. There exists an open set T in Y such that Cl(S) =Cl(T) which is also open in Y. By precontinuity of f there exists an open set Q in X containing y such that $Q \subseteq Cl(f^{-1}(\mathrm{Cl}(S)))$. By openness of f, $Cl(f^{-1}(Cl(\mathrm{S})))$=Cl($f^{-1}(S)$). Hence

$$\emptyset \neq W \cap Q \subseteq Int(Cl(f^{-1}(S))).$$

Therefore $W\cap$ Int(Cl($f^{-1}Cl(S)$))) $\neq \emptyset$ that means $x \in$Cl(Int(Cl(f^{-1} (S)))). □

By Theorems 4.6, 4.9 and 4.12 the following hold.

THEOREM 4.13. *Let X satisfy the closure property and Y be extremally disconnected. If $f : X \to Y$ is open, the following are equivalent.*

(a) *f is weakly (θ, s)-continuous.*
(b) *f is weakly quasi-continuous.*
(c) *f is quasi- (θ, s)-continuous.*
(d) *f is precontinuous.*

5. s-Quasi-continuous functions

DEFINITION 5.1. *A function $f : X \to Y$ is called weakly s-quasi-continuous at x if for each open subset V of Y containing $f(x)$ and for each $S \in SO(X, x)$, there exists a non empty $R \in SO(X)$ such that $R \subseteq S$ and $R \subseteq f^{-1}(Cl(V))$.*

f is said to be weakly s-quasi-continuous, if it is weakly s-quasi-continuous at each x of X.

It is easy to see that the weak s-quasi-continuity implies the weak quasi-continuity, but the same function of example 4.2 shows that the converse of both cases does not hold.

THEOREM 5.2. *If a function $f : X \to Y$ is weakly quasi-continuous and X is extremally disconnected, then f is weakly s-quasi-continuous.*

Proof. Let $x \in X$ and V be an open set in Y containing $f(x)$. Let $S \in$SO(X, x). There exists an open T in X such that Cl(T) =Cl(S) which is also open in X. Hence there exists an non empty open set $G \subset$Cl(T) such that $G \subset f^{-1}$(Cl(V)). From $G \subset$Cl(S) follows

$$\emptyset \neq G \cap S \subset f^{-1}(Cl(V)) \cap S$$

where $G \cap S$ is semi-open. □

THEOREM 5.3. *Let X be extremally disconnected. Then the following are equivalent.*

(a) *$f : X \to Y$ is weakly quasi-continuous.*
(b) *f is weakly s-quasi-continuous.*

Due to theorem 4.7, if the condition of openness of a function f is added in theorem 5.3, we have

THEOREM 5.4. *Let $f : X \to Y$ be open and X be extremally disconnected. Then the following are equivalent.*

(a) *f is weakly (θ, s)-continuous.*
(b) *f is weakly quasi-continuous.*
(c) *f is quasi-(θ, s)-continuous.*
(d) *f is weakly s-quasi-continuous.*

THEOREM 5.5. *If the conditions of the theorem 4.11 are satisfied in 5.4, then the following are equivalent.*

(a) *f is weakly (θ, s)-continuous.*
(b) *f is weakly quasi-continuous.*
(c) *f is quasi-(θ, s)-continuous.*
(d) *f is weakly s-quasi-continuous.*
(e) *f is precontinuous.*

THEOREM 5.6. *If a function $f : X \to Y$ is weakly s-quasi-continuous and Y is extremally disconnected, there exists an open $T \subseteq X$ such that $Cl(T) = Cl(f^{-1}(Cl(S)))$ for all $S \in SO(Y)$.*

Proof. Let $S \in$SO(Y). Define

$$\mathcal{R} = \bigcup\{R | R \in SO(X), R \subset f^{-1}(Cl(S))\}$$

and it is enough to show that Cl$(\mathcal{R})$ =Cl$(f^{-1}($Cl$(S)))$. Let $x \in$Cl$(f^{-1}($ Cl$(S)))$ and $x \in U$ which is open in X. Then $U \cap f^{-1}($Cl$(S)) \neq \emptyset$. Hence there exists $y \in U$ such that $f(y) \in$Cl(S). Since Cl(S) is open in Y and containing $f(y)$, there exists non empty $R \in$SO(X) such that $R \subset U \cap f^{-1}($Cl$(S))$. From this follows $R \subset \mathcal{R}$. Thus $R \subset U \cap \mathcal{R}$ which is not empty. Hence $x \in$Cl$(\mathcal{R})$. The other inclusion is obvious. □

By Theorem 4.11 the next theorem holds.

THEOREM 5.7. *Let X satisfy the closure equality and Y be extremally disconnected. Then the following are equivalent.*

(a) $f : X \to Y$ *is weakly quasi-continuous.*
(b) f *is weakly s-quasi-continuous.*

Theorem 5.8. *Let $f : X \to Y$ be open, X satisfy the closure property and Y be extremally disconnected. Then the following are equivalent.*

(a) f *is weakly (θ, s)-continuous.*
(b) f *is weakly quasi-continuous.*
(c) f *is quasi-(θ, s)-continuous.*
(d) f *is weakly s-quasi-continuous.*
(e) f *is precontinuous.*

Remark 5.9. *Here a table is established to clarify the whole relationships between the functions considered in this paper as follows:*

X	Y	f	Equivalence	Theorem
		open	w.(θ, s)-, w.q.-,q.-(θ, s)-	4.7
CE+ED		open	w.(θ, s)-,w.q.-,q.-(θ, s)-, prec.	4.11
CE	ED	open	w.(θ, s)-,w.q.-,q.-(θ, s)-, prec.	4.13
ED			w.q.-,w.s-q.-	5.3
ED		open	w.(θ, s)-, w.q.-,q.-(θ, s)-,w.s-q.-	5.4
CE+ED		open	w.(θ, s)-,w.q.-,q.-(θ, s)-, prec.,w.s-q.-	5.5
CE	ED		w.q.-, w.s-q.-	5.7
CE	ED	open	w.(θ, s)-,w.q.-,q.-(θ, s)-, prec.,w.s-q.-	5.8

CE=closure equality, ED=extremally disconnected

References

[1] G. Di Maio and T. Noiri, *On s-closed spaces*, Indian J. pure appl. Math. **18** (1978), 226–233.
[2] J.E. Joseph, M.H. Kwack, *On S-closed Spaces*, Proc. Amer. Math. Soc. **80** (1980), 341–348.
[3] N. Levin, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41.
[4] S. Kim, *On weakened forms of (θ, s)-continuity*, Korean J. Math. **14** (2) (2006), 249–258.
[5] S.N. Maheswari and R. Prasad, *On s-regular spaces*, Glasnik Mat. **10** (1975), 347–350.
[6] T. Noiri, S. Jafari, *Properties of (θ, s)-continuous functions*, Topology and its Applications, **123** (2002), 167–179.
[7] T. Thomsen, *S-closed spaces*, Proc. Amer.Math. Soc. **60** (1976), 335–338.

Department of Mathematics
Hankuk University of Foreign Studies
Yonginshi, Kyunggido, 449-791
Korea
E-mail: mathwook@hufs.ac.kr