Korean J. Math. **24** (2016), No. 1, pp. 51–63
http://dx.doi.org/10.11568/kjm.2016.24.1.51

REGULARIZED EQUILIBRIUM PROBLEMS IN BANACH SPACES

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Abstract. In this works, we consider a class of *regularized equilibrium problems* in Banach spaces. By using the auxiliary principle techniques to suggest some iterative schemes for *regularized equilibrium problems* and proved the convergence of these iterative methods required either pseudoaccretivity or partially relaxed strongly accretivity.

1. Introduction

Equilibrium problems have been extensively studied in recent years, the origin of which can be traced back to Takahashi [14] and Blum and Oettli [3]. It is well known that vector equilibrium problem provides a unified model for several classes of problems, *for example* vector variational inequalities, vector complementarity problems, vector optimization problems, vector saddle point problems and variational inequality problems *see* [2, 6–8]. We remark that the almost all results concerning the solutions of iterative scheme for solving the variational inequalities and related problems are being considered in the setting of convex sets. Consequently the techniques are based on the projection of operator over

Received December 22, 2015. Revised January 6, 2016. Accepted January 25, 2016.
2010 Mathematics Subject Classification: 49J40, 47H06.
Key words and phrases: Regularized equilibrium problems, Regularized variational inequalities, uniformly r-prox regular sets, nonconvex sets, iterative methods, convergence analysis, Banach spaces.

convex sets, which may not hold in general, when the sets are nonconvex. It is known that the unified prox-regular sets are nonconvex and convex sets as special cases, *see* [5, 9, 11–13].
A class of set valued equilibrium problems in nonconvex sets are known as *regularized equilibrium problems.* These *regularized equilibrium problems* are more general and included the classical equilibrium problems and related optimizations. Since the under lying set is a nonconvex set, it is not possible to extend the usual projection and resolvent techniques for solving *regularized equilibrium problems.* Fortunately these difficulties can be overcome by using the auxiliary principle techniques, which is mainly due to Glowinski *et al.* [8]. Noor [10] has used this techniques to developed some iterative schemes for solving various classes of equilibrium problem and variational inequality problem.

In this paper we prove the auxiliary principle techniques can be used to suggest a class of iterative methods for solving the *regularized equilibrium problems* in uniformly smooth Banach spaces. We also prove that the convergence of these new methods either require pseudoaccretivity or partially relaxed strongly accretivity.

2. Preliminaries

Let B be a real finite dimensional Banach space with dual space B^*. Let $\|\cdot\|$ be a norm and $\langle\cdot,\cdot\rangle$ a duality pairing. Let Ω be a nonempty convex closed subset of B and $R = (-\infty, +\infty)$. The normalized duality mapping $J_q : B \longrightarrow 2^{B^*}$ is defined by

$$J_q(u) = \left\{f^* \in B^* : \langle u, f^*\rangle = \|f^*\|\|u\|^q, \|f^*\| = \|u\|^{q-1}\right\}, \ \forall u \in B$$

where $q > 1$ is a constant. In particular, $J = J_2$ is a usual normalized duality mapping. It is well known that j_q is a single valued if B is smooth and

$$J_q(u) = \|u\|^{q-2}J(u), \quad \text{for all } u \neq 0.$$

In the sequel, we always assume that B is a real Banach space such that j_q is a single valued. If B is a Hilbert space than j become the identity mapping on B. The modulus of smoothness of B is the function $\rho_B : [0, \infty) \longrightarrow [0, \infty)$ is defined by

$$\rho_B(t) = \sup\{\frac{1}{2}(\|u + v\| + \|u - v\|) - 1 : \|u\| = 1, \|v\| \leq t\}.$$

A Banach space B is called uniformly smooth if

$$\lim_{t \longrightarrow 0} \frac{\rho_B(t)}{t} = 0.$$

B is called q-uniformly smooth if there exists a constant $c > 0$ such that

$$\rho_B(t) < ct^q, \ q > 1.$$

A Banach space B is said to be uniformly convex if given $\epsilon > 0$ there exists $\delta > 0$ such that for all $u, v \in B$ with $\|u\| \leq 1, \|v\| \leq 1$ and $\|u - v\| \geq \epsilon, \|\frac{1}{2}(u + v)\| \leq 1 - \delta$.

LEMMA 2.1. [15] *A real Banach space B is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $u, v \in B$,*

$$\|u + v\|^q = \|u\|^q + q\langle v, j_q(u)\rangle + c_q\|v\|^q.$$

DEFINITION 2.2. The proximal normal cone of Ω at u is given by

$$N^P(\Omega; u) = \{\zeta \in B : u \in P_\Omega(u + \alpha\zeta)\},$$

where $\alpha > 0$ is a constant and

$$P_\Omega[u] = \{u^* \in \Omega : d_\Omega(u) = \|u - u^*\|\}.$$

Here $d_\Omega(\cdot)$ is the usual distance function to the subset of Ω, that is,

$$d_\Omega(u) = \inf_{v \in \Omega} \|u - v\|.$$

LEMMA 2.3. [13] *Let Ω be a nonempty closed subset in B. Then $\zeta \in N^P(\Omega; u)$ if and only if there exists a constant $\alpha > 0$ such that*

$$\langle \zeta, j_q(v - u)\rangle \leq \alpha\|v - u\|^q \ \forall v \in \Omega.$$

DEFINITION 2.4. The Clarke normal cone, denoted by $N^C(\Omega; u)$ is defined as

$$N^C(\Omega; u) = \overline{co}[N^P(\Omega; u)]$$

where $\overline{co}\mathcal{A}$ means the closure of the convex hull of $\mathcal{A}$.

DEFINITION 2.5. [12] For any $r \in (0, +\infty]$, a subset Ω is said to be normalized uniformly prox-regular (or uniformly r-prox-regular) if and only if every nonzero proximal normal to Ω can be realized by an r-ball, *i.e.* for all $u \in \Omega$ and $0 \neq \zeta \in N^P(\Omega; u)$ with $\|\zeta\| = 1$,

$$\langle \zeta, j_q(v - u)\rangle \leq \frac{1}{2r}\|v - u\|^q, \ v \in \Omega.$$

REMARK 2.6. It is clear that if $r = \infty$, then uniformly r-prox regularity of Ω is equivalent to the convexity of Ω.
It is known that if Ω is a uniformly r-prox regular set then the proximal normal cone $N^P(\Omega; u)$ is closed as set valued mapping. Thus we have $N^C(\Omega; u) = N^P(\Omega; u)$. For sake of simplicity we denote

$$N(\Omega; u) = N^C(\Omega; u) = N^P(\Omega; u)$$

and take $\gamma = \frac{1}{2r}$. Clearly for $r = \infty, \gamma = 0$.

From now onward, the set Ω is uniformly r-prox regular set unless otherwise specified.
Assume that $T : B \longrightarrow 2^{B^*}$ is a set valued mapping where 2^{B^*} is the power set of dual space B and $h : B \times B \times B \longrightarrow R$ is a nonlinear continuous function and $\varphi : B \times B \longrightarrow R \bigcup \{+\infty\}$. We consider the problem of finding $u \in \Omega, x \in T(u)$ such that

$$h(x, u, v) + \varphi(v, u) - \varphi(u, u) + \gamma \|v - u\|^q \geq 0, v \in \Omega, \tag{2.1}$$

is called the *regularized equilibrium problems* in Banach spaces.

REMARK 2.7. If B is a real Hilbert spaces, then (2.1) reduces to a problem for finding $u \in \Omega, x \in T(u)$ such that

$$h(x, u, v) + \varphi(v, u) - \varphi(u, u) + \gamma \|v - u\|^2 \geq 0, v \in \Omega. \tag{2.2}$$

If $\gamma = 0$ then the uniformly prox regular set Ω becomes the convex set Ω and consequently problem (2.2) reduces to finding $u \in \Omega, x \in T(u)$ such that

$$h(x, u, v) + \varphi(v, u) - \varphi(u, u) \geq 0, v \in \Omega, \tag{2.3}$$

is known as *equilibrium problems*.
Again we note that if T is single valued mapping then the problem (2.2) reduces to the following problem of finding $u \in \Omega$ such that

$$h(Tu, u, v) + \varphi(v, u) - \varphi(u, u) + \gamma \|v - u\|^2 \geq 0, v \in \Omega \tag{2.4}$$

is known as *regularized mixed quasi equilibrium problems* and studied by Noor [10].

DEFINITION 2.8. A set valued mapping $T : B \longrightarrow 2^{B^*}$ is said to be

(i) accretive if

$$\langle x - y, j_q(u - v) \rangle \geq 0, \quad \forall u, v \in B, x \in T(u), y \in T(v),$$

(ii) β-strongly accretive if there exists a constant $\beta > 0$ such that

$$\langle x - y, j_q(u - v)\rangle \geq \beta\|u - v\|^q, \ \forall u, v \in B, x \in T(u), y \in T(v),$$

(iii) σ-$\mathcal{H}$-Lipschitz continuous mapping if there exists a constant $\sigma > o$ such that

$$\|x - y\| \leq \mathcal{H}(T(u), T(v)) \leq \sigma\|u - v\|, \forall u, v \in B$$

where $\mathcal{H}$ is a Hausdorff metric.

Definition 2.9. Let Ω be a closed convex subset of a Banach space B. A real valued bifunction $h : \Omega \times \Omega \longrightarrow R$ is said to be

(i) accretive if

$$h(u, v) + h(v, u) \leq 0, \quad \forall u, v \in \Omega,$$

(ii) α-strongly accretive if there exists a constant $\alpha > 0$ such that

$$h(u, v) + h(v, u) \leq -\alpha\|u - v\|^q, \quad \forall u, v \in \Omega.$$

Remark 2.10. Clearly strong accretivity of h implies that accretivity of h.

Definition 2.11. The function $h : B \times B \times B \longrightarrow R$ with respect to the mapping $T : B \longrightarrow 2^{B^*}$ is said to be:

(i) jointly pseudoaccretive if

$$\begin{gathered} h(x, u, v) + \varphi(v, u) - \varphi(u, u) \geq 0, \forall x \in T(u) \\ \Rightarrow -h(y, v, u) + \varphi(v, u) - \varphi(u, u) \geq 0, \forall y \in T(v), u, v \in \Omega. \end{gathered}$$

(ii) partially relaxed strongly jointly accretive if there exists a constant $\alpha > 0$ such that

$$h(x, u, v) + h(y, v, w) \leq \alpha\|w - u\|^q, \ \forall u, v, w \in \Omega, x \in T(u), y \in T(v).$$

Remark 2.12. We note that if B is a real Hilbert space then the Definition 2.11 is similar to Definition 2.4 of Noor [10].
Again, if $w = u$, partially relaxed strongly jointly accretivity reduces to

$$h(x, u, v) + h(y, v, u) \leq 0, \forall u, v \in \Omega, x \in T(u), y \in T(v)$$

is known as jointly accretivity of $h(\cdot, \cdot, \cdot)$.

Definition 2.13. [4] The bifunction $\varphi : B \times B \longrightarrow R\bigcup\{+\infty\}$ is called skew - symmetric if and only if

$$\varphi(u, u) - \varphi(u, v) - \varphi(v, u) - \varphi(v, v) \geq 0, \forall u, v \in B.$$

Clearly if the skew-symmetric bifunction $\varphi(\cdot,\cdot)$ is bilinear then

$$\varphi(u,u)-\varphi(u,v)-\varphi(v,u)-\varphi(v,v)=\varphi(u-v,u-v)\geq 0, \forall u,v\in B.$$

The skew - symmetric bifunctions have the properties which can be considered analog monotonicity of gradient and nonnegativity of a second derivative for the convex function. For properties and applications of skew-symmetric bifunction, we refer to [1].

3. Main results

By using the auxiliary principle techniques to defined iterative methods for solving the *regularized equilibrium problems* (2.1).
For given $u\in\Omega$ where Ω is a prox-regular set in B, consider the problems of finding $w\in\Omega, z\in T(w)$ such that

$$\rho h(z,w,v)+\langle w-u, j_q(v-w)\rangle \geq -\rho\gamma\|v-w\|^q+\rho\{\varphi(u,u)-\varphi(v,u)\}, \forall v\in\Omega, \tag{3.1}$$

where $\rho>0$ is a constant. Equation (3.1) is called the *auxiliary uniformly regularized equilibrium problems* in Banach spaces.

Remark 3.1. If B is a real Hilbert space, then we define the auxiliary equation for given $u\in\Omega$ where Ω is a prox-regular set in B, consider the problems of finding $w\in\Omega, z\in T(w)$ such that

$$\rho h(z,w,v)+\langle w-u, v-w\rangle \geq -\rho\gamma\|v-w\|^2+\rho\{\varphi(u,u)-\varphi(v,u)\}, \forall v\in\Omega, \tag{3.2}$$

where $\rho>0$ is a constant.
Again, if $w=u$ then w is a solution of (2.1) and also (2.2).

On the basis of (3.1) we suggest the following iterative techniques for solving (2.1).

Algorithm 3.2. For a given $u_0\in\Omega$ such that $x_0\in T(u_0)$, compute the approximate solution $u_{n+1}\in\Omega, x_{n+1}\in T(u_{n+1})$ by the iterative scheme

$$\begin{aligned}&\rho h(x_{n+1},u_{n+1},v)+\langle u_{n+1}-u_n, j_q(v-u_{n+1})\rangle \\ &\geq -\rho\gamma\|u_{n+1}-v\|^q \;+\rho\{\varphi(u_{n+1},u_{n+1})-\varphi(v,u_{n+1})\},\ \forall v\in\Omega.\end{aligned} \tag{3.3}$$

Algorithm 3.2 is called the proximal point algorithm for solving *regularized equilibrium problems* in Banach spaces B.
We note that if B is a real Hilbert space, then we suggest the following algorithm on basis of (3.2).

ALGORITHM 3.3. For a given $u_0 \in \Omega$ such that $x_0 \in T(u_0)$, compute the approximate solution $u_{n+1} \in \Omega, x_{n+1} \in T(u_{n+1})$ by the iterative scheme

$$\begin{aligned} &\rho h(x_{n+1}, u_{n+1}, v) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \\ &\geq -\rho\gamma\|u_{n+1} - v\|^2 \ + \rho\{\varphi(u_{n+1}, u_{n+1}) - \varphi(v, u_{n+1})\},\ \forall v \in \Omega. \end{aligned} \tag{3.4}$$

We note that if $\gamma = 0$ then the prox regular set Ω becomes the standard convex set Ω and consequently Algorithm 3.2 reduces to:

ALGORITHM 3.4. For a given $u_0 \in \Omega$ such that $x_0 \in T(u_0)$ compute the approximate solution $u_{n+1} \in \Omega, x_{n+1} \in T(u_{n+1})$ by the iterative scheme

$$\begin{aligned} &\rho h(x_{n+1}, u_{n+1}, v) + \langle u_{n+1} - u_n\ ,\ j_q(v - u_{n+1})\rangle \\ &\qquad \geq \rho\{\varphi(u_{n+1}, u_{n+1}) - \varphi(v, u_{n+1})\}, \forall v \in \Omega. \end{aligned} \tag{3.5}$$

Again we note that if B is a real Hilbert space, then Algorithm 3.4 becomes:

ALGORITHM 3.5. For a given $u_0 \in \Omega$ such that $x_0 \in T(u_0)$ compute the approximate solution $u_{n+1} \in \Omega, x_{n+1} \in T(u_{n+1})$ by the iterative scheme

$$\begin{aligned} &\rho h(x_{n+1}, u_{n+1}, v) + \langle u_{n+1} - u_n\ ,\ v - u_{n+1}\rangle \\ &\qquad \geq \rho\{\varphi(u_{n+1}, u_{n+1}) - \varphi(v, u_{n+1})\}, \forall v \in \Omega. \end{aligned} \tag{3.6}$$

We now consider the convergence criteria of Algorithm 3.2.

THEOREM 3.6. *Let $u \in \Omega$ be a solution of* (2.1) *and let u_{n+1} be the approximate solution obtained from Algorithm 3.2. Let $T : \Omega \longrightarrow 2^{B^*}$ be the set valued mapping and the trifunction $h(\cdot, \cdot, \cdot)$ is jointly pseudoaccretive and the bifunction $\varphi(\cdot, \cdot)$ is skew-symmetric, then*

$$(1 - 2q\rho\gamma)\|u - u_{n+1}\|^q \leq \|u - u_n\|^q - c_q\|u_n - u_{n+1}\|^q, \tag{3.7}$$

where $c_q > 0, q > 0$.

Proof. Let $u \in \Omega$ be a solution of (2.1), then

$$h(x,u,v) + \gamma\|v-u\|^q \geq \varphi(u,u) - \varphi(v,u), v \in \Omega, x \in T(u). \tag{3.8}$$

Now take $v = u_{n+1}$ in (3.8) we have

$$h(x,u,u_{n+1}) + \gamma\|u_{n+1}-u\|^q \geq \varphi(u,u) - \varphi(u_{n+1},u), \tag{3.9}$$

which implies that

$$-h(x_{n+1},u_{n+1},u) + \gamma\|u_{n+1}-u\|^q \geq \varphi(u,u) - \varphi(u_{n+1},u), \tag{3.10}$$

since $h(\cdot,\cdot,\cdot)$ is a pseudoaccretive mapping.
Taking $v = u$ in (3.3) we get

$$\begin{aligned} &\rho h(x_{n+1},u_{n+1},u) + \langle u_{n+1}-u_n, j_q(u-u_{n+1})\rangle \\ &\qquad \geq -\rho\gamma\|u-u_{n+1}\|^q + \rho\{\varphi(u_{n+1},u_{n+1}) - \varphi(u,u_{n+1})\} \end{aligned} \tag{3.11}$$

which can be written as

$$\begin{aligned} &\langle u_{n+1}-u_n, j_q(u-u_{n+1})\rangle \\ &\quad \geq -\rho h(x_{n+1},u_{n+1},u) - \rho\gamma\|u-u_{n+1}\|^q \\ &\qquad + \rho\{\varphi(u_{n+1},u_{n+1}) - \varphi(u,u_{n+1})\} \\ &\quad \geq -\rho\gamma\|u-u_{n+1}\|^q + \rho\{\varphi(u,u) - \varphi(u,u_{n+1}) - \varphi(u_{n+1},u) \\ &\qquad + \varphi(u_{n+1},u_{n+1})\} - \rho\gamma\|u-u_{n+1}\|^q \end{aligned} \tag{3.12}$$

where we have used (3.11) and fact that the function $\varphi(\cdot,\cdot)$ is skew-symmetric. Now from Lemma 2.1, we have

$$\langle u_{n+1}-u_n\,,\, j_q(u-u_{n+1})\rangle = \frac{1}{q}\big\{\|u-u_n\|^q - \|u-u_{n+1}\|^q - c_q\|u_{n+1}-u_n\|^q\big\}. \tag{3.13}$$

Combining (3.12) and (3.13) we have

$(1-2q\rho\gamma)\|u-u_{n+1}\|^q \leq \|u-u_n\|^q - c_q\|u_n-u_{n+1}\|^q,$ for $c_q > 0, q > 0$,

the required result (3.7). □

From Algorithm 3.3 we have the following result.

COROLLARY 3.7. *Let $u \in \Omega$ be a solution of (2.2) and let u_{n+1} be the approximate solution obtained from Algorithm 3.3. Let $T : \Omega \longrightarrow 2^B$ be the set valued mapping and the trifunction $h(\cdot,\cdot,\cdot)$ is jointly pseudomonotone and the bifunction $\varphi(\cdot,\cdot)$ is skew- symmetric, then*

$$(1-4\rho\gamma)\|u-u_{n+1}\|^2 \leq \|u-u_n\|^2 - \|u_n-u_{n+1}\|^2. \tag{3.14}$$

THEOREM 3.8. *Let B be a finite dimensional Banach space and $T : \Omega \longrightarrow 2^{B^*}$ a set valued mapping. If $2q\rho\gamma < 1$ then the sequences $\{u_n\}_{n=0}^{\infty}$ and $\{x_n\}_{n=0}^{\infty}$ given by Algorithm 3.2 converge to a solution of u and x of* (2.1)*, respectively.*

Proof. Let $u \in \Omega$ be a solution of (2.1). From (3.7) it follows that the sequence $\{\|u - u_n\|\}_{n=0}^{\infty}$ is nonincreasing and consequentially $\{u_n\}_{n=0}^{\infty}$ is bounded. Furthermore we have

$$\sum_{n=0}^{\infty} c_q \|u_{n+1} - u_n\|^q \leq \|u_0 - u\|^q, \text{ for } q > 0, c_q > 0,$$

which implies that

$$\lim_{n \longrightarrow \infty} \|u - u_{n+1}\| = 0. \tag{3.15}$$

Let $\hat{u}$ be the limit point of $\{u_n\}_{n=0}^{\infty}$; a subsequence $\{u_{n_j}\}_{j=0}^{\infty}$ of $\{u_n\}_{n=0}^{\infty}$, converges to $\hat{u} \in B$. Replacing u_{n+1} by u_{n_j} in (3.3), taking the limit $n_j \longrightarrow \infty$ and using (3.15) we have

$$h(\hat{x}, \hat{u}, v) + \gamma\|v - \hat{u}\|^q \geq \varphi(\hat{u}, \hat{u}) - \varphi(v, \hat{u}), \forall v \in \Omega \tag{3.16}$$

which implies that $\hat{u}$ solve the *regularized equilibrium problems* (2.1) and

$$\|u_{n+1} - u\|^q \leq \|u_n - u\|^q.$$

Thus it follows from the above inequality that $\{u_n\}_{n=0}^{\infty}$ has exactly one limit point $\hat{u}$ and

$$\lim_{n \longrightarrow \infty} u_n = \hat{u},$$

the required results. □

Again from Algorithm 3.3, we have the following result.

COROLLARY 3.9. Let B be a finite dimensional Hilbert space and $T : \Omega \longrightarrow 2^{B}$ a set valued mapping. If $4\rho\gamma < 1$ then the sequences $\{u_n\}_{n=0}^{\infty}$ and $\{x_n\}_{n=0}^{\infty}$ given by Algorithm 3.3 converge to a solution of u and x of (2.2), *respectively.*

Now we again suggest another iterative methods, the convergence of which require only the partially relaxed strongly accretivity which is a weaker condition of cocoercivity.

For a given $u \in \Omega$, consider the problem of finding $w \in \Omega, x \in T(u)$ such that

$$\rho h(x,u,v)+\langle w-u, j_q(v-w)\rangle \geq -\rho\gamma\|v-w\|^q+\rho\{\varphi(w,w)-\varphi(v,w)\}, \forall v \in \Omega \tag{3.17}$$

is called the *auxiliary uniformly regularized equilibrium problems.*
If B is a real Hilbert space then we have the following:
For a given $u \in \Omega$, consider the problem of finding $w \in \Omega, x \in T(u)$ such that

$$\rho h(x,u,v)+\langle w-u, v-w\rangle \geq -\rho\gamma\|v-w\|^2+\rho\{\varphi(w,w)-\varphi(v,w)\}, \forall v \in \Omega. \tag{3.18}$$

REMARK 3.10. We remark that problems (3.1) and (3.17) are quite different, also (3.2) and (3.18). Again we point out, if $w = u$ then w is a solution of the *regularized equilibrium problems* (2.1).

ALGORITHM 3.11. For a given $u_0 \in \Omega$ such that $x_0 \in T(u_0)$, compute the approximate solution $u_{n+1} \in \Omega, x_{n+1} \in T(u_{n+1})$ by the iterative scheme

$$\begin{aligned}&\rho h(x_n,u_n,v) + \langle u_{n+1}-u_n\ ,\ j_q(v-u_{n+1})\rangle \\ &\quad\geq -\rho\gamma\|v-u_{n+1}\|^q + \rho\{\varphi(u_{n+1},u_{n+1}) - \varphi(v,u_{n+1})\}, \forall v \in \Omega.\end{aligned} \tag{3.19}$$

If B is a real Hilbert space, we have following algorithm:

ALGORITHM 3.12. For a given $u_0 \in \Omega$ such that $x_0 \in T(u_0)$, compute the approximate solution $u_{n+1} \in \Omega, x_{n+1} \in T(u_{n+1})$ by the iterative scheme

$$\begin{aligned}&\rho h(x_n,u_n,v) + \langle u_{n+1}-u_n\ ,\ v-u_{n+1}\rangle \\ &\quad\geq -\rho\gamma\|v-u_{n+1}\|^2 + \rho\{\varphi(u_{n+1},u_{n+1}) - \varphi(v,u_{n+1})\}, \forall v \in \Omega.\end{aligned} \tag{3.20}$$

REMARK 3.13. If $\gamma = 0$, the prox regular set Ω becomes a convex set Ω, then Algorithm 3.11 and Algorithm 3.12 reduces to following algorithms.

ALGORITHM 3.14. For a given $u_0 \in \Omega$ such that $x_0 \in T(u_0)$, compute the approximate solution $u_{n+1} \in \Omega, x_{n+1} \in T(u_{n+1})$ by the iterative scheme

$$\begin{aligned}&\rho h(x_n,u_n,v) + \langle u_{n+1}-u_n\ ,\ j_q(v-u_{n+1})\rangle \\ &\quad\geq \rho\{\varphi(u_{n+1},u_{n+1}) - \varphi(v,u_{n+1})\}, \forall v \in \Omega.\end{aligned} \tag{3.21}$$

ALGORITHM 3.15. For a given $u_0 \in \Omega$ such that $x_0 \in T(u_0)$ compute the approximate solution $u_{n+1} \in \Omega, x_{n+1} \in T(u_{n+1})$ by the iterative scheme

$$\rho h(x_n, u_n, v)+\langle u_{n+1}-u_n, v-u_{n+1}\rangle \geq \rho\{\varphi(u_{n+1}, u_{n+1})-\varphi(v, u_{n+1})\}, \forall v \in \Omega. \tag{3.22}$$

THEOREM 3.16. Let B be a finite dimensional Banach space and $T : \Omega \longrightarrow 2^{B^*}$ a set valued mapping. Let the trifunction $h(\cdot,\cdot,\cdot)$ be partially relaxed strongly jointly accretive with constant $\alpha > 0$ and the bifunction $\varphi(\cdot,\cdot)$ be skew - symmetric. If u_{n+1} is the approximate solution obtained from Algorithm 3.11 and $u \in \Omega, x \in T(u)$ is a solution of (2.1), then

$$(1 - 2q\gamma\rho)\|u - u_{n+1}\|^q \leq \|u - u_n\|^q - (c_q - q\rho\alpha)\|u_n - u_{n+1}\|^q. \tag{3.23}$$

Proof. Let $u \in \Omega, x \in T(u)$ be a solution of (2.1). Then

$$h(x, u, v) + \gamma\|v - u\|^q \geq \varphi(u, u) - \varphi(v, u), \forall v \in \Omega. \tag{3.24}$$

Taking $v = u_{n+1}$ in (3.24) we have

$$h(x, u, u_{n+1}) + \gamma\|u_{n+1} - u\|^q \geq \varphi(u, u) - \varphi(u_{n+1}, u). \tag{3.25}$$

Letting $v = u$ in (3.19) we obtain

$$\begin{aligned}&\rho h(x_n, u_n, u) + \langle u_{n+1} - u_n \ , \ j_q(u - u_{n+1})\rangle \\ &\qquad \geq -\rho\gamma\|u - u_{n+1}\|^q + \rho\{\varphi(u_{n+1}, u_{n+1}) - \varphi(u, u_{n+1})\}\end{aligned}$$

which implies that

$$\begin{aligned}\langle u_{n+1} - u_n \ , \ j_q(u - u_{n+1})\rangle &\geq -\rho h(x_n, u_n, u) - \rho\gamma\|u - u_{n+1}\|^q \\ &\quad + \rho\{\varphi(u_{n+1}, u_{n+1}) - \varphi(u, u_{n+1})\}. \end{aligned} \tag{3.26}$$

From (3.25) and (3.26), we have

$$\begin{aligned}&\langle u_{n+1} - u_n \ , \ j_q(u - u_{n+1})\rangle \\ &\quad \geq -\rho\{h(x_n, u_n, u) + h(x, u, u_{n+1})\} - \rho\gamma\|u - u_{n+1}\|^q + \rho\{\varphi(u, u) \\ &\qquad - \varphi(u, u_{n+1}) - \varphi(u_{n+1}, u) + \varphi(u_{n+1}, u_{n+1})\} - \rho\gamma\|u - u_{n+1}\|^q \\ &\quad \geq -\alpha\rho\|u_n - u_{n+1}\|^q - \rho\gamma\|u - u_{n+1}\|^q - \rho\gamma\|u - u_{n+1}\|^q, \end{aligned} \tag{3.27}$$

where $h(\cdot,\cdot,\cdot)$ is partially relaxed strongly jointly accretive with constant $\alpha > 0$ and $\varphi(\cdot,\cdot)$ is skew-symmetric.
Combining (3.13) and (3.27), we obtain the required results (3.23). □

From Algorithm 3.11, we have the following result.

Corollary 3.17. Let B be a finite dimensional Hilbert space and $T : \Omega \longrightarrow 2^B$ a set valued mapping. Let the trifunction $h(\cdot,\cdot,\cdot)$ be partially relaxed strongly jointly monotone with constant $\alpha > 0$ and the bifunction $\varphi(\cdot,\cdot)$ be skew - symmetric. If u_{n+1} is the approximate solution obtained from Algorithm 3.12 and $u \in \Omega, x \in T(u)$ is a solution of (2.2), then

$$(1-4\gamma\rho)\|u-u_{n+1}\|^2 \leq \|u-u_n\|^2 - (1-2\rho\alpha)\|u_n-u_{n+1}\|^2.$$

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