Korean J. Math. **24** (2016), No. 1, pp. 41–49
http://dx.doi.org/10.11568/kjm.2016.24.1.41

THE HYERS-ULAM STABILITY OF A QUADRATIC FUNCTIONAL EQUATION WITH INVOLUTION IN PARANORMED SPACES

Chang Il Kim and Chang Hyeob Shin*

Abstract. In this paper, using fixed point method, we prove the Hyers-Ulam stability of the following functional equation

$$\begin{aligned} & f(x+y+z)+f(\sigma(x)+y+z)+f(x+\sigma(y)+z)+f(x+y+\sigma(z)) \\ = \;& 4f(x)+4f(y)+4f(z) \end{aligned}$$

with involution in paranormed spaces.

1. Introduction and Preliminaries

In 1940, Ulam [19] proposed the following problem concerning the stability of group homomorphism: *Let G_1 be a group and let G_2 a meric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?*

Hyers [7] solved the Ulam's problem for the case of approximately additive functions in Banach spaces. Since then, the stability of several functional equations have been extensively investigated by several mathematicians [2, 3, 6, 8, 9, 12, 14–16].

Received December 22, 2015. Revised January 6, 2016. Accepted January 25, 2016.

2010 Mathematics Subject Classification: 39B82, 39B52.

Key words and phrases: quadratic functional equation, involution, fixed point method, paranormed space.

∗Corresponding Author.

Let X and Y be real vector spaces. For an additive mapping $\sigma : X \longrightarrow X$ with $\sigma(\sigma(x)) = x$ for all $x \in X$, σ is called *an involution* of X [1, 18]. Stetkær [18] introduced the following quadratic functional equation with involution

$$f(x+y) + f(x+\sigma(y)) = 2f(x) + 2f(y) \tag{1.1}$$

and solved the general solution and Belaid et al. [1] established generalized Hyers-Ulam stability in Banach spaces for this functional equation. Jung and Lee [11] investigated the Hyers-Ulam-Rassias stability of (1.1) in a complete β-normed space, using fixed point method.

The concept of statistical convergence for sequences of real numbers was introduced by Fast [5] and Steinhaus [17] independently and since then several generalizations and applications of this notion have been investigated by various authors. This notion was defined in normed spaces by Kolk [13].

Definition 1.1. Let X be a vector space. A paranorm $P : X \longrightarrow [0, \infty)$ is a function on X such that
(1) $P(0) = 0$;
(2) $P(-x) = P(x)$;
(3) $P(x + y) \leq P(x) + P(y)$ (triangle inequality);
(4) If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_n x_n - tx) \to 0$ (continuity of multiplication).
The pair (X, P) is called *a paranormed space* if P is *a paranorm* on X.
The paranorm is called *total* if, in addition, we have
(5) $P(x) = 0$ implies $x = 0$. A *Frěchet space* is a total and complete paranormed space.

Let (X, d) be a generalized metric space. An operator $T : X \longrightarrow X$ satisfies a Lipschitz condition with the Lipschitz constant L if there exists a constant $L \geq 0$ such that $d(Tx, Ty) \leq Ld(x, y)$ for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the operator T is called a *strictly contractive operator*. Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity.

We recall a fundamental result in fixed point theories.

Theorem 1.2. [4] *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with some Lipschitz constant L with $0 < L < 1$. Then for each given element $x \in X$,*

either $d(J^n x, J^{n+1}x) = \infty$ *for all nonnegative integers* n *or there exists a positive integer* n_0 *such that*
(1) $d(J^n x, J^{n+1}x) < \infty$ *for all* $n \geq n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* x^* *of* J ;
(3) x^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$ and
(4) $d(y, y^*) \leq \dfrac{1}{1-L}\, d(y, Jy)$ *for all* $y \in Y$.

In 1996, Issac and Rassias [10] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorem with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors.

In this paper, using fixed point method, we prove the generalized Hyers-Ulam stability of the following functional equation

$$\begin{aligned} &f(x+y+z) + f(\sigma(x)+y+z) + f(x+\sigma(y)+z) \\ &+ f(x+y+\sigma(z)) = 4f(x) + 4f(y) + 4f(z) \end{aligned} \tag{1.2}$$

Throughout this paper, we assume that X is a Frěchet space and Y is a Banach space.

2. The generalized Hyers-Ulam stability for (1.2)

Using the fixed point methods, we will prove the generalized Hyers-Ulam stability of the quadratic functional equation (1.2) with involution σ in paranormed spaces. For a given mapping $f : X \longrightarrow Y$, we define the difference operator $Df : X^3 \longrightarrow Y$ by

$$\begin{aligned} &Df(x,y,z) = f(x+y+z) + f(\sigma(x)+y+z) \\ + \quad &f(x+\sigma(y)+z) + f(x+y+\sigma(z)) - 4f(x) - 4f(y) - 4f(z) \end{aligned}$$

for all $x, y, z \in X$.

Lemma 2.1. *Let* $f : X \longrightarrow Y$ *be a mapping. Then* f *satisfies (1.2) if and only if* f *satisfies (1.1).*

Proof. Suppose that f satisfies (1.2). Letting $x = y = z = 0$ in (1.2), we have $f(0) = 0$. Letting $y = z = 0$ in (1.2), we have $f(x) = f(\sigma(x))$ for all $x \in X$. Letting $z = 0$ in (1.2), we have

$$f(x+y) + f(x+\sigma(y)) = 2f(x) + 2f(y)$$

for all $x, y \in X$.

Assume that f satisfies (1.1). We have

$$\begin{aligned} & f(x+y+z) + f(\sigma(x)+y+z) + f(x+\sigma(y)+z) + f(x+y+\sigma(z)) \\ = \; & 2f(x+y) + 2f(z) + 2f(x+\sigma(y)) + 2f(z) \\ = \; & 4f(x) + 4f(y) + 4f(z) \end{aligned}$$

for all $x, y, z \in X$. □

THEOREM 2.2. *Assume that $\phi : X^3 \longrightarrow [0, \infty)$ is a mapping and there exists a real number L with $0 < L < 1$ such that*

$$\begin{aligned} & \phi(2x, 2y, 2z) \le 2L\phi(x, y, z), \\ & \phi(x+\sigma(x), y+\sigma(y), z+\sigma(z)) \le 2L\phi(x, y, z) \end{aligned} \tag{2.1}$$

for all $x, y, z \in X$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$\|Df(x, y, z)\| \le \phi(x, y, z) \tag{2.2}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \longrightarrow Y$ with involution such that

$$\|f(x) - Q(x)\| \le \frac{1}{8(1-L)}\phi(x, x, 0) \tag{2.3}$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d in S defined by

$$d(g, h) = inf\{c \in [0, \infty) \mid \|g(x) - h(x)\| \le c\ \phi(x, x, 0)\ for\ all\ x \in X\}$$

for all $g, h \in S$. Then (S, d) is a complete metric space(See [11]). Define a mapping $J : S \longrightarrow S$ by

$$Jg(x) = \frac{1}{4}\{g(2x) + g(x+\sigma(x))\}$$

for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g, h) \le c$ for some non-negative real number c. Then by (2.1), we have

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= \frac{1}{4}\ \|g(2x) + g(x+\sigma(x)) - h(2x) - h(x+\sigma(x))\| \\ &\le \frac{1}{4}\Big[\|g(2x) - h(2x)\| + \|g(x+\sigma(x)) - h(x+\sigma(x))\|\Big] \\ &\le cL\phi(x, x, 0) \end{aligned}$$

for all $x \in X$. Hence we have $d(Jg, Jh) \leq Ld(g, h)$ for any $g, h \in S$ and so J is a strictly contractive mapping.

Putting $y = x$ and $z = 0$ in (2.2) and dividing both sides by 8, we get

$$\|Jf(x) - f(x)\| = \left\|\frac{1}{4}\{f(2x) + f(x + \sigma(x))\} - f(x)\right\| \leq \frac{1}{8}\phi(x, x, 0)$$

for all $x \in X$ and hence

$$d(Jf, f) \leq \frac{1}{8} < \infty. \tag{2.4}$$

By Theorem 1.2, there exists a mapping $Q : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, Q) \to 0$ as $n \to \infty$. By induction, we can easily show that

$$(J^n f)(x) = \frac{1}{2^{2n}}\Big\{f(2^n x) + (2^n - 1)f\Big(2^{n-1}\big(x + \sigma(x)\big)\Big)\Big\}$$

for all $x \in X$ and $n \in \mathbb{N}$. Thus for each fixed $x \in X$, we have

$$Q(x) = \lim_{n\longrightarrow\infty} \frac{1}{2^{2n}}\Big\{f(2^n x) + (2^n - 1)f\Big(2^{n-1}\big(x + \sigma(x)\big)\Big)\Big\}. \tag{2.5}$$

From (2.2) and (2.5), we have

$$\begin{aligned}
&\|DQ(x, y, z)\| \\
\leq\ & \lim_{n\longrightarrow\infty} \frac{1}{2^{2n}}\big[\phi(2^n x, 2^n y, 2^n z) \\
&+(2^n - 1)\phi\big(2^{n-1}(x + \sigma(x)), 2^{n-1}(y + \sigma(y)), 2^{n-1}(z + \sigma(z))\big)\big] \\
\leq\ & \lim_{n\longrightarrow\infty} \frac{1}{2^{2n}}\big[2^n L^n \phi(x, y, z) + (2^n - 1)2^n L^n \phi(x, y, z)\big] = 0
\end{aligned}$$

for all $x, y \in X$. Hence Q satisfies (1.2) and by Lemma 2.1, Q is a quadratic mapping with involution. By (4) in Theorem 1.2 and (2.4), Q satisfies (2.3).

Assume that $Q_1 : X \longrightarrow Y$ is a quadratic mapping with involution satisfying (2.3). We know that Q_1 is a fixed point of J. Due to (3) in Theorem 1.2, we get $Q = Q_1$. This proves the uniqueness of Q. □

Corollary 2.3. *Let r be a positive real number $r < 1$, and let $f : X \longrightarrow Y$ be a mapping such that*

$$\|Df(x, y, z)\| \leq P(x)^r + P(y)^r + P(z)^r \tag{2.6}$$

for all $x, y, z \in X$ and suppose that $x + \sigma(x) \leq 2x$. Then there exists a unique quadratic mapping $Q : X \longrightarrow Y$ with involution such that

$$\|f(x) - Q(x)\| \leq \frac{P(x)^r}{2(2-2^r)} \tag{2.7}$$

for all $x \in X$.

Proof. Taking $\phi(x, y, z) = P(x)^r + P(y)^r + P(z)^r$ for all $x, y, z \in X$ and $L = 2^{r-1}$ in Theorem 2.2, we get desired result. □

THEOREM 2.4. *Assume that $\phi : Y^3 \longrightarrow [0, \infty)$ is a mapping and there exists a real number L with $0 < L < 1$ such that*

$$\begin{aligned} &\phi(x, y, z) \leq \frac{L}{8}\phi(2x, 2y, 2z), \\ &\phi(x + \sigma(x), y + \sigma(y), z + \sigma(z)) \leq 2\phi(2x, 2y, 2z) \end{aligned} \tag{2.8}$$

for all $x, y, z \in Y$. Let $f : Y \longrightarrow X$ be a mapping such that $f(0) = 0$ and

$$P(Df(x, y, z)) \leq \phi(x, y, z) \tag{2.9}$$

for all $x, y, z \in Y$. Then there exists a unique quadratic mapping $Q : Y \longrightarrow X$ with involution such that

$$P(4f(x) - Q(x)) \leq \frac{L}{2(1-L)}\phi(x, x, 0) \tag{2.10}$$

for all $x \in Y$.

Proof. Consider the set $S = \{g \mid g : Y \longrightarrow X\}$ and the generalized metric d in S defined by $d(g, h) = inf\{c \in [0, \infty)|\ P(g(x) - h(x)) \leq c\ \phi(x, x, 0)$ for all $x \in Y\}$. Then (S, d) is a complete metric space. Define a mapping $J : S \longrightarrow S$ by

$$Jg(x) = 2\Big\{2g\Big(\frac{x}{2}\Big) - g\Big(\frac{x + \sigma(x)}{4}\Big)\Big\}$$

for all $x \in Y$ and all $g \in S$. Let $g, h \in S$ and $d(g, h) \leq c$ for some non-negative real number c. Then by (2.8), we have

$$\begin{aligned} &P(Jg(x) - Jh(x)) \\ \leq\ & 2\Big[2P\Big(g\Big(\frac{x}{2}\Big) - h\Big(\frac{x}{2}\Big)\Big) + P\Big(g\Big(\frac{x + \sigma(x)}{4}\Big) - h\Big(\frac{x + \sigma(x)}{4}\Big)\Big)\Big] \\ \leq\ & 2\Big[2c\phi\Big(\frac{x}{2}, \frac{x}{2}, 0\Big) + c\phi\Big(\frac{x + \sigma(x)}{4}, \frac{x + \sigma(x)}{4}, 0\Big)\Big] \\ \leq\ & cL\phi(x, x, 0) \end{aligned}$$

for all $x \in Y$. Hence $d(Jg, Jh) \leq Ld(g, h)$ for any $g, h \in S$ and so J is a strictly contractive mapping.

Putting $x = \frac{x}{2}$, $y = \frac{x}{2}$ and $z = 0$ in (2.9), we get

$$P\Big(2f(x) + 2f\Big(\frac{x+\sigma(x)}{2}\Big) - 8f\Big(\frac{x}{2}\Big)\Big) \leq \phi\Big(\frac{x}{2}, \frac{x}{2}, 0\Big) \tag{2.11}$$

for all $x \in Y$. By (2.11), we get

$$P\Big(4f(x) + 4f\Big(\frac{x+\sigma(x)}{2}\Big) - 16f\Big(\frac{x}{2}\Big)\Big) \leq 2\phi\Big(\frac{x}{2}, \frac{x}{2}, 0\Big) \tag{2.12}$$

and putting $x = \frac{x+\sigma(x)}{4}$ and $y = \frac{x+\sigma(x)}{4}$, $z = 0$ in (2.9), we get

$$\begin{aligned} &P\Big(4f\Big(\frac{x+\sigma(x)}{2}\Big) - 8f\Big(\frac{x+\sigma(x)}{4}\Big)\Big) \\ &\leq \phi\Big(\frac{x+\sigma(x)}{4}, \frac{x+\sigma(x)}{4}, 0\Big) \end{aligned} \tag{2.13}$$

for all $x \in Y$. Combining (2.12) and (2.13), by (2.8), we deduce that

$$\begin{aligned} P\big(J4f(x) - 4f(x)\big) &= P\Big(16f\Big(\frac{x}{2}\Big) - 8f\Big(\frac{x+\sigma(x)}{4}\Big) - 4f(x)\Big) \\ &\leq \frac{L}{2}\phi(x, x, 0) \end{aligned}$$

for all $x \in Y$ and hence

$$d(J4f, 4f) \leq \frac{L}{2} < \infty. \tag{2.14}$$

By Theorem 1.2, there exists a mapping $Q : Y \longrightarrow X$ which is a fixed point of J such that $d(J^n 4f, Q) \to 0$ as $n \to \infty$. By induction, we can easily show that

$$(J^n 4f)(x) = 4^n f\Big(\frac{x}{2^n}\Big) + 2^n(1 - 2^n) f\Big(\frac{x+\sigma(x)}{2^{n+1}}\Big)$$

for each $n \in \mathbb{N}$. Thus for each fixed $x \in Y$, we have

$$Q(x) = \lim_{n\longrightarrow\infty} 4^n f\Big(\frac{x}{2^n}\Big) + 2^n(1 - 2^n) f\Big(\frac{x+\sigma(x)}{2^{n+1}}\Big). \tag{2.15}$$

It follows from (2.9) and (2.15) that

$$\begin{aligned} P\big(DQ(x, y, z)\big) &\leq \lim_{n\longrightarrow\infty} 4^n \Big\{\phi\Big(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\Big) \\ &+\Big(\frac{1}{2^n} - 1\Big)\phi\Big(\frac{x+\sigma(x)}{2^{n+1}}, \frac{y+\sigma(y)}{2^{n+1}}, \frac{z+\sigma(z)}{2^{n+1}}\Big)\Big\} = 0 \end{aligned}$$

for all $x, y, z \in Y$. Hence Q satisfies (1.2) and Q is a quadratic mapping with involution. By (4) in Theorem 1.2 and (2.14), Q satisfies (2.10).

Assume that $Q_1 : Y \longrightarrow X$ is a quadratic mapping with involution satisfying (2.10). We know that Q_1 is a fixed point of J. Due to (3) in Theorem 1.2, we get $Q = Q_1$. This proves the uniqueness of Q. $\square$

COROLLARY 2.5. *Let r, θ be positive real numbers with $r > 3$, and let $f : Y \longrightarrow X$ be a mapping such that*

$$P\big(Df(x,y,z)\big) \le \theta(\|x\|^r + \|y\|^r + \|z\|^r) \text{ and } \|x + \sigma(x)\| \le 2\|x\| \tag{2.16}$$

for all $x, y, z \in Y$. Then there exists a unique quadratic mapping $Q : Y \longrightarrow X$ such that

$$P\big(4f(x) - Q(x)\big) \le \frac{2^{3-r}}{1 - 2^{3-r}}\theta\|x\|^r \tag{2.17}$$

for all $x \in Y$.

Proof. Taking $\phi(x,y,z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in Y$ and $L = 2^{3-r}$ in Theorem 2.4, we get desired result. $\square$

References

[1] B. Boukhalene, E. Elqorachi, and Th. M. Rassias, *On the generalized Hyers-Ulam stability of the quadratic functional equation with a general involution*, Nonlinear Funct. Anal. Appl. **12** (2) (2007), 247–262.

[2] B. Boukhalene, E. Elqorachi, and Th. M. Rassias, *On the Hyers-Ulam stability of approximately pexider mappings*, Math. Ineqal. Appl. **11** (2008), 805–818.

[3] S. Czerwik, *Functional equations and Inequalities in several variables*, World Scientific, New Jersey, London, 2002.

[4] J. B. Diaz, Beatriz Margolis *A fixed point theorem of the alternative, for contractions on a generalized complete metric space* Bull. Amer. Math. Soc. **74** (1968), 305–309.

[5] H. Fast, *Sur la Convergence statistique*, Colloq. Math. **2** (1951), 241–244.

[6] G. L. Forti, *Hyers-Ulam stability of functional equations in several variables*, Aequationes Math. **50** (1995), 143–190.

[7] D. H. Hyers, *On the stability of linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222–224.

[8] D. H. Hyers, G. Isac, and T. M. Rassias, *Stability of functional equations in several variables*, Birkhäuser, Boston, 1998.

[9] D. H. Hyers, T. M. Rassias, *Approximate homomorphisms*, Aequationes Math. **44** (1992), 125–153.

[10] G. Isac, Th. M. Rassias, *Stability of ψ-additive mappings: Applications to nonlinear analysis*, Internat. J. Math. & Math. Sci. **19** (1996), 219-228.

[11] S. M. Jung, Z. H. Lee, *A fixed point approach to the stability of quadratic functional equation with involution*, Fixed Point Theory Appl. 2008.

[12] S. M. Jung, *On the Hyers-Ulam stability of the functional equation that have the quadratic property*, J. Math. Anal. Appl. **222** (1998), 126–137.

[13] E. Kolk, *The statistical convergence in Banach spaces*, Tartu Ul. Toime. **928** (1991), 41–52.

[14] M. S. Moslehian, T. M. Rassias, *Stability of functional equations in non-Archimedean spaces*, Applicable Analysis and Discrete Mathematics, **1** (2007), 325–334.

[15] M. S. Moslehian, GH. Sadeghi, *Stability of two types of cubic functional equations in non-Archimedean spaces*, Real Analysis Exchange **33** (2) (2007/2008), 375–384.

[16] F. Skof, *Approssimazione di funzioni δ-quadratic su dominio restretto*, Atti. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **118** (1984), 58–70.

[17] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math. **2** (1951), 73–74.

[18] H. Stetkær, *Functional equations on abelian groups with involution*, Aequationes Math. **54** (1997), 144–172.

[19] S. M. Ulam, *A collection of mathematical problems*, Interscience Publ., New York, 1960.

Chang Il Kim
Department of Mathematics Education
Dankook University
Yongin 448-701, Korea
E-mail: kci206@hanmail.net

Chang Hyeob Shin
Department of Mathematics
Soongsil University
Seoul 156-743, Korea
E-mail: seashin@hanmail.net