Korean J. Math. **23** (2015), No. 4, pp. 607–618
http://dx.doi.org/10.11568/kjm.2015.23.4.607

DISTANCE TWO LABELING ON THE SQUARE OF A CYCLE

Xiaoling Zhang

Abstract. An $L(2,1)$-labeling of a graph G is a function f from the vertex set $V(G)$ to the set of all non-negative integers such that $|f(u)-f(v)| \geq 2$ if $d(u,v)=1$ and $|f(u)-f(v)| \geq 1$ if $d(u,v)=2$. The λ-number of G, denoted $\lambda(G)$, is the smallest number k such that G admits an $L(2,1)$-labeling with $k = \max\{f(u)|u \in V(G)\}$. In this paper, we consider the square of a cycle and provide exact value for its λ-number. In addition, we also completely determine its edge span.

1. Introduction

The notion of $L(2,1)$-labeling was proposed by Griggs and Yeh [6], which arose from a variation of the channel assignment problem introduced by Hale [8]. Suppose we are given a number of transmitters or stations. The $L(2,1)$-labeling problem is to assign frequencies (nonnegative integers) to the transmitters so that "close" transmitters must receive different frequencies and "very close" transmitters must receive frequencies that are at least two frequencies apart.

Received September 2, 2015. Revised November 26, 2015. Accepted November 30, 2015.

2010 Mathematics Subject Classification: 05C15.

Key words and phrases: Channel assignment; $L(2,1)$-labeling; square of a cycle; λ-number; edge span.

Research supported by Science Foundation of the Fujian Province, China (No. 2015J05013).

To formulate the problem in graphs, the transmitters are represented by the vertices of a graph; two vertices are "very close" if they are adjacent in the graph and "close" if they are of distance two in the graph. More precisely, an $L(2,1)$-*labeling* of a graph G is a function f from the vertex set $V(G)$ to the set of all non-negative integers such that $|f(u)-f(v)| \geq 2$ if $d(u,v)=1$ and $|f(u)-f(v)| \geq 1$ if $d(u,v)=2$. The λ-*number* of G, denoted $\lambda(G)$, is the smallest number k such that G admits an $L(2,1)$-labeling with $k = \max\{f(u) | u \in V(G)\}$. If an $L(2,1)$-labeling uses labels in the set $\{0,1,\cdots,k\}$, it will be called an k-$L(2,1)$-*labeling*.

The $L(2,1)$-labeling problem has been widely studied over the past decades [4, 5, 16]. Griggs and Yeh showed that the general problem of determining λ-number of a graph is NP-complete; Moreover, this problem remains NP-complete even for graphs with diameter two [6]. So it is not possible to compute λ-number of a graph in polynomial time unless P = NP. Therefore, the problem has been studied for many special classes of graphs, such as regular grids [1, 2], product graphs [10, 14], trees [4, 18], planar graphs [17], generalized flowers [11], permutation and bipartite permutation graphs [15] and so on. For more details, one may refer to the surveys [3, 19].

The r-*th power of a graph* G, denoted G^r, is a graph on the same vertex set such that two vertices are joined by an edge if and only if their distance in G is at most r. In particular, we also call the 2-th power as the square. In [13], Kohl studied the $L(d,1)$-labeling of r-th power of a cycle for all $r \in \mathbb{N}^+$ and $d \geq 3$. However, he did not obtain the exact λ-number of r-th power of a cycle even for $r=2$. Hence the problem of determining λ-number of r-th power of a cycle is clearly welcome.

A lot of variants other than $L(2,1)$-labeling problem have been also introduced, e.g., optimize the number of effectively used colors [8], consider the color set as a cyclic interval [9], use a more general model in which the labels and separations are real numbers [7], minimize the edge span [20], and so on. The $L(2,1)$ edge span of a graph G, denoted $\beta(G)$, is defined to be the minimum of $\beta(G,f)$ over all the $L(2,1)$-labelings f of G, where $\beta(G,f) = \max\{|f(u)-f(v)| : uv \in E(G)\}$.

In this paper, we provide exact λ-number for the square of a cycle. In addition, we also completely determine its edge span. The main results are the following two theorems, which will be proved in Sections 2 and

3 respectively.

THEOREM 2.6 *Let $n \geq 4$. We have*

$$\lambda(C_n^2) = \begin{cases} 6, & \text{if } n = 4 \text{ or } n \equiv 0 (\text{mod } 7), \\ 8, & \text{if } n \in \{5, 9, 10, 11, 17\}, \\ 7, & \text{otherwise.} \end{cases}$$

THEOREM 3.3 *Let $n \geq 4$. Then $\beta(C_n^2) = 5$ if and only if the system of equations and an inequality has a non-negative integer solution. Otherwise, $\beta(C_n^2) = 6$.*

$$\begin{cases} 5x_3 = 2x_1 + 3x_2, \\ n = x_1 + x_2 + x_3, \\ x_1 \geq x_2 + x_3 + 2. \end{cases}$$

2. The λ-number of the square of a cycle

Let $C_n = v_1 v_2 \cdots v_n v_1$ be a cycle of length n and C_n^2 be the square of C_n. Throughout this article, we always think that subscripts are taken modulo n. For simplicity, we refer to $v_{(i+k)(\text{mod } n)}$ as $v_{(i+k)}$.

The following result constitutes a useful lower bound.

LEMMA 2.1. [6] *Let G be a k-regular graph with $k \geq 2$. Then $\lambda(G) \geq k + 2$.*

Given an k-$L(2,1)$-labeling f of C_n^2, let $L_i = \{v \in V(C_n^2) | f(v) = i\}$ and l_i be the cardinality of L_i. It is not hard to see that $l_i \leq \lfloor \frac{n}{5} \rfloor$ for $0 \leq i \leq k$ and $\sum_{i=0}^{k} l_i = n$.

LEMMA 2.2. *Let $n \in \{5, 9, 10, 11, 17\}$. Then $\lambda(C_n^2) = 8$.*

Proof. Clearly, $\lambda(C_5^2) = \lambda(K_5) = 8$, where K_5 is a complete graph with 5 vertices.

For the case $n = 9$, the labels on $V(C_9^2)$ are pairwisely distinct since $d(u,v) \leq 2$ for $u, v \in V(C_9^2)$. This gives that $\lambda(C_9^2) \geq 8$. Furthermore, $[0, 4, 6, 8, 3, 1, 5, 7, 2]$ is an 8-$L(2,1)$-labeling of C_9^2, which implies $\lambda(C_9^2) \leq 8$. Therefore $\lambda(C_9^2) = 8$.

For the case $n = 10$, let f be an 7-$L(2,1)$-labeling of C_{10}^2.

Firstly, we find that $l_i \leq \lfloor \frac{10}{5} \rfloor = 2$ for each label i. Next, if $l_i = 2$, then $l_{i-1} = l_{i+1} = 0$ (if $i-1$ or $i+1$ exist). Thus we have $l_i + l_{i+1} \leq 2$ for $0 \leq i \leq 6$. So $\sum_{i=0}^{7} l_i \leq 8$, which is a contradiction to $\sum_{i=0}^{7} l_i = 10$. This shows that $\lambda(C_{10}^2) \geq 8$. On the other hand, $[0,2,4,6,8,0,2,4,6,8]$ is an 8-$L(2,1)$-labeling of C_{10}^2. Hence $\lambda(C_{10}^2) = 8$.

Now we consider the case $n = 11$. Let f be an 7-$L(2,1)$-labeling of C_{11}^2.

Then $l_i \leq \lfloor \frac{11}{5} \rfloor = 2$ for each label i. Furthermore, if $l_i = 2$ for $1 \leq i \leq 6$, then $l_{i-1} + l_{i+1} \leq 1$; And if $l_i = 2$ and $l_{i\pm1} = 1$, then $l_{i\pm2} \leq 1$. Thus $l_{i-1} + l_i + l_{i+1} \leq 3$ for $1 \leq i \leq 6$; $l_0 + l_1 + l_2 \leq 4$ and $l_5 + l_6 + l_7 \leq 4$. So $\sum_{i=0}^{7} l_i \leq 10$, which is again a contradiction. Therefore $\lambda(C_{11}^2) \geq 8$. Since $[0,2,5,7,3,0,8,5,1,3,6]$ is an 8-$L(2,1)$-labeling of C_{11}^2, this implies $\lambda(C_{11}^2) \leq 8$. Therefore $\lambda(C_{11}^2) = 8$.

Finally, we treat the case when $n = 17$. Let f be an 7-$L(2,1)$-labeling of C_{17}^2.

Now, it is straightforward to check that the following facts hold:

Fact 1. $l_i \leq \lfloor \frac{17}{5} \rfloor = 3$ for each label i.

Fact 2. If $l_i = 3$, then $l_{i-1} \leq 2$ and $l_{i+1} \leq 2$ (if $i-1$ or $i+1$ exist).

Fact 3. If $l_i = 3$ for $1 \leq i \leq 6$, then $l_{i-1} + l_{i+1} \leq 2$.

Fact 4. If $l_i = 3$ and $l_{i\pm1} = 2$, then $l_{i\pm2} \leq 1$.

By Fact 1 and Fact 2, we have $|\{i : l_i = 3\}| \leq 4$.

Now let $|\{i : l_i = 3\}| = 4$. Then $\{i : l_i = 3\} \cap \{0,7\} \neq \varnothing$ and there must exist some label $i \in \{0,1,\cdots,7\} \backslash \{i : l_i = 3\}$ such that $l_i = 2$. Otherwise, $\sum_{i=0}^{7} l_i \leq 4 \cdot 3 + 4 \cdot 1 = 16$. Without loss of generality, we assume that $0 \in \{i : l_i = 3\}$. The only possible cases are $\{i : l_i = 3\} = \{0,2,4,6\}$, $\{0,2,4,7\}$, $\{0,2,5,7\}$ or $\{0,3,5,7\}$. According to Fact 3 and Fact 4, we can check that it is impossible for $\{0,2,4,6\}$ or $\{0,2,5,7\}$. Now if $\{i : l_i = 3\} = \{0,2,4,7\}$ or $\{0,3,5,7\}$, then we leave $(l_0, l_1, \cdots, l_7) = (3,1,3,1,3,1,2,3)$ or $(l_0, l_1, \cdots, l_7) = (3,2,1,3,1,3,1,3)$. Now, we will prove that above two cases are impossible.

For $(l_0, l_1, \cdots, l_7) = (3,1,3,1,3,1,2,3)$, without loss of generality, let $f(v_1) = 7$. Then $f(v_6) = f(v_{11}) = 7$ or $f(v_6) = f(v_{12}) = 7$. But $l_6 = 2$, so $f(v_1) = f(v_6) = f(v_{12}) = 7$ and $f(v_9) = f(v_{15}) = 6$. Since $l_4 = 3$, we have $f(v_2) = 5$ or $f(v_4) = 5$. Actually, $f(v_2) = 5$ and $f(v_4) = 5$ are symmetrical in this case. So we only need to consider $f(v_2) = 5$. Thus, $f(v_5) = f(v_{10}) = f(v_{16}) = 4, f(v_{13}) = 3$ or $f(v_5) = f(v_{11}) = f(v_{16}) =$

$4, f(v_8) = 3$. Now there are no proper positons for the label 2 due to $l_2 = 3$. Therefore, it is impossible for $(l_0, l_1, \cdots, l_7) = (3, 1, 3, 1, 3, 1, 2, 3)$. An similar argument can be made for $(l_0, l_1, \cdots, l_7) = (3, 2, 1, 3, 1, 3, 1, 3)$.

This implies $|\{i : l_i = 3\}| \leq 3$. On the other hand, $|\{i : l_i = 3\}| \geq 1$, otherwise $\sum_{i=0}^{7} l_i \leq 2 \cdot 8 = 16$. Now, let x_k be the cardinality of $\{i : l_i = k\}$. Then we have

$$\begin{cases} x_0 + x_1 + x_2 + x_3 = 8, \\ x_1 + 2x_2 + 3x_3 = 17, \\ 1 \leq x_3 \leq 3. \end{cases} \tag{1}$$

Thus the system of equations and an inequality in (1) have the following solutions: $(x_0, x_1, x_2, x_3) = (0, 0, 7, 1)$, $(1, 0, 4, 3)$, $(0, 1, 5, 2)$ or $(0, 2, 3, 3)$. It follows from Fact 3 and Fact 4 that all the solutions are impossible. Hence $\lambda(C_{17}^2) \geq 8$. On the other hand, $[0, 2, 4, 6, 8, 0, 2, 4, 6, 8, 0, 2, 4, 6, 1, 3, 5]$ is an 8-$L(2, 1)$-labeling of C_{17}^2. Therefore, $\lambda(C_{17}^2) = 8$. □

LEMMA 2.3. *Let f be an $L(2, 1)$-labeling of C_n^2. Then the following two statements will not occur:*

(i) There are three consecutive labels or two pairs of consecutive labels on four consecutive vertices on C_n.

(ii) There are five consecutive labels on five consecutive vertices on C_n.

Proof. (i) Suppose that there are four consecutive vertices, say v_i, $v_{(i+1)}, v_{(i+2)}$ and $v_{(i+3)}$, such that $\{a, a+1, a+2\} \subseteq \{f(v_i), f(v_{(i+1)}), f(v_{(i+2)}), f(v_{(i+3)})\}$, where $a \in \mathbb{N}$. If $f(v_i) = a$, this implies $f(v_{(i+3)}) = a + 1$. But there is no proper position for $a + 2$ on $v_{(i+1)}$ and $v_{(i+2)}$, a contradiction. If $f(v_{(i+1)}) = a$, then there is no proper position for $a+1$ on $v_i, v_{(i+2)}$ and $v_{(i+3)}$, again a contradiction. By the symmetry of v_i and $v_{(i+3)}$, $v_{(i+1)}$ and $v_{(i+2)}$, we have the result follow.

Similarly, let $a, a+1$ and $b, b+1$ be two pairs of consecutive labels. And $\{f(v_i), f(v_{i+1}), f(v_{i+2}), f(v_{i+3})\} = \{a, a+1, b, b+1\}$. In this case, if $f(v_i) = a$, then $f(v_{(i+3)}) = a+1$, but we leave $\{f(v_{(i+1)}), f(v_{(i+2)})\} = \{b, b+1\}$, a contradiction. If $f(v_{(i+1)}) = a$, then there is no proper position for $a + 1$, a contradiction.

(ii) Suppose that there are five consecutive vertices, say $v_i, v_{(i+1)}, v_{(i+2)}$, $v_{(i+3)}$ and $v_{(i+4)}$, such that $\{a, a+1, a+2, a+3, a+4\} = \{f(v_i), f(v_{i+1}),$

$f(v_{i+2}), f(v_{i+3}), f(v_{(i+4)})\}$, where $a \in \mathbb{N}$. By the symmetry of v_i and $v_{(i+4)}$, $v_{(i+1)}$ and $v_{(i+3)}$, we only need to consider the following cases.

Case 1. $f(v_i) = a$, then we derive $f(v_{(i+3)}) = a + 1$ or $f(v_{(i+4)}) = a + 1$. Now, if $f(v_{(i+3)}) = a + 1$, then there is no proper position for $a + 2$. If $f(v_{(i+4)}) = a + 1$, then $f(v_{(i+2)}) = a + 2$, thus we leave $\{f(v_{(i+1)}), f(v_{(i+3)})\} = \{a + 3, a + 4\}$, a contradiction.

Case 2. $f(v_{(i+1)}) = a$, then $f(v_{(i+4)}) = a + 1$. But now there is no proper position for $a + 2$, a contradiction.

Case 3. $f(v_{(i+2)}) = a$, then there is no proper position for $a + 2$, a contradiction.

Therefore we complete the proof. □

LEMMA 2.4. *Let f be an 6-$L(2,1)$-labeling of C_n^2 such that $f(v_i) = f(v_j) = k$, where $0 \leq k \leq 6$. Then $|i - j| \geq 7$.*

Proof. Notice that $|i - j| \geq 5$ by the definition of $L(2, 1)$-labeling.

If $|i-j| = 5$, we may assume $j = i+5$. In the case, if $k = 0$ or 6, then $\{(f(v_{(i+1)}), f(v_{(i+2)}), f(v_{(i+3)}), f(v_{(i+4)})\} \subseteq \{2, 3, 4, 5, 6\}$ or $\{0,1,2,3,4\}$. If $k \in \{1, 2, 3, 4, 5\}$, then $\{f(v_{(i+1)}), f(v_{(i+2)}), f(v_{(i+3)}), f(v_{(i+4)})\} = \{0, 1, \cdots, 6\}\backslash\{k-1, k, k+1\}$. For the two cases, there always exist three consecutive labels or two pairs of consecutive labels on $v_{(i+1)}, v_{(i+2)}, v_{(i+3)}$ and $v_{(i+4)}$. But this is impossible in view of Lemma 2.3. Therefore, we have $|i - j| \geq 6$.

Next, if $|i - j| = 6$, suppose that $j = i + 6$. We have the following three cases.

Case 1. $k = 0$. In the case, if $f(v_{(i+3)}) = 1$, then we obtain a labeling $[0, 5, 3, 1, 6, 4, 0]$ on $v_i, v_{(i+1)}, \cdots, v_{(i+6)}$. Thus $f(v_{(i+7)}) = 2$ and $f(v_{(i+8)}) = 5$, but now no label can be assigned to the vertex $v_{(i+9)}$. If $f(v_{(i+3)}) \neq 1$, then $\{f(v_{(i+1)}), f(v_{(i+2)}), f(v_{(i+3)}), f(v_{(i+4)}), f(v_{(i+5)})\} = \{2, 3, 4, 5, 6\}$. According to Lemma 2.3, it is impossible. By symmetry, we can show for $k = 6$.

Case 2. $k = 1$. If $f(v_{(i+3)}) = 0$, then we have a labeling $[1, 5, 3, 0, 6, 4, 1]$ on $v_i, v_{(i+1)}, \cdots, v_{(i+6)}$, but now no label can be assigned to the vertex $v_{(i+7)}$. If $f(v_{(i+3)}) \neq 0$, then $\{f(v_{(i+1)}), f(v_{(i+2)}), f(v_{(i+3)}), f(v_{(i+4)}), f(v_{(i+5)})\} = \{2, 3, 4, 5, 6\}$, again a contradiction to Lemma 2.3. By symmetry, it is proved similarly for $k = 5$.

Case 3. $k \in \{2, 3, 4\}$. If $f(v_{(i+3)}) = k - 1$ or $k + 1$, then $\{f(v_{(i+1)}), f(v_{(i+2)}), f(v_{(i+4)}), f(v_{(i+5)})\} \subseteq \{0, 1, \cdots, 6\}\backslash\{k, k - 1, k, k + 1\}$ or

$\{0, 1, \cdots, 6\} \backslash \{k-1, k, k+1, k+2\}$. Otherwise, $\{f(v_{(i+1)}), f(v_{(i+2)}), f(v_{(i+3)}), f(v_{(i+4)}), f(v_{(i+5)})\} \subseteq \{0, 1, \cdots, 6\} \backslash \{k-1, k, k+1\}$. Both of the cases are impossible.

Hence $|i-j| \geq 7$. □

Given an 6-$L(2,1)$-labeling f of C_n^2, then by Lemma 2.4, it is easy to see that $l_i \leq \lfloor \frac{n}{7} \rfloor$ for $0 \leq i \leq 6$.

THEOREM 2.5. *If $n \neq 0 \pmod 7$ and $n \geq 8$, then $\lambda(C_n^2) \geq 7$.*

Proof. Without loss of generality , we assume that $n = 7k + i$, where $1 \leq i \leq 6$.

Suppose for contradiction that there is an 6-$L(2,1)$-labeling f of C_n^2. Then by Lemma 2.4, $l_i \leq \lfloor \frac{n}{7} \rfloor = k$ for $0 \leq i \leq 6$. This implies $7k+i \leq 7k$, a contradiction.

Hence $\lambda(C_n^2) \geq 7$ when $n \neq 0 \pmod 7$ and $n \geq 8$. □

THEOREM 2.6. *Let $n \geq 4$. We have*

$$\lambda(C_n^2) = \begin{cases} 6, & \text{if } n = 4 \text{ or } n \equiv 0 \pmod 7, \\ 8, & \text{if } n \in \{5, 9, 10, 11, 17\}, \\ 7, & \text{otherwise.} \end{cases}$$

Proof. For $n = 4$, $\lambda(C_4^2) = \lambda(K_4) = 6$, where K_4 is a complete graph with 4 vertices. If $n = 0 \pmod 7$, without loss of generality, let $n = 7k$. Then we can repeat the sequence $[4, 2, 0, 5, 3, 1, 6]$ k times. This implies that $\lambda(C_n^2) \leq 6$. On the other hand, $\lambda(C_n^2) \geq 6$ due to Lemma 2.1, by the fact that C_n^2 is a 4-regular graph. Thus we conclude that $\lambda(C_n^2) = 6$, if $n = 4$ or $n = 0 \pmod 7$.

If $n \neq 0 \pmod 7$, we may assume $n = 7k + i$, where $1 \leq i \leq 6$. We have two cases as follows.

Case 1. If $k \geq i$, then we take the sequence $[4, 2, 0, 5, 7, 3, 1, 6]$ i times and follow by the sequence $[4, 2, 0, 5, 3, 1, 6]$ $(k-i)$ times repeated. Thus $\lambda(C_n^2) \leq 7$. Furthermore, $\lambda(C_n^2) \geq 7$ follows by Theorem 2.5. Therefore $\lambda(C_n^2) = 7$.

Case 2. If $k < i$, we can rewrite $n = 7k+i = 6k+(k+i) = j \pmod 6$, where $0 \leq j \leq 5$.

Subcase 2.1. If $k + i \geq 6$, then $j \leq k$. We repeat the sequence $[1, 3, 7, 0, 4, 6]$ $(k-j+1)$ times and $[1, 3, 5, 7, 0, 4, 6]$ j times. Again by Theorem 2.5, this gives that $\lambda(C_n^2) = 7$.

For all $n \geq 4$	the label pattern
$n \equiv 0 \pmod 7$	$A, \cdots, A, A$
$n \equiv 1 \pmod 7$	$A, \cdots, A, A_1$
$n \equiv 2 \pmod 7$	$A, \cdots, A, A_2$
$n \equiv 3 \pmod 7$	$B, \cdots, B, A_3$
$n \equiv 4 \pmod 7$	$A, \cdots, A, A_4$
$n \equiv 5 \pmod 7$	$A, \cdots, A, A_5$
$n \equiv 6 \pmod 7$	$A, \cdots, A, A_6$

TABLE 1. The $L(2,1)$-labeling of C_n^2 with edge span 6 for different cases of n.

Subcase 2.2. If $k + i < 6$, then we have $n \in \{4, 5, 9, 10, 11, 17\}$. When $n \in \{5, 9, 10, 11, 17\}$, the result is already proved in Lemma 2.2.

This completes the proof of Theorem 2.6. □

3. The $L(2,1)$ edge span of the square of a cycle

The main objective of this section is to determine the $L(2,1)$ edge span of the square of a cycle. Firstly, we establish the lower and upper bound.

LEMMA 3.1. $5 \leq \beta(C_n^2) \leq 6$ *for all* $n \geq 4$.

Proof. It is clear that $\beta(C_n^2) \geq \beta(C_3) = 4$ since C_3 is an induced subgraph of C_n^2. Now suppose that C_n^2 admits an $L(2,1)$-labeling f with edge span 4. Without loss of generality, let $f(v_1) = 0$. In this case, if $f(v_2) = 2$, then $f(v_n) = 4$. However, no label can be assigned to the vertex v_{n-1}, a contradiction. An similar argument can be made for $f(v_2) = 4$. If $f(v_2) = 3$, then no label can be assigned to the vertex v_n. Hence $\beta(C_n^2) \geq 5$.

Next, we show that $\beta(C_n^2) \leq 6$.

Firstly, let $A = [0, 2, 4, 6, 1, 3, 5], B = [4, 2, 0, 5, 3, 8, 6]$,
$A_1 = [8, 10, 14, 12, 9, 7, 3, 5], A_2 = [8, 10, 13, 15, 11, 9, 7, 3, 5], A_3 = [11, 9, 7]$,
$A_4 = [8, 10, 12, 15, 17, 13, 11, 9, 7, 3, 5]$,
$A_5 = [8, 10, 12, 14, 18, 16, 13, 11, 9, 7, 3, 5]$,
$A_6 = [8, 10, 12, 14, 17, 19, 15, 13, 11, 9, 7, 3, 5]$.

Now we give an $L(2,1)$-labeling of C_n^2 with edge span 6, as shown in Table 1.

Therefore the lemma is proved. □

To determine the exact value of $\beta(C_n^2)$, we need to use a consequence in [21]. Firstly, we gives some notations in the following.

Let G be an undirected simple graph. An *orientation* $\overrightarrow{G}$ of G is an assignment of directions to each edge of G. In this sense, we also call G the *underlying graph* of $\overrightarrow{G}$. Let $\overrightarrow{G}$ be an orientation of a graph G and let $W = u_1u_2\cdots u_l$ be a trail in G. For an edge $e_i = u_iu_{i+1}$ $(i = 1, 2, \cdots, l-1)$, we call e_i a *forward edge* (resp., *backward edge*) of W if the direction of e_i in $\overrightarrow{G}$ is from u_i to u_{i+1} (resp., u_{i+1} to u_i). Denote by W^+ and W^- the set of the forward edges and backward edges of W, respectively.

A *k-tension* [12] on G is an ordered pair $(\overrightarrow{G}, \phi)$, where $\overrightarrow{G}$ is an orientation of G and $\phi : E(\overrightarrow{G}) \mapsto \{0, 1, \cdots, k-1\}$ is a map such that $\sum_{e\in C^+} \phi(e) = \sum_{e\in C^-} \phi(e)$ for every cycle C in G. In particular, if ϕ is an integer-valued function then $(\overrightarrow{G}, \phi)$ is called an *integer tension*. An integer tension $(\overrightarrow{G}, \phi)$ is a *nowhere-zero k-tension* if $0 < \phi(e) < k$ for every $e \in E(\overrightarrow{G})$.

The authers in [21] established a connection between the $L(2,1)$-labeling and integer tension of a graph. This connection provides us with an effective way to minimize the edge span.

Lemma 3.2. [21] *Let G be a simple graph and let k be a positive integer. Then G admits an $L(2,1)$-labeling with edge span $k-1$ if and only if G admits a k-tension $(\overrightarrow{G}, \phi)$ satisfying the following conditions:*
(i) For any edge e of G, $2 \le \phi(e) \le k-1$;
(ii) For any two adjacent arcs $e_1 = ux$ and $e_2 = xv$ where u and v are not adjacent, if x is the common head or the common tail of e_1 and e_2, then $|\phi(e_1) - \phi(e_2)| \ge 1$; and if x is the head of one in $\{e_1, e_2\}$ and the tail of the other, then $\phi(e_1) + \phi(e_2) \ge 1$.

In view of Lemma 3.2, we have the following main result.

Theorem 3.3. *Let $n \ge 4$. Then $\beta(C_n^2) = 5$ if and only if the system of equations and an inequality in (2) has a non-negative integer solution. Otherwise, $\beta(C_n^2) = 6$.*

$$(2) \qquad \begin{cases} 5x_3 = 2x_1 + 3x_2, \\ n = x_1 + x_2 + x_3, \\ x_1 \geq x_2 + x_3 + 2. \end{cases}$$

Proof. Firstly, we prove the necessity. Let $(\overrightarrow{C_n^2}, \phi)$ be a 6-tension of C_n^2 satisfying the two conditions of Lemma 3.2.

Claim 1. $\phi(v_i v_{(i+1)}) \neq 4$ for $i = 1, 2, \cdots, n$.

Suppose to the contrary that there exists some i such that $\phi(v_i v_{(i+1)}) = 4$. Then $\phi(v_{(i-1)} v_{(i+1)}) = \phi(v_{(i+1)} v_{(i+2)}) = 2$ and the two adjacent arcs $v_{(i-1)} v_{(i+1)}$ and $v_{(i+1)} v_{(i+2)}$ have $v_{(i+1)}$ as their common head or tail, a contradiction to Lemma 3.2.

Claim 2. There do not exist two adjacent arcs $v_i v_{(i+1)}$ and $v_{(i+1)} v_{(i+2)}$ with weight 3 and 5, respectively.

If $\phi(v_i v_{(i+1)}) = 3, \phi(v_{(i+1)} v_{(i+2)}) = 5$, then $\phi(v_{(i+k)} v_{(i+k+1)}) = 3$ for $k = 0, 2, \cdots$ and $\phi(v_{(i+k)} v_{(i+k+1)}) = 5$ for $k = 1, 3, \cdots$. Moreover, those arcs with weight 3 and 5 have the same orientation respectively. Therefore, $\sum_{e \in C_n^+} \phi(e) \neq \sum_{e \in C_n^-} \phi(e)$, where $C_n = v_1 v_2 \cdots v_n$, a contradiction to the definition of tension.

Claim 3. If $\phi(v_i v_{(i+1)}) = 5$ and $\phi(v_{(i+1)} v_{(i+2)}) = 2$, then $\phi(v_{(i+2)} v_{(i+3)}) \neq 3$.

Let $\phi(v_i v_{(i+1)}) = 5, \phi(v_{(i+1)} v_{(i+2)}) = 2$ and $\phi(v_{(i+2)} v_{(i+3)}) = 3$. Then we know that $\phi(v_i v_{(i+1)}) = \phi(v_{(i+1)} v_{(i+3)}) = 5$ and the two adjacent arcs $v_i v_{(i+1)}$ and $v_{(i+1)} v_{(i+3)}$ have $v_{(i+1)}$ as their common head or tail. This contradicts with Lemma 3.2.

Let x_1, x_2, x_3 be the number of arcs in $\{v_i v_{(i+1)} | i = 1, 2, \cdots, n\}$ with weight 2, 3 and 5 , respectively. Thus Claim 1 implies that $5x_3 = 2x_1 + 3x_2$ and $n = x_1 + x_2 + x_3$. Furthermore, we have $x_1 \geq x_2 + x_3 + 2$ by Claim 2 and Claim 3.

Secondly, we prove the sufficiency. Suppose that the system of equations and an inequality in (2) has a non-negative integer solution. Then we can give an ordered pair $(\overrightarrow{C_n^2}, \phi)$ by the following three steps:

Step 1.

$$\phi(v_i v_{(i+1)}) = \begin{cases} 3, & \text{if } i \in \{2x_3 + 2, 2x_3 + 4, \cdots, 2x_3 + 2x_1\}, \\ 5, & \text{if } i \in \{1, 3, \cdots, 2x_3 - 1\}, \\ 2, & \text{otherwise.} \end{cases}$$

Step 2. We assign each edge $v_iv_{(i+1)}$ an orientation such that the orientations on those edges with weight 5 have the same orientation which are opposite to those edges with weight 2 and 3.

Step 3. For those edges $v_iv_{(i+2)}$ $(i = 1, 2, \cdots, n)$, we assign the weight and the orientation to each edge such that $\sum_{e\in C_i^+} \phi(e) = \sum_{e\in C_i^-} \phi(e)$, where $C_i = v_iv_{(i+1)}v_{(i+2)}v_i$.

Obviously, ϕ is well-defined since the system of equations and an inequality in (2) holds. Moreover, we can check that the ordered pair $(\overrightarrow{C_n^2}, \phi)$ is a 6-tension satisfying the two conditions of Lemma 3.2. Therefore the result follows. □

COROLLARY 3.4 *Let $n \geq 37$. Then $\beta(C_n^2) = 5$.*

Proof. Let $a \in \mathbb{Z}$ such that $0 \leq a \leq 6$ and $a \equiv 5n (\mathrm{mod}\ 7)$. Since $5n - 8a \equiv 0 (\mathrm{mod}\ 7)$ and $5n - 8a \geq 5 \cdot 37 - 8 \cdot 6 > 0, 5n - 8a = 7b$ for some $b \in \mathbb{Z}^+$. Let $x_1 = b, x_2 = a$ and $x_3 = n - a - b$. Then $x_1, x_2 \geq 0$, $x_3 = n - a - b = n - a - \frac{5n-8a}{7} = \frac{2n+a}{7} \geq 0$. Also

$$5x_3 = 5n - 5a - 5b = 8a + 7b - 5a - 5b = 2b + 3a = 2x_1 + 3x_2,$$

$$n = b + a + n - a - b = x_1 + x_2 + x_3$$

and

$$x_1 = b = \frac{5n - 8a}{7} = \frac{2n}{7} + \frac{3n - 8a}{7} \geq \frac{2n}{7} + \frac{111 - 48}{7} \geq \frac{2n}{7} + \frac{63}{7}$$
$$> \frac{2n}{7} + \frac{43}{7} + 2 = n - \frac{5n - 8a}{7} + 2 = n - b + 2 = x_2 + x_3 + 2.$$

Thus the system of equations and an inequality in Theorem 3.3 has a non-negative integer solution if $n \geq 37$. Hence $\beta(C_n^2) = 5$ if $n \geq 37$. □

ACKNOWLEDGEMENTS

The author would like to thank the anonymous reviewers for their valuable comments and suggestions.

References

[1] T. Calamoneri and R. Petreschi, *$L(h, 1)$-labeling subclasses of planar graphs*, J. Parall. Distrib. Comput. **64** (2004), 414–426.

[2] T. Calamoneri, *Exact solution of a class of frequency assignment problems in cellular networks and other regular grids*, Theor. Comput. Sci. **8** (2006), 141–158.

[3] T. Calamoneri, *The $L(h, k)$-labelling problem: an updated survey and annotated bibliography*, Computer J. **54** (2011), 1344–1371.

[4] G.J. Chang and D. Kuo, *The $L(2,1)$-labeling problem on graphs*, SIAM J. Discrete Math. **9** (1996), 309–316.
[5] J.P. Georges, D.W. Mauro and M.A. Whittlesey, *Relating path coverings to vertex labellings with a condition at distance two*, Discrete Math. **135** (1994), 103–111.
[6] J.R. Griggs and R.K. Yeh, *Labeling graph with a condition at distance two*, SIAM J. Discrete Math. **5** (1992), 586–595.
[7] J.R. Griggs, *Real number channel assignments with distance conditions*, SIAM J. Discrete Math. **20** (2006), 302–327.
[8] W.K. Hale, *Frequency assignment: Theory and applications*, Proc. IEEE **68** (1980), 1497–1514.
[9] J. van den Heuvel, R.A. Leese and M.A. Shepherd, *Graph labelling and radio channel assignment*, J. Graph Theory **29** (1998), 263–283.
[10] L.-H. Huang and G.J. Chang, *$L(h,k)$-labelings of Hamming graphs*, Discrete Math. **309** (2009), 2197–2201.
[11] N. Karst, J. Oehrlein, D.S. Troxell and J. Zhu, *The minimum span of $L(2,1)$-labelings of generalized flowers*, Discrete Appl. Math. **181** (2015), 139–151.
[12] M. Kochol, *Tension polynomials of graphs*, J. Graph Theory **3** (2002), 137–146.
[13] A. Kohl, *Knotenfärbungen mit Abstandsbedingungen*, Dissertation, TU Bergakademie Freiberg, Germany, 2006(in German).
[14] D. Korže and A. Vesel, *$L(2,1)$-labeling of strong products of cycles*, Inform. Process Lett. **94** (2005),183–190.
[15] S. Paul, M. Pal and A. Pal, *$L(2,1)$-Labeling of Permutation and Bipartite Permutation Graphs*, Math. Comput. Sci. **9** (2015),113–123.
[16] Z. Shao and R.K. Yeh, *The $L(2,1)$-labeling and operations of graphs*, IEEE Trans. Circuits Syst. I Fund. Theory Appl. **52** (2005), 668–671.
[17] Z. Shao and R.K. Yeh, *The $L(2,1)$-labeling on planar graphs*, Appl. Math. Lett. **20** (2007), 222–226.
[18] W.F. Wang, *The $L(2,1)$-labelling of trees*, Discrete Appl. Math. **154** (2007), 598–603.
[19] R.K. Yeh, *A survey on labeling graphs with a condition at distance two*, Discrete Math. **306** (2006), 1217–1231.
[20] R.K. Yeh, *The edge span of distance two labelings of graphs*, Taiwanese J. Math. **4** (2000), 675–683.
[21] X.L. Zhang and J.G. Qian, *$L(p,q)$-labeling and integer tension of a graph embedded on torus*, J. Combin. Optim. 2014, Published on-line, DOI: 10.1007/s10878-014-9714-4.

Xiaoling Zhang
College of Mathematics and Computer Science
Quanzhou Normal University
Quanzhou 362000, Fujian, P.R. China
E-mail: xml000999@163.com