Korean J. Math. **20** (2012), No. 4, pp. 451–461
http://dx.doi.org/10.11568/kjm.2012.20.4.451

SOME INFINITE SERIES IDENTITIES

Sung-Geun Lim

Abstract. B.C. Berndt has established many relations between various infinite series using a transformation formula for a large class of functions, which comes from a more general class of Eisenstein series. In this paper, continuing his study, we find some infinite series identities.

1. Introduction

B.C. Berndt [3, 4] derived a transformation formula for a large class of functions which comes from a more general class of Eisenstein series and found various infinite series identities. In this paper, using his methods, we derive some new results about infinite series and identities.

At first, we introduce some notations and definitions. Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$, the upper half-plane. For a complex number w, we choose the branch of the argument defined by $-\pi \leq \arg w < \pi$. Let $e(w) = e^{2\pi i w}$. For a positive integer N, let λ_N denote the characteristic function of the integers modulo N, i.e.,

$$\lambda_N(m) = \begin{cases} 1, & \text{if } m \equiv 0 \pmod{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $V\tau = V(\tau) = \frac{a\tau+b}{c\tau+d}$ denotes a modular transformation with $c > 0$ and $c \equiv 0 \pmod{N}$ for every $\tau \in \mathbb{H}$.(We say that V corresponds to a

Received October 15, 2012. Revised December 12, 2012. Accepted December 15, 2012.

2010 Mathematics Subject Classification: 34A25, 11M36.

Key words and phrases: Infinite series, Modular transformation, Eisenstein series.

matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.) Let $r = (r_1, r_2)$ and $h = (h_1, h_2)$ denote real vectors, and define the associated vectors R and H by

$$R = (R_1, R_2) = (ar_1 + cr_2, br_1 + dr_2)$$

and

$$H = (H_1, H_2) = (dh_1 - bh_2, -ch_1 + ah_2)$$

.

For any real x, y and complex s with $\mathrm{Re}(s) > 1$, let

$$\psi(x, y, s) := \sum_{n+y>0} \frac{e(nx)}{(n+y)^s}.$$

For a real number x, $[x]$ denotes the greatest integer less than or equal to x and $\{x\} := x - [x]$. For $\tau \in \mathbb{H}$ and an arbitrary complex numbers s, define

$$\begin{aligned} &A_N(\tau, s; r, h) : \\ &= \sum_{Nm+r_1>0} \sum_{n-h_2>0} \frac{e\left(Nmh_1 + ((Nm + r_1)\tau + r_2)(n - h_2)\right)}{(n-h_2)^{1-s}}. \end{aligned}$$

Let

$$H_N(\tau, s; r, h) := A_N(\tau, s; r, h) + e\left(\frac{s}{2}\right) A_N(\tau, s; -r, -h).$$

The following thoerem is a twist version of Berndt's theorem in [4].

Theorem 1.1. *([4]) Let $Q = \{\tau \in \mathbb{H} \mid \mathrm{Re}(\tau) > -d/c\}$, $\varrho_N = c\{R_2\} - Nd\{R_1/N\}$ and $c = c'N$. Then for $\tau \in Q$ and all s,*

$$\begin{aligned} &(c\tau + d)^{-s} H_N(V\tau, s; r, h) \\ &= H_N(\tau, s; R, H) \\ &\quad -\lambda_N(r_1) e(-r_1 h_1)(c\tau + d)^{-s}\Gamma(s)(-2\pi i)^{-s}\left(\psi(h_2, r_2, s) + e\left(\frac{s}{2}\right)\psi(-h_2, -r_2, s)\right) \\ &\quad +\lambda_N(R_1) e(-R_1 H_1)\Gamma(s)(-2\pi i)^{-s}\left(\psi(H_2, R_2, s) + e\left(-\frac{s}{2}\right)\psi(-H_2, -R_2, s)\right) \\ &\quad +(2\pi i)^{-s} L_N(\tau, s; R, H), \end{aligned}$$

where

$$\begin{aligned} &L_N(\tau, s; R, H) : \\ &= \sum_{j=1}^{c'} e(-H_1(Nj + N[R_1/N] - c) - H_2([R_2] + 1 + [(Njd + \varrho_N)/c] - d)) \\ &\qquad \cdot \int_C u^{s-1} \frac{e^{-(c\tau+d)(Nj - N\{R_1/N\})u/c}}{e^{-(c\tau+d)u} - e(cH_1 + dH_2)} \frac{e^{\{(Njd+\varrho_N)/c\}u}}{e^u - e(-H_2)} du, \end{aligned}$$

where C is a loop beginning at $+\infty$, proceeding in the upper half-plane, encircling the origin in the positive direction so that $u = 0$ is the only zero of

$$(e^{-(c\tau+d)u} - e(cH_1 + dH_2))(e^u - e(-H_2))$$

lying "inside" the loop, and then returning to $+\infty$ in the lower half plane. Here, we choose the branch of u^s with $0 < \arg u < 2\pi$.

If s is an integer, then we can evaluate the integration in Theorem 1.1 by using the residue theorem. Note that after evaluation of $L_N(\tau, s; R, H)$ for an integer s, the transformation formula in Theorem 1.1 will be valid for all $\tau \in \mathbb{H}$ by analytic continuation. We shall use the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \ (|t| < 2\pi)$$

for the Bernoulli polynomials $B_n(x)$, $n \geq 0$. The n-th Bernoulli number B_n, $n \geq 0$, is defined by $B_n = B_n(0)$. Put $\bar{B}_n(x) = B_n(\{x\})$, $n \geq 0$. Recall that $B_{2n+1} = 0$, $n \geq 1$, and that $B_{2n+1}(1/2) = 0$, $n \geq 0$. We often use the following formulas [1];

$$B_n(1-x) = (-1)^n B_n(x),$$

$$\sum_{j=0}^{c-1} B_n\left(\frac{j}{c} + x\right) = c^{1-n} B_n(cx),$$

$$B_n\left(\frac{1}{2}\right) = -(1 - 2^{1-n}) B_n, \ n \geq 0.$$

For the zeta function $\zeta(s)$, we see in [1] that

$$\zeta(2M) = \frac{2^{2M-1}|B_{2M}|\pi^{2M}}{(2M)!}, \ M > 0,$$

and

$$\zeta(1-2M) = -\frac{B_{2M}}{2M}, \ M > 0.$$

Let $\zeta(s, x)$ be the Hurwitz zeta-function. Here, for brevity, we set

$$D_1(h_2, r_2, s) := -(c\tau + d)^{-s}\Gamma(s)(-2\pi i)^{-s}\left(\psi(h_2, r_2, s) + e\left(\frac{s}{2}\right)\psi(-h_2, -r_2, s)\right),$$

and

$$D_2(H_2, R_2, s) := \Gamma(s)(-2\pi i)^{-s}\left(\psi(H_2, R_2, s) + e\left(-\frac{s}{2}\right)\psi(-H_2, -R_2, s)\right).$$

We have two lemmas as follows.

Lemma 1.2. *Let n be an arbitrary integer and assume that r_2, R_2 are not integers. If $n < -1$, then*

$$\lim_{s\to -n} D_1(0, r_2, s) = -(2\pi i)^n \Gamma(-n)(c\tau + d)^n ((-1)^n \zeta(-n, \{r_2\}) + \zeta(-n, 1 - \{r_2\})),$$
$$\lim_{s\to -n} D_2(0, R_2, s) = (2\pi i)^n \Gamma(-n)((-1)^n \zeta(-n, \{R_2\}) + \zeta(-n, 1 - \{R_2\})).$$

If $n = -1, 0$, then

$$\lim_{s\to 1} D_1(0, r_2, s) = (2i)^{-1}(c\tau + d)^{-1}(\cot(\pi r_2) - i),$$
$$\lim_{s\to 1} D_2(0, r_2, s) = -(2i)^{-1}(\cot(\pi r_2) + i),$$

$$\lim_{s\to 0} D_1(0, r_2, s) = \log\left(1 - e^{-2\pi i r_2}\right),$$
$$\lim_{s\to 0} D_2(0, R_2, s) = -\log\left(1 - e^{-2\pi i R_2}\right) - 2\pi i \bar{B}_1(R_2).$$

If $n > 0$, then

$$\lim_{s\to -n} D_1(0, r_2, s) = (-1)^{n+1}(c\tau + d)^n \psi(-r_2, 0, 1 + n),$$
$$\lim_{s\to -n} D_2(0, R_2, s) = \psi(R_2, 0, 1 + n).$$

Proof. We only give a proof for the case of $s = 1$. The others are immediate consequences of facts in [2], pp. 501–502 and an elementary calculus. From [10], we see that

$$\lim_{s\to 1}(\zeta(s, x) - \frac{1}{s-1}) = -\psi_0(x),$$

where ψ_0 is the digamma function, i.e., $\psi_0(x) = \frac{d}{dx}\log\Gamma(x)$. It is known [1] that ψ_0 satisfies

$$\psi_0(1 - x) - \psi_0(x) = \pi\cot(\pi x).$$

Use now the expansions at $s = 1$,

$$\zeta(s, x) = \frac{1}{s-1} - \psi_0(x) + \cdots,$$
$$e^{-\pi i s} = -1 + \pi(s - 1)i + \cdots,$$

to conclude that

$$\lim_{s\to 1}\left(\zeta(s, x) + e\left(\frac{s}{2}\right)\zeta(s, 1 - x)\right) = \pi\cot(\pi x) - \pi i.$$

□

LEMMA 1.3. *Let n be an arbitrary integer and assume that r_2, R_2 are integers. Then*

$$\lim_{s\to 0}(D_1(0,r_2,s)+D_2(0,R_2,s))=\pi i-\log(c\tau+d).$$

If $n\neq 0$, then

$$\lim_{s\to -n} D_1(0,r_2,s)=(-1)^{n+1}(c\tau+d)^n\zeta(1+n),$$
$$\lim_{s\to -n} D_2(0,R_2,s)=\zeta(1+n).$$

This last lemma is obtained by the similar way in the proof of Lemma 1.2.

2. Infinite series

In this section, we find some infinite series from Theorem 1.1 under a modular transformation. Let N be a positive integer, and let r_1 and r_2 be arbitrary real numbers. Put

$$r=\left(r_1,\frac{r_2}{N}\right),\ h=(0,0),\ s=-n,\ \tau=\frac{N-1}{N}+\frac{1}{N}z,\ V\tau=\frac{1}{N}-\frac{1}{N}\frac{1}{z}$$

for Re $z\geq 0$ and for any integer n. Here, V is a modular transformation corresponding to the matrix

$$\begin{pmatrix} 1 & -1 \\ N & -N+1 \end{pmatrix}.$$

Then $(R_1,R_2)=(r_1+r_2,-r_1-r_2+r_2/N)$. Now we let $N\mid r_1$ and $N\nmid(r_1+r_2)$. By Theorem 1.1, we see that

$$z^nH_N(V\tau,-n;r,0)=H_N(\tau,-n;R,0)+(2\pi i)^nL_N(\tau,-2n;R,0)$$
$$+\lim_{s\to -n} D_1\left(0,\frac{r_2}{N},s\right).$$

We also obtain the following results regarding $H_N(\tau,s;r,h)$. Let n be an arbitrary integer. For $N|r_1$,

$$H_N(V\tau,-2n;r,0)=2\sum_{k=1}^{\infty}\frac{\cos(2\pi r_2k/N)}{k^{2n+1}(e^{2\pi ik/z}-1)} \tag{2.1}$$

and

$$H_N(V\tau,-2n-1;r,0)=2i\sum_{k=1}^{\infty}\frac{\sin(2\pi r_2k/N)}{k^{2n+2}(e^{2\pi ik/z}-1)}. \tag{2.2}$$

For $N \nmid R_1 = r_1 + r_2$,

$$\begin{aligned} &H_N(\tau, -2n; R, 0) \\ &= \sum_{k=1}^{\infty} \frac{\cosh(\pi ik(-2r_1/N + (2\{(r_1+r_2)/N\} - 1)z))}{k^{2n+1} \sinh(-\pi ikz)} \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &H_N(\tau, -2n-1; R, 0) \\ &= \sum_{k=1}^{\infty} \frac{\sinh(\pi ik(-2r_1/N + (2\{(r_1+r_2)/N\} - 1)z))}{k^{2n+2} \sinh(-\pi ikz)}. \end{aligned} \tag{2.4}$$

Next, we have

$$\begin{aligned} &L_N(\tau, -n; R, 0) \\ &= -2\pi i \sum_{k=0}^{n+2} \frac{\bar{B}_k((r_1+r_2)/N)\bar{B}_{n+2-k}(\varrho_N/N)}{k!(n+2-k)!} z^{k-1}. \end{aligned} \tag{2.5}$$

Since $\varrho_N/N = -[-r_2 + r_2/N] - N[r_2/N] + [r_2/N]$, we have $\{\varrho_N/N\} = 0$. Using Lemma 1.2, Lemma 1.3 and (2.1)–(2.5), it follows that

$$\begin{aligned} z^{2n} \sum_{k=1}^{\infty} \frac{2\cos(2\pi r_2 k/N)}{k^{2n+1}(e^{2\pi ik/z} - 1)} &= \sum_{k=1}^{\infty} \frac{\cosh(\pi izk(2\{r_2/N\} - 1))}{k^{2n+1} \sinh(-\pi izk)} \\ &\quad -(2\pi i)^{2n+1} \sum_{k=0}^{2n+2} \frac{\bar{B}_k(r_2/N)B_{2n+2-k}}{k!(2n+2-k)!} z^{k-1} + K_0(n), \end{aligned} \tag{2.6}$$

where

$$K_0(n) := \begin{cases} -z^{2n}(2\pi i)^{2n}(-2n-1)!(\zeta(-2n, \{r_2/N\}) + \zeta(-2n, 1 - \{r_2/N\})), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n < 0, \\ \log(1 - e^{-2\pi ir_2/N}), & \text{if } n = 0, \\ -z^{2n}\psi(-r_2/N, 0, 2n+1), & \text{if } n > 0, \end{cases}$$

and

$$\begin{aligned} z^{2n+1} \sum_{k=1}^{\infty} \frac{2i\ \sin(2\pi r_2 k/N)}{k^{2n+2}(e^{2\pi ik/z} - 1)} &= \sum_{k=1}^{\infty} \frac{\sinh(\pi izk(2\{r_2/N\} - 1))}{k^{2n+2} \sinh(-\pi izk)} \\ &\quad -(2\pi i)^{2n+2} \sum_{k=0}^{2n+3} \frac{\bar{B}_k(r_2/N)B_{2n+3-k}}{k!(2n+3-k)!} z^{k-1} + \mathcal{K}_0(n), \end{aligned} \tag{2.7}$$

where

$$\mathcal{K}_0(n) := \begin{cases} (2\pi iz)^{2n+1}(-2n-2)!(\zeta(-2n-1,\{r_2/N\}) - \zeta(-2n-1, 1-\{r_2/N\})), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n < -1, \\ (2\pi iz)^{-1}\lim_{s\to 1}(\zeta(s,\{r_2/N\}) + e(s/2)\zeta(s, 1-\{r_2/N\})), \quad \text{if } n = -1, \\ z^{2n+1}\psi(-r_2/N, 0, 2n+2), \quad \text{if } n \geq 0. \end{cases}$$

THEOREM 2.1. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$ and let $0 < \gamma < 1$. Then, for any integer n,*

$$\begin{aligned} \alpha^{-n}\sum_{k=1}^{\infty}\frac{2\cos(2\pi\gamma k)}{k^{2n+1}(e^{2\alpha k}-1)} &= (-\beta)^{-n}\sum_{k=1}^{\infty}\frac{\cosh((1-2\gamma)\beta k)}{k^{2n+1}\ \sinh(\beta k)} - 2^{2n+1}\sum_{k=0}^{n+1}\frac{B_{2k}(\gamma)B_{2n+2-2k}}{(2k)!(2n+2-2k)!}\alpha^{n-k+1}(-\beta)^k \\ &\quad + i\frac{(-\alpha)^{-n}(2\pi)^{2n+1}B_{2n+1}(\gamma)}{2(2n+1)!} + K_1(n), \end{aligned} \tag{2.8}$$

where

$$K_1(n) := \begin{cases} -2^{2n}(-\beta)^n(-2n-1)!(\zeta(-2n,\gamma) + \zeta(-2n, 1-\gamma)), & \text{if } n < 0, \\ \log(1-e^{-2\pi i\gamma}), & \text{if } n = 0, \\ -\alpha^{-n}\psi(-\gamma, 0, 2n+1), & \text{if } n > 0. \end{cases}$$

Proof. Put $z = \pi i/\alpha$ and let $\{r_2/N\} = \gamma$ in (2.6). Use $B_1 = -1/2$ and $B_{2k+1} = 0$ for $k > 0$. □

THEOREM 2.2. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$ and let $0 < \gamma < 1$. Then, for any integer n,*

$$\begin{aligned} \alpha^{-n-1/2}\sum_{k=1}^{\infty}\frac{2i\ \sin(2\pi\gamma k)}{k^{2n+2}(e^{2\alpha k}-1)} &= (-\beta)^{-n-1/2}\sum_{k=1}^{\infty}\frac{\sinh((1-2\gamma)\beta k)}{k^{2n+2}\ \sinh(\beta k)} \\ &\quad - 2^{2n+2}\pi i\sum_{k=0}^{n+1}\frac{B_{2k+1}(\gamma)B_{2n+2-2k}}{(2k+1)!(2n+2-2k)!}\alpha^{n-k+\frac{1}{2}}(-\beta)^k \\ &\quad + \frac{(-1)^{n+1}(2\pi)^{2n+2}\alpha^{-n-1/2}B_{2n+2}(\gamma)}{2(2n+2)!} + \mathcal{K}_1(n), \end{aligned} \tag{2.9}$$

where

$$\mathcal{K}_1(n) := \begin{cases} 2^{2n+1}(-\beta)^{n+1/2}(-2n-2)!(\zeta(-2n-1,\gamma) - \zeta(-2n-1, 1-\gamma)), & \text{if } n < -1, \\ -\alpha^{1/2}(1+i\cot(\pi\gamma))/2, & \text{if } n = -1, \\ \alpha^{-n-1/2}\psi(-\gamma, 0, 2n+2), & \text{if } n \geq 0. \end{cases}$$

Proof. Put $z = \pi i/\alpha$ and let $\{r_2/N\} = \gamma$ in (2.7). For $n = -1$, apply Lemma 1.2. □

Theorem 2.1 for $n \geq 0$ and Theorem 2.2 for $n \geq 1$ have been given by Berndt [3] in different forms employing

$$\frac{\cosh((1-2\gamma)\beta k)}{\sinh(\beta k)} = \frac{2\cosh(2\beta k\gamma)}{e^{2\beta k}-1} + e^{-2\beta k\gamma}. \tag{2.10}$$

Some special cases of Theorem 2.1 and Theorem 2.2 have been given by other authors. In case of $n = -1$, Schlömilch [7, 8] established (2.8) and (2.9) in different forms using (2.10). For $n = -1$ and $\alpha = \beta = \pi$, (2.8) was given by Watson [9]. Lagrange [5] has given a proof for (2.9) in case of $n = -1$ and a proof of (2.8) in case of $n = 0$. With replacing α by α^2, β by β^2, and letting $\gamma = t/\beta$, (2.9) in cases of $n = -1$ and $n = 0$ are found in Ramanujan's Notebooks [6]. By equating the imaginary parts and the real parts of (2.8) and (2.9), respectively, we have the Fourier series of the Bernoulli polynomials [3], i.e., for any positive integer M,

$$B_{2M-1}(\gamma) = \frac{2(2M-1)!(-1)^M}{(2\pi)^{2M-1}} \sum_{k=1}^{\infty} \frac{\sin(2\pi\gamma k)}{k^{2M-1}},$$

$$B_{2M}(\gamma) = \frac{2(2M)!(-1)^{M-1}}{(2\pi)^{2M}} \sum_{k=1}^{\infty} \frac{\cos(2\pi\gamma k)}{k^{2M}}.$$

Let $\left(\frac{\cdot}{3}\right)$ be the Legendre symbol.

Proposition 2.3. *Let α, $\beta > 0$ with $\alpha\beta = \pi^2$. Then, for any integer n,*

$$\begin{aligned}\sqrt{3}\alpha^{-n-1/2}\sum_{k=1}^{\infty}\left(\frac{k}{3}\right)\frac{k^{-2n-2}}{e^{2\alpha k}-1} &= (-1)^{n+1}\beta^{-n-1/2}\sum_{k=1}^{\infty}\frac{\sinh(\beta k/3)}{k^{2n+2}\,\sinh(\beta k)}\\ &\quad -2^{2n+2}\pi\sum_{k=0}^{n+1}\frac{B_{2k+1}(1/3)B_{2n+2-2k}}{(2k+1)!(2n+2-2k)!}\alpha^{n-k+1/2}(-\beta)^k + \mathcal{K}_4(n),\end{aligned}$$

where

$$\mathcal{K}_4(n) := \begin{cases} -3^{-2n-3/2}\alpha^{-n-1/2}B_{-2n-1}(1/3)/(2n+1), & \text{if } n < -1,\\ -\sqrt{3}\alpha^{1/2}/6, & \text{if } n = -1,\\ -2^{-1}3^{-2n-3/2}\alpha^{-n-1/2}(\zeta(2n+2,1/3)-\zeta(2n+2,2/3)), & \text{if } n \geq 0.\end{cases}$$

Proof. Put $\gamma = 1/3$ in Theorem 2.2. For $n < -1$, we obtain

$$\begin{aligned}\zeta\left(-2n-1,\frac{1}{3}\right) - \zeta\left(-2n-1,\frac{2}{3}\right)\\ = \frac{(-1)^n(6\pi)^{-2n-1}B_{-2n-1}(1/3)}{\sqrt{3}(-2n-1)!}.\end{aligned} \tag{2.11}$$

For $n \geq 0$, we see

$$\begin{aligned}
&\operatorname{Im}\left(\psi\left(-\frac{1}{3}, 0, 2n+2\right)\right) \\
&= -\frac{\sqrt{3}}{2}\left(\sum_{k \equiv 1 \pmod 3} \frac{1}{k^{2n+2}} - \sum_{k \equiv 2 \pmod 3} \frac{1}{k^{2n+2}}\right) \\
&= -\frac{3^{-2n-3/2}}{2}\left(\zeta\left(2n+2, \frac{1}{3}\right) - \zeta\left(2n+2, \frac{2}{3}\right)\right).
\end{aligned}$$

Equate the imaginary parts in Theorem 2.2. □

COROLLARY 2.4. *Let α, $\beta > 0$ with $\alpha\beta = \pi^2$. Then*

$$\sqrt{3}\alpha^{1/2}\left(\sum_{k=1}^{\infty}\left(\frac{k}{3}\right)\frac{1}{e^{2\alpha k}-1} + \frac{1}{6}\right) = \beta^{1/2}\left(\sum_{k=1}^{\infty}\frac{\sinh(\beta k/3)}{\sinh(\beta k)} + \frac{1}{6}\right).$$

Proof. Put $n = -1$ in Proposition 2.3. □

COROLLARY 2.5. *For any positive integer M,*

$$\sum_{k=1}^{\infty}\left(\frac{k}{3}\right)\frac{k^{2M}}{e^{2\pi k}-1} = \frac{(-1)^M}{\sqrt{3}}\sum_{k=1}^{\infty}\frac{k^{2M}\ \sinh(\pi k/3)}{\sinh(\pi k)} + \frac{3^{2M}B_{2M+1}(1/3)}{2M+1}.$$

Proof. Put $n = -M-1$ and let $\alpha = \beta = \pi$ in Proposition 2.3. □

Corollary 2.5 should be compared with Proposition 4.24 and Proposition 4.25 in [3], p. 185.

PROPOSITION 2.6. *Let α, $\beta > 0$ with $\alpha\beta = \pi^2$. Then, for any integer n,*

$$\begin{aligned}
&\sqrt{3}\alpha^{-n-1/2}\sum_{k=1}^{\infty}\left(\frac{k}{3}\right)\frac{(-1)^{k+1}k^{-2n-2}}{e^{2\alpha k}-1} \\
&= (-1)^{n+1}\beta^{-n-1/2}\sum_{k=1}^{\infty}\frac{k^{-2n-2}\ \sinh(2\beta k/3)}{\sinh(\beta k)} \\
&\qquad -2^{2n+2}\pi\sum_{k=0}^{n+1}\frac{B_{2k+1}(1/6)B_{2n+2-2k}}{(2k+1)!(2n+2-2k)!}\alpha^{n-k+1/2}(-\beta)^k + \mathcal{K}_5(n),
\end{aligned}$$

where

$$\mathcal{K}_5(n) := \begin{cases} -3^{-2n-3/2}(2^{-2n-1}+1)\alpha^{-n-1/2}B_{-2n-1}(1/3)/(2n+1), & \text{if } n < -1, \\ -\frac{\sqrt{3}}{2}\alpha^{1/2}, & \text{if } n = -1, \\ -\frac{\sqrt{3}}{2}6^{-2n-2}\alpha^{-n-\frac{1}{2}}f_n, & \text{if } n \geq 0 \end{cases}$$

and $f_n := \zeta(2n+2, 1/3) - \zeta(2n+2, 2/3) + \zeta(2n+2, 1/6) - \zeta(2n+2, 5/6)$.

Proof. Put $\gamma = 1/6$ in Theorem 2.2. For $n < -1$, we have

$$\zeta\left(-2n-1, \frac{1}{6}\right) - \zeta\left(-2n-1, \frac{5}{6}\right) = \frac{(-1)^n (2^{-2n-1}+1)(6\pi)^{-2n-1} B_{-2n-1}(1/3)}{\sqrt{3}(-2n-1)!}.$$

For $n \geq 0$, we see

$$\begin{aligned}\operatorname{Im}\left(\psi\left(-\frac{1}{6}, 0, 2n+2\right)\right) &= \frac{\sqrt{3}}{2}\left(\sum_{k\equiv 1 \pmod 3} \frac{(-1)^k}{k^{2n+2}} - \sum_{k\equiv 2 \pmod 3} \frac{(-1)^k}{k^{2n+2}}\right)\\ &= -\frac{\sqrt{3}}{2} 6^{-2n-2}\left(\zeta\left(2n+2, \frac{1}{3}\right) - \zeta\left(2n+2, \frac{2}{3}\right) + \zeta\left(2n+2, \frac{1}{6}\right) - \zeta\left(2n+2, \frac{5}{6}\right)\right).\end{aligned}$$

Equate the imaginary parts in Theorem 2.2. □

Corollary 2.7. *Let α, $\beta > 0$ with $\alpha\beta = \pi^2$. Then*

$$\sqrt{3}\alpha^{1/2}\left(\sum_{k=1}^{\infty}\left(\frac{k}{3}\right)\frac{(-1)^{k+1}}{e^{2\alpha k}-1} + \frac{1}{2}\right) = \beta^{1/2}\left(\sum_{k=1}^{\infty}\frac{\sinh(2\beta k/3)}{\sinh(\beta k)} + \frac{1}{3}\right).$$

Proof. Put $n = -1$ in Proposition 2.6. □

Corollary 2.8. *For any positive integer M,*

$$\begin{aligned}&\sum_{k=1}^{\infty}\left(\frac{k}{3}\right)\frac{(-1)^k\, k^{2M}}{e^{2\pi k}-1}\\ &= \frac{(-1)^M}{\sqrt{3}}\sum_{k=1}^{\infty}\frac{k^{2M}\ \sinh(2\pi k/3)}{\sinh(\pi k)} + \frac{3^{2M}(2^{2M+1}+1)B_{2M+1}(1/3)}{2M+1}.\end{aligned}$$

Proof. Put $n = -M-1$ and let $\alpha = \beta = \pi$ in Proposition 2.6. □

Remark 2.9. In case of $N \mid r_1$ and $N \mid (r_1 + r_2)$, we have

$$\begin{aligned}z^{2n} H_N(V\tau, -2n; r, 0) &= H_N(\tau, -2n; R, 0) + (2\pi i)^{2n} L_N(\tau, -2n; R, 0)\\ &\quad + \lim_{s\to -2n}\left(D_1\left(0, \frac{r_2}{N}, s\right) + D_2\left(0, -r_1 - r_2 + \frac{r_2}{N}, s\right)\right).\end{aligned}$$

Hence employing Lemma 1.3, we obtain

$$\begin{aligned}&z^{2n}\sum_{k=1}^{\infty}\frac{1}{k^{2n+1}}\frac{2}{e^{2\pi ik/z}-1}\\ &= \sum_{k=1}^{\infty}\frac{1}{k^{2n+1}}\frac{2}{e^{-2\pi ikz}-1} - (2\pi i)^{2n+1}\sum_{k=0}^{2n+2}\frac{B_k B_{2n+2-k}}{k!(2n+2-k)!}z^{k-1} + J(n),\end{aligned}\tag{2.12}$$

where

$$J(n) := \begin{cases} \left(1 - z^{2n}\right)\zeta(1+2n), & \text{if } n \neq 0,\\ \pi i - \log z, & \text{if } n = 0.\end{cases}$$

For $z = \pi i/\alpha$, (2.12) was fully studied by Berndt [3]. In fact, many authors containing Ramanujan established various infinite series which come from (2.12) and these are written very well in [3]. For example, 'Ramanujan's Formula for $\zeta(2M+1)$' that is stated twice in Ramanujan's Notebooks [6](vol. I, p.259, no.15; vol. II, p.177, no.21) can be obtained from (2.12).

Acknowledgement. The author thanks B.C. Berndt for his suggestions and comments throughout this work.

References

[1] M. Abramowitz and I.A. Stegun, editor, *Handbook of mathematical functions*, New York 1965.

[2] B.C. Berndt, *Generalized Dedekind eta-functions and generalized Dedekind sums*, Trans. Amer. Math. Soc. **178** (1973), 495–508.

[3] B.C. Berndt, *Modular transformations and generalizations of several formulae of Ramanujan*, Rocky mountain J. Math. **7** (1) (1977), 147–189.

[4] B.C. Berndt, *Analytic Eisenstein series, theta-functions, and series relations in the spirit of Ramanujan*, J. Reine Angew. Math. **304** (1978), 332–365.

[5] J. Lagrange, *Une formule sommatoire et ses applications*, Bull. Sci. Math. (2) **84** (1960), 105–110.

[6] S. Ramanujan, *Notebooks of Srinivasa Ramanujan* (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.

[7] O. Schlömilch, *Ueber einige unendliche Reichen*, Berichte über die Verh. d. Konige Sachsischen Gesell. Wiss. zu Leipzig **29** (1877), 101–105.

[8] O. Schlömilch, *Compendium der hoheren Analysis*, Zweiter Band, 4th ed., Friedrich Vieweg und Sohn, Braunschweig, 1895.

[9] G.N. Watson, *Theorems stated by Ramanujan II*, J. Lond. Math. Soc. **3** (1928), 216–225.

[10] E.T. Whittaker and G. N. Watson, *A course in modern analysis*, 4th ed. Cambridge, England, 1990.

Department of Mathematics
Mokwon University
Mokwon Gil 21, Seo-gu
Daejeon, 302-318, Republic of Korea
E-mail: `sglimj@mokwon.ac.kr`