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INCLUSION PROPERTIES OF A CLASS OF FUNCTIONS INVOLVING THE DZIOK-SRIVASTAVA OPERATOR

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Abstract. In this work, we first introduce a class of analytic functions involving the Dziok-Srivastava linear operator that generalizes the class of uniformly starlike functions with respect to symmetric points. We then establish the closure of certain well-known integral transforms under this analytic function class. This behaviour leads to various radius results for these integral transforms. Some of the interesting consequences of these results are outlined. Further, the lower bounds for the ratio between the functions $f(z)$ in the class under discussion, their partial sums $f_m(z)$ and the corresponding derivative functions $f'(z)$ and $f'_m(z)$ are determined by using the coefficient estimates.

1. Introduction

Let $\mathcal{A}$ denote the class of all *normalized* analytic functions f defined in the open unit disk

$$\mathbb{D} = \{z : z \in \mathbb{C} \qquad \text{and} \qquad |z| < 1\},$$

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which satisfy the normalization condition:

$$f(0) = 0 = f'(0) - 1$$

and whose Taylor-Maclaurin series expansion is given as follows:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{D}). \tag{1.1}$$

Also let $\mathcal{S}$ be the class of functions in $\mathcal{A}$ which are univalent in $\mathbb{D}$.

A function $f \in \mathcal{S}$ is said to be in the class k-$\mathcal{US}^*(\xi)$ of k-uniformly starlike functions of order ξ and in the class k-$\mathcal{UCV}(\xi)$ of k-uniformly convex functions of order ξ if, for $k \geqq 0$ and $0 \leqq \xi < 1$, we have (see, for example, [4])

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \xi \qquad (z \in \mathbb{D})$$

and

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| + \xi \qquad (z \in \mathbb{D}),$$

respectively. These classes were introduced by Bharati et al. [4] and the geometric characterization for k-$\mathcal{UCV}(0) \equiv k$-$\mathcal{UCV}$ was given by Kanas et al. [14, 15]. We note also that, by the Alexander type integral transform [7], $f \in k$-$\mathcal{UCV}(\xi) \Longleftrightarrow zf' \in k$-$\mathcal{US}^*(\xi)$.

Particular values of k and ξ give interesting and useful subclasses of the univalent function class $\mathcal{S}$. For instance, if we set $k = 1$ and $\xi = 0$ in k-$\mathcal{UCV}(\xi)$, then we get the class $\mathcal{UCV}$ considered by Goodman [13] with the two-variable analytic characterization. The corresponding class 1-$\mathcal{US}^*(0) := \mathcal{US}^*$ was introduced by Rønning [22], who also gave the one-variable analytic characterization for both the classes $\mathcal{UCV}$ and $\mathcal{US}^*$ (see also [18]).

When $k = 0$, the classes k-$\mathcal{UCV}(\xi)$ and k-$\mathcal{US}^*(\xi)$ provide the analytic characterization for the well-known classes $\mathcal{C}(\xi)$ and $\mathcal{S}^*(\xi)$ of convex functions of order ξ in $\mathbb{D}$ and starlike functions of order ξ in $\mathbb{D}$ ($0 \leqq \xi < 1$), respectively. We observe that $\mathcal{C}(0) \equiv \mathcal{C}$ and $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ are the classes of functions in $\mathcal{S}$ that map the unit disk $\mathbb{D}$ onto domains that are, respectively, convex and starlike with respect to the origin.

Suppose next that

$$\alpha_j \in \mathbb{C} \quad (j = 1, \cdots, r) \quad \text{and} \quad \beta_j \in \mathbb{C} \setminus \{0, -1, -2, \cdots\} \quad (j = 1, \cdots, s)$$

are complex parameters. Then the generalized hypergeometric function ${}_rF_s$ is defined by the infinite series as follows:

$$ {}_rF_s(\alpha_1, \cdots, \alpha_r; \beta_1, \cdots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^n}{n!} \qquad (z \in \mathbb{D}), \tag{1.2} $$

where

$$ r \leqq s+1 \qquad (r, s \in \mathbb{N}_0 := \{0, 1, 2, \cdots\} = \mathbb{N} \cup \{0\}) $$

and the Pochhammer symbol $(\lambda)_n$ used on the right-hand side of (1.2) is given by

$$ (\lambda)_0 = 1 \qquad \text{and} \qquad (\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1) = \lambda(\lambda+1)_{n-1} \quad (n \in \mathbb{N}). $$

By convoluting the generalized hypergeometric function $z\,{}_rF_s(\alpha_1, \cdots, \alpha_r; \beta_1, \cdots, \beta_s; z)$ with the function $f(z) \in \mathcal{A}$, having series representation of the form (1.1), Dziok and Srivastava [8] (see, for more details, [1,9,17,20, 27,30]) introduced the Dziok-Srivastava linear operator $H_s^r(\alpha_1, \cdots, \alpha_r; \beta_1, \cdots, \beta_s; z)$ which is defined as follows:

$$ \begin{aligned} &H_s^r(\alpha_1, \cdots, \alpha_r; \beta_1, \cdots, \beta_s; z) f(z) \\ &:= \Big(z\,{}_rF_s(\alpha_1, \cdots, \alpha_r; \beta_1, \cdots, \beta_s; z) \Big) * f(z) \ = z + \sum_{n=2}^{\infty} \psi_n a_n z^n, \end{aligned} \tag{1.3} $$

where

$$ \psi_n = \frac{(\alpha_1)_{n-1} \cdots (\alpha_r)_{n-1}}{(n-1)!(\beta_1)_{n-1} \cdots (\beta_s)_{n-1}}. \tag{1.4} $$

Here the symbol $*$ means the Hadamard product (or convolution) which is given by the following representation:

$$ (f_1 * f_2)(z) = \sum_{n=0}^{\infty} a_{n,1} a_{n,2}\, z^n, \qquad \left(f_j(z) = \sum_{n=0}^{\infty} a_{n,j} z^n \quad (j = 1, 2); \ z \in \mathbb{D} \right). $$

We remark in passing that the existing literature on *Geometric Function Theory* also contains systematic investigations of various analytic function classes associated with a *further* generalization of the Dziok-Srivastava operator, which is known as the Wright-Srivastava operator defined by using the Fox-Wright generalized hypergeometric function (see, for details, [16] and [26]; see also [30] and the references cited therein including [16] and [26]).

Now, from the well-known result in [10], the following relation holds true:

$$\begin{aligned}\alpha_1 H_s^r(\alpha_1+1,\alpha_2,\cdots,\alpha_r;\beta_1,\cdots,\beta_s;z)f(z)\\ =z[H_s^r(\alpha_1,\alpha_2,\cdots,\alpha_r;\beta_1,\cdots,\beta_s;z)f(z)]'\\ +(\alpha_1-1)H_s^r(\alpha_1,\cdots,\alpha_r;\beta_1,\cdots,\beta_s;z)f(z).\end{aligned}$$

For convenience, we write

$$H_s^r(\alpha_1)f(z)=H_s^r(\alpha_1,\cdots,\alpha_r;\beta_1,\cdots,\beta_s;z)f(z)\quad (r\leqq s+1, r,s\in\mathbb{N}_0). \tag{1.5}$$

For the present work, using the Dziok-Srivastava linear operator defined by (1.3), the following modified class of normalized analytic functions is introduced.

Definition 1.1. For $0\leqq\mu<1$ and $\kappa\geqq 0$, the function $f\in\mathcal{A}$ whose series expansion is of the form (1.1) is said to be in $\mathcal{S}_s^r(\lambda,\kappa,\mu)$ if

$$\Re\left(\frac{2zF'(z)}{F(z)-F(-z)}\right)>\kappa\left|\frac{2zF'(z)}{F(z)-F(-z)}-1\right|+\mu\qquad z\in\mathbb{D}, \tag{1.6}$$

with

$$F(z):=\lambda[H_s^r(\alpha_1+1)f(z)]+(1-\lambda)[H_s^r(\alpha_1)f(z)],\quad 0\leqq\lambda\leqq 1, \tag{1.7}$$

where $r,s\in\mathbb{N}_0$ satisfies $r\leqq s+1$.

The particular case $\alpha_1=1$, $\alpha_2=1$ and $\beta_1=1$ in (1.5) gives interesting and well-known geometric classes. In this case

$$H_1^2(\alpha_1)=\frac{z}{1-z}.$$

For example, we note that the class $\mathcal{S}_1^2(0,0,0)\equiv\mathcal{S}_s^*$ for $\alpha_1=1$, $\alpha_2=1$ and $\beta_1=1$ where the analytic characterization for the class $\mathcal{S}_s^*$ is given as follows:

$$\Re\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)>0\qquad (z\in\mathbb{D}),$$

which is known as the class of starlike functions with respect to symmetrical points. This class was defined and studied by Sakaguchi [23]. For the study of some related classes see [2,3,11,21,28,29]. Similarly, $\mathcal{C}_s$ is the subclass of $\mathcal{S}$ consisting of the functions that are convex with respect to symmetric points and satisfy the following analytic characterization:

$$\Re\left(\frac{2(zf'(z))'}{f'(z)+f'(-z)}\right) > 0 \qquad (z \in \mathbb{D}).$$

The class $\mathcal{C}_s$ was discussed earlier by several authors (see [28,29] and the references therein) and can be obtained by choosing $\alpha_1 = 1$, $\alpha_2 = 1$ and $\beta_1 = 1$ such that

$$H_1^2(\alpha_1+1) = \frac{z}{(1-z)^2}.$$

Using our notation, we can write $\mathcal{S}_1^2(1,0,0) \equiv \mathcal{C}_s$ for $\alpha_1 = 1$, $\alpha_2 = 1$ and $\beta_1 = 1$.

The sequence $\{f_m\}$ of partial sums of the function f, whose series expansion is of the form (1.1), is defined by

$$f_m(z) = z + \sum_{n=2}^{m} a_n z^n \qquad (z \in \mathbb{D}).$$

Following the earlier work by Silverman [24] and Silvia [25] (see also [12, 19]), it would be interesting to obtain the lower bounds of

$$\Re\left(\frac{f(z)}{f_m(z)}\right), \quad \Re\left(\frac{f_m(z)}{f(z)}\right), \quad \Re\left(\frac{f'(z)}{f'_m(z)}\right) \qquad \text{and} \qquad \Re\left(\frac{f'_m(z)}{f'(z)}\right)$$

for the class $\mathcal{S}_s^r(\lambda,\kappa,\mu)$ and to determine the sharpness.

Our present investigation is organized as follows. In Section 2, the coefficient bounds are obtained for the class $\mathcal{S}_s^r(\lambda,\kappa,\mu)$ and verification of this class in terms of the Taylor coefficients is also given. In Section 3, radii results for $f \in \mathcal{S}_s^r(\lambda,\kappa,\mu)$ are obtained by using its coefficient estimate to establish the fact that various integral operators map the function $f \in \mathcal{S}_s^r(\lambda,\kappa,\mu)$ to various subclasses of $\mathcal{S}$. For functions $f(z)$ in $\mathcal{S}_s^r(\lambda,\kappa,\mu)$, considering the real part, the lower bound for the ratio between the function $f(z)$, its partial sum $f_m(z)$ and also between the corresponding derivatives $f'(z)$ and $f'_m(z)$ are determined in Section 4.

2. Coefficient Conditions

The main interest of the section is to find certain equivalent conditions for the class $\mathcal{S}_s^r(\lambda,\kappa,\mu)$ using the Taylor coefficients of functions in this class.

Before proceeding further, we make an observation regarding the inequality (1.6) which gives the analytic characterization for the class $\mathcal{S}_s^r(\lambda, \kappa, \mu)$ and write it as a remark to use it in the sequel.

REMARK 2.1. *Using the fact that*

$$\Re(\upsilon) > \kappa|\upsilon - 1| + \mu \Longleftrightarrow \Re\left(\upsilon(1+\kappa e^{i\theta}) - \kappa e^{i\theta}\right) > \mu \qquad (-\pi < \theta < \pi),$$

(1.6) *can be rewritten as follows:*

$$\Re\left(\frac{2(1+\kappa e^{i\theta})zF'(z) - \kappa e^{i\theta}[F(z) - F(-z)]}{F(z) - F(-z)}\right) > \mu \tag{2.1}$$

for the function $F(z)$ *defined in* (1.7). *Further the series expansion of the function* $F(z)$ *is*

$$F(z) = z + \sum_{n=2}^{\infty}\left(\lambda(\varphi_n - 1) + 1\right)\psi_n a_n z^n, \tag{2.2}$$

where φ_n *and* ψ_n *are defined in* (2.5) *and* (1.4), *respectively. If we assume*

$$\begin{aligned} X(z) &:= 2(1+\kappa e^{i\theta})zF'(z) - \kappa e^{i\theta}[F(z) - F(-z)] \\ &= 2z + \sum_{n=2}^{\infty}\left(2n(1+\kappa e^{i\theta}) - \kappa e^{i\theta}[1-(-1)^n]\right)(\lambda[\varphi_n - 1] + 1)\psi_n a_n z^n \end{aligned}$$

and

$$Y(z) := [F(z) - F(-z)] = 2z + \sum_{n=2}^{\infty}[1-(-1)^n](\lambda[\varphi_n - 1] + 1)\psi_n a_n z^n,$$

then (2.1) *is equivalent to*

$$|X(z) + (1-\mu)Y(z)| \geqq |X(z) - (1+\mu)Y(z)| \quad (0 \leqq \mu < 1). \tag{2.3}$$

The coefficient estimate for the function belonging to the class $\mathcal{S}_s^r(\lambda, \kappa, \mu)$ is given in the following lemma. For proving this result, the technique applied in [20] is used.

LEMMA 2.1. *For* $0 \leqq \lambda \leqq 1$, $0 \leqq \mu < 1$ *and* $\kappa \geqq 0$, *the function* $f \in \mathcal{S}_s^r(\lambda, \kappa, \mu)$ *if and only if*

$$\sum_{n=2}^{\infty}\left(2n(1+\kappa) - (\mu+\kappa)[1-(-1)^n]\right)(\lambda[\varphi_n - 1] + 1)\psi_n|a_n| \leqq 2(1-\mu), \tag{2.4}$$

where φ_n is defined by

$$\varphi_n = \frac{(\beta_1 \cdots \beta_s)[(\alpha_1 + n - 1) \cdots (\alpha_r + n - 1)]}{(\alpha_1 \cdots \alpha_r)[(\beta_1 + n - 1) \cdots (\beta_s + n - 1)]} \tag{2.5}$$

and ψ_n is given by (1.4)*. The result is sharp.*

Proof. Let f is of the form (1.1) and satisfies the inequality (2.4), then to show that $f \in \mathcal{S}_s^r(\lambda, \kappa, \mu)$ from Remark 2.1, it is enough to prove inequality (2.3).

From (2.2), we get

$$\begin{aligned} &|X(z) + (1-\mu)Y(z)| \geqq 2(2-\mu)|z| \\ &- \sum_{n=2}^{\infty} \Big(2n(1+\kappa) + (1-\mu-\kappa)[1-(-1)^n] \Big(\lambda[\varphi_n - 1] + 1)\psi_n |a_n| \cdot |z^n|. \end{aligned} \tag{2.6}$$

Similarly, we find that

$$\begin{aligned} &|X(z) - (1+\mu)Y(z)| \leqq 2\mu|z| \\ &+ \sum_{n=2}^{\infty} \Big(2n(1+\kappa) - (1+\mu+\kappa)[1-(-1)^n] \Big(\lambda[\varphi_n - 1] + 1)\psi_n |a_n| \cdot |z^n|. \end{aligned} \tag{2.7}$$

Upon subtracting (2.6) from (2.7), we obtain

$$\begin{aligned} &|X(z) + (1-\mu)Y(z)| - |X(z) - (1+\mu)Y(z)| \geqq 4(1-\mu)|z| \\ &- \sum_{n=2}^{\infty} \Big(4n(1+\kappa) - 2(\mu+\kappa)[1-(-1)^n] \Big) (\lambda[\varphi_n - 1] + 1)\psi_n |a_n| \cdot |z^n|, \end{aligned}$$

which is true for all values of $|z| < 1$. Using (2.4) and letting $z \to 1^-$ in the above expression, we get

$$\begin{aligned} &|X(z) + (1-\mu)Y(z)| - |X(z) - (1+\mu)Y(z)| \geqq 4(1-\mu) \\ &- \sum_{n=2}^{\infty} \Big(4n(1+\kappa) - 2(\mu+\kappa)[1-(-1)^n] \Big) (\lambda[\varphi_n - 1] + 1)\psi_n |a_n| \geq 0 \end{aligned}$$

which proves inequality (2.3) and hence $f \in \mathcal{S}_s^r(\lambda, \kappa, \mu)$.

Conversely, we suppose that $f \in \mathcal{S}_s^r(\lambda, \kappa, \mu)$ and we deduce (2.4). Now $f \in \mathcal{S}_s^r(\lambda, \kappa, \mu)$ is equivalent to assuming (2.1). Choosing z along positive real axis with $0 \leqq |z| = r < 1$, it is easy to see that (2.1) is equivalent to

$$\Re\left(\frac{2r+\sum_{n=2}^{\infty}\left(2n(1+\kappa e^{i\theta})-\kappa e^{i\theta}[1-(-1)^n]\right)(\lambda[\varphi_n-1]+1)\psi_n a_n r^n}{2r+\sum_{n=2}^{\infty}[1-(-1)^n](\lambda[\varphi_n-1]+1)\psi_n a_n r^n}\right) \geqq \mu.$$

Since

$$\Re\left(-e^{i\theta}\right) > -1, \quad (\pi < \theta < \pi),$$

the above inequality reduces to

$$\Re\left(\frac{2(1-\mu)+\sum_{n=2}^{\infty}\left(2n(1+\kappa)-(\kappa+\mu)[1-(-1)^n]\right)(\lambda[\varphi_n-1]+1)\psi_n a_n r^{n-1}}{2+\sum_{n=2}^{\infty}[1-(-1)^n](\lambda[\varphi_n-1]+1)\psi_n a_n r^{n-1}}\right) \geqq 0.$$

Now, letting $r \to 1^-$ gives (2.4) and the result is proved. $\square$

We now provide another result that have conditions on the Taylor coefficients that suffices the corresponding function to be in $\mathcal{S}_s^r(\lambda, \kappa, \mu)$, which requires the following lemma.

Lemma 2.2. *A function $f(z)$ is in the class $\mathcal{S}_s^r(\lambda, \kappa, \mu)$ if and only if*

$$1 + \sum_{n=2}^{\infty} D_n L_n z^{n-1} \neq 0, \tag{2.8}$$

where

$$D_n := \frac{2n(x+1)(1+\kappa e^{i\theta}) - [1-(-1)^n]((x+1)(\mu+\kappa)e^{i\theta} + (x-1)(1-\mu))}{2[2-\mu(1-x)-\mu e^{i\theta}(1+x)]}$$

and

$$L_n = [\lambda(\varphi_n - 1) + 1]\psi_n a_n.$$

Proof. From Remark 2.1 we see that $f \in \mathcal{S}_s^r(\lambda, \kappa, \mu)$ is equivalent to $\Re p(z) > 0$ where

$$p(z) = \frac{\left(\dfrac{2(1+\kappa e^{i\theta})zF'(z) - (\mu+\kappa)e^{i\theta}[F(z)-F(-z)]}{F(z)-F(-z)}\right)}{1-\mu}.$$

Since

$$\Re\big(p(z)\big) > 0 \Longleftrightarrow p(z) \neq \frac{x-1}{x+1} \qquad (|x|=1;\ z \in \mathbb{D}),$$

we have

$$\frac{\left(\dfrac{2(1+\kappa e^{i\theta})zF'(z) - (\mu+\kappa)e^{i\theta}[F(z)-F(-z)]}{F(z)-F(-z)}\right)}{1-\mu} \neq \frac{x-1}{x+1} \tag{2.9}$$

for $|x| = 1$ $(x \neq 1)$ and $z \in \mathbb{D}$. For $z = 0$, we observe that

$$p(0) = 1 \neq \frac{x-1}{x+1}.$$

Simplifying (2.9) gives

$$\begin{aligned} 2(x+1)(1+\kappa e^{i\theta})zF'(z) - [(x+1)(\mu+\kappa)e^{i\theta} \\ + (x-1)(1-\mu)][F(z)-F(-z)]) \neq 0. \end{aligned}$$

This last equation can be further simplified as follows:

$$\begin{aligned} 2\left[2-\mu(1-x)-\mu e^{i\theta}(1+x)\right] z + \sum_{n=2}^{\infty}\Big(& 2n(x+1)\left(1+\kappa e^{i\theta}\right) \\ -[1-(-1)^n]\left[(x+1)(\mu+\kappa)e^{i\theta} + (x-1)(1-\mu)\right] & \Big) \\ \cdot[\lambda(\varphi_n - 1)+1]\psi_n a_n z^n & \neq 0, \end{aligned}$$

which is equivalent to (2.8). □

THEOREM 2.1. *If $f \in \mathcal{A}$ satisfies the inequality:*

$$\begin{aligned} &\sum_{n=2}^{\infty}\left(\left|\sum_{m=1}^{n}\left(\sum_{p=1}^{m} 2p(1+\kappa e^{i\theta})(-1)^{m-p}\binom{\delta}{m-p}L_p\right)\binom{\gamma}{n-m}\right|\right. \\ &\quad \left. + \left|\sum_{m=1}^{n}\left(\sum_{p=1}^{m}(1-(-1)^p)(\mu+\kappa)e^{i\theta}(-1)^{m-p}\binom{\delta}{m-p}L_p\right)\binom{\gamma}{n-m}\right|\right) \\ &\quad \leqq 2(1-\mu), \end{aligned}$$

where

$$L_p = [\lambda(\varphi_p - 1) + 1]\psi_p a_p$$

and $\delta, \gamma \in \mathbb{R}$, *then* $f \in \mathcal{S}_s^r(\lambda, \kappa, \mu)$.

Proof. In order to show that $f \in \mathcal{S}_s^r(\lambda, \kappa, \mu)$, it suffices to obtain the condition (2.8) which has the equivalent form

$$\begin{aligned}(1-z)^\delta(1+z)^\gamma &\left(1 + \sum_{n=2}^{\infty} D_n L_n z^{n-1}\right) \\ &= 1 + \sum_{n=2}^{\infty}\left(\sum_{m=1}^{n}\left(\sum_{p=1}^{m}(-1)^{m-p}\binom{\delta}{m-p} D_p L_p\right)\binom{\gamma}{n-m}\right) z^{n-1} \\ &\neq 0.\end{aligned}$$

Thus the function $f(z)$ should satisfy the following inequality:

$$\sum_{n=2}^{\infty}\left|\left(\sum_{m=1}^{n}\left(\sum_{p=1}^{m}(-1)^{m-p}\binom{\delta}{m-p} D_p L_p\right)\binom{\gamma}{n-m}\right)\right| \leqq 1.$$

In light of the hypothesis, this inequality can be obtained by a direct computation. □

REMARK 2.2. *Substituting* $\delta = 0$ *and* $\gamma = 0$ *in Theorem 2.1 provides that if the function* $f(z)$ *satisfies the inequality*

$$\sum_{n=2}^{\infty}\Big(2n(1+\kappa) + [1 - (-1)^n](\mu + \kappa)\Big)(\lambda[\varphi_n - 1] + 1)\psi_n |a_n| \leqq 2(1-\mu),$$

then $f \in \mathcal{S}_s^r(\lambda, \kappa, \mu)$, *which is the sufficiency part of Lemma* 2.1. *Hence, up to sufficiency,* Theorem 2.1 *is more general than Lemma* 2.1.

3. Radius Results for a Family of Integral Operators

For the non-negative and real-valued integrable function $\eta(t)$ satisfying the condition:

$$\int_0^1 \eta(t)dt = 1,$$

the integral transform for the function $f \in \mathcal{A}$ is defined as follows:

$$V_\eta\big(f(z)\big) = \int_0^1 \eta(t)\,\frac{f(zt)}{t}\,dt. \tag{3.1}$$

In this section, it is proved that the class $\mathcal{S}_s^r(\lambda,\kappa,\mu)$ is closed under various integral operators (see also [5, 17]) which are associated with several particular cases of $\eta(t)$. We start with the following. If $\eta(t)$ is chosen as

$$\eta(t) = \frac{(c+1)^\delta}{\Gamma(\delta)} t^c \left[\log\left(\frac{1}{t}\right)\right]^{\delta-1} \qquad (c > -1;\ \delta \geqq 0)$$

in (3.1), then the integral operator obtained is given by

$$V_\eta\big(f(z)\big) := F_{c,\delta}\big(f(z)\big),$$

which reduces to the Komatu integral operator given by (see [6] and the references therein)

$$F_{c,\delta}\big(f(z)\big) = \frac{(c+1)^\delta}{\Gamma(\delta)} \int_0^1 t^c \left[\log\left(\frac{1}{t}\right)\right]^{\delta-1} f(zt)dt. \tag{3.2}$$

It is important to observe that, whenever we set $\delta = 1$ in (3.2), the Komatu integral operator $F_{c,\delta}\big(f(z)\big)$ reduces further to the Bernardi integral operator denoted by $\mathcal{B}_c\big(f(z)\big)$. Various results for closure of the Komatu operator are available in the literature for the subclasses of the univalent function class $\mathcal{S}$. But the results corresponding to the generalized hypergeometric transform given by (1.3) is (presumably) new in this direction.

Theorem 3.1. *Assume that $f \in \mathcal{S}_s^r(\lambda,\kappa,\mu)$. Then $F_{c,\delta}\big(f(z)\big)$ is in the class k-$\mathcal{US}^*(\rho)$ in the disk $|z| < r_1$, where*

$$r_1 = \inf_n \left(\frac{(1-\rho)\Big(2n(1+\kappa) - (\mu+\kappa)[1-(-1)^n]\Big)(\lambda[\varphi_n - 1]+1)\psi_n}{2[(n-1)(k+1)+1-\rho](1-\mu)} \cdot \left(\frac{c+n}{c+1}\right)^\delta \right)^{1/(n-1)}$$

for $k \geqq 0$, $0 \leqq \rho < 1$, $c > -1$, $\delta \geqq 0$ and $n \geqq 2$. The result is sharp for the function given by (4.1).

Proof. In order to show that $F_{c,\delta}\big(f(z)\big)$ is in the class k-$\mathcal{US}^*(\rho)$ for $f(z) \in \mathcal{S}_s^r(\lambda,\kappa,\mu)$, it is sufficient to derive the following inequality:

$$(k+1)\left|\frac{z\big(F_{c,\delta}(f(z))\big)'}{F_{c,\delta}\big(f(z)\big)} - 1\right| \leqq 1-\rho \qquad (|z| < r_1)$$

or, equivalently,

$$\sum_{n=2}^{\infty} \frac{[(n-1)(k+1)+1-\rho]}{1-\rho} \left(\frac{c+1}{c+n}\right)^{\delta} a_n |z|^{n-1} \leqq 1 \qquad (|z| < r_1).$$

Using Lemma 2.1, the above condition is satisfied if

$$\frac{(n-1)(k+1)+(1-\rho)}{1-\rho} \left(\frac{c+1}{c+n}\right)^{\delta} |z|^{n-1}$$
$$\leqq \frac{\Big(2n(1+\kappa)-(\mu+\kappa)[1-(-1)^n]\Big)(\lambda[\varphi_n-1]+1)\psi_n}{2(1-\mu)}.$$

An easy calculation provides the required result. Consider the function $f_n(z)$ given by

$$f_n(z) = z + \frac{2(1-\mu)}{\Big(2n(1+\kappa)-(\mu+\kappa)[1-(-1)^n]\Big)(\lambda[\varphi_n-1]+1)\psi_n} z^n$$

satisfying the hypothesis of Theorem 3.1. Then

$$\frac{z\big(F_{c,\delta}\big(f_n(z)\big)\big)'}{F_{c,\delta}\big(f_n(z)\big)} - 1 = \frac{(n-1)A_n z^{n-1}}{1+A_n z^{n-1}},$$

where

$$A_n = \frac{2(1-\mu)}{\Big(2n(1+\kappa)-(\mu+\kappa)[1-(-1)^n]\Big)(\lambda[\varphi_n-1]+1)\psi_n} \left(\frac{c+1}{c+n}\right)^{\delta}.$$

For $|z| = r_1$, we get

$$(k+1)\left|\frac{(n-1)A_n r_1^{n-1}}{1+A_n r_1^{n-1}}\right| = 1-\rho,$$

which shows that the radius r_1 for $f_n(z)$ is sharp. □

Corollary 3.1. *Assume that* $f \in \mathcal{S}_s^r(\lambda, \kappa, \mu)$. *Then* $F_{c,\delta}\big(f\big) \in k-\mathcal{UCV}(\rho)$ *in the disk* $|z| < r_2$, *where*

$$r_2 = \inf_n \left(\frac{(1-\rho)\Big(2n(1+\kappa)-(\mu+\kappa)[1-(-1)^n]\Big)(\lambda[\varphi_n-1]+1)\psi_n}{2n[(n-1)(k+1)+1-\rho](1-\mu)} \cdot \left(\frac{c+n}{c+1}\right)^{\delta}\right)^{1/(n-1)}$$

for $k \geqq 0$, $0 \leqq \rho < 1$, $c > -1$, $\delta \geqq 0$ *and* $n \geqq 2$.

Proof. It is well known that

$$zf' \in k\text{–}\mathcal{US}^*(\rho) \Longleftrightarrow f \in k\text{–}\mathcal{UCV}(\rho).$$

Since Theorem 3.1 is true for the class k-$\mathcal{US}^*(\rho)$, upon replacing $f(z)$ by $zf'(z)$ in Theorem 3.1, we get the required result. $\square$

EXAMPLE 3.1. Consider $k = 0$ in Theorem 3.1. Then $F_{c,\delta}(\mathcal{S}_s^r(\lambda, \kappa, \mu)) \in \mathcal{S}^*(\rho)$ in the disk $|z| < r_3$, where

$$r_3 = \inf_n \left(\frac{(1-\rho)\left(2n(1+\kappa) - (\mu+\kappa)[1-(-1)^n]\right)(\lambda[\varphi_n - 1] + 1)\psi_n}{2(n-\rho)(1-\mu)} \cdot \left(\frac{c+n}{c+1}\right)^{\delta}\right)^{1/(n-1)}$$

for $c > -1$, $\delta \geqq 0$ and $n \geqq 2$.

REMARK 3.1. Since $zf' \in \mathcal{S}^*(\rho) \Longleftrightarrow f \in \mathcal{C}(\rho)$, if we replace $f(z)$ by $zf'(z)$ in Example 3.1, we get $F_{c,\delta}(\mathcal{S}_s^r(\lambda, \kappa, \mu)) \in \mathcal{C}(\rho)$ in the disk $|z| < r_4$, where

$$r_4 = \inf_n \left(\frac{(1-\rho)\left(2n(1+\kappa) - (\mu+\kappa)[1-(-1)^n]\right)(\lambda[\varphi_n - 1] + 1)\psi_n}{2n(n-\rho)(1-\mu)} \cdot \left(\frac{c+n}{c+1}\right)^{\delta}\right)^{1/(n-1)}$$

for $c > -1$, $\delta \geqq 0$ and $n \geqq 2$.

For $a, b, c > 0$, if $\eta(t)$ in (3.1) has the following particular value:

$$\eta(t) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} t^{b-1}(1-t)^{c-a-b} \cdot {}_2F_1(c-a, 1-a; c-a-b+1; 1-t),$$

then the resulting integral operator: $V_\eta\big(f(z)\big) := H_{a,b,c}\big(f(z)\big)$ is known as the Hohlov operator. We note that

$$H_{a,b,c}\big(f(z)\big) = z\, {}_2F_1(a, b; c; z) * f(z), \tag{3.3}$$

which unifies several well-known operators such as the Carlson-Shaffer operator ($a = 1$) and the Bernardi operator ($a = 1$; $b = c'+1$; $c = c'+2$). We note also that, for $c' = 0$ and $c' = 1$, the Bernardi operator reduces, respectively, to the Alexander operator and the Libera operator. Hence the Hohlov operator and the Komatu operator are two different generalizations of the Bernardi integral operator. Furthermore, it is interesting to observe that for $r = 2$ and $s = 1$, (1.3) reduces to the Hohlov operator $H_{a,b,c}\big(f(z)\big)$. However, a proper representation of (1.3) in terms of the function $\eta(t)$ given by (3.1) is not available in the literature.

THEOREM 3.2. *Assume that* $f \in \mathcal{S}_s^r(\lambda, \kappa, \mu)$. *Then* $H_{a,b,c}(f) \in k\text{–}\mathcal{US}^*(\rho)$ *in the disk* $|z| < r_{11}$, *where*

$$r_{11} = \inf_n \Bigg[\frac{(1-\rho)\,(2n(1+\kappa) - (\mu+\kappa)(1-(-1)^n))\,(\lambda[\varphi_n - 1] + 1)\psi_n}{2((n-1)(k+1) + 1 - \rho)(1-\mu)} \cdot \left(\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right) \Bigg]^{1/(n-1)}$$

for $k \geqq 0$, $0 \leqq \rho < 1$, $0 < b \leqq 1$ *and* $0 < a \leqq c$. *The result is sharp for the function given by* (4.1).

Proof. To prove that $H_{a,b,c}\big(f\big) \in k\text{–}\mathcal{US}^*(\rho)$ for $f \in \mathcal{S}_s^r(\lambda, \kappa, \mu)$, it is sufficient to obtain

$$(k+1) \left| \frac{z\Big(H_{a,b,c}\big(f(z)\big)\Big)'}{H_{a,b,c}\big(f(z)\big)} - 1 \right| \leqq 1 - \rho \qquad)(|z| < r_{11})$$

or, equivalently,

$$\sum_{n=2}^{\infty} \frac{((n-1)(k+1) + 1 - \rho)}{1-\rho} \left(\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right) a_n |z|^{n-1} \leqq 1 \qquad (|z| < r_{11}).$$

Using Lemma 2.1, the above condition is satisfied if

$$\frac{(n-1)(k+1) + (1-\rho)}{1-\rho} \left(\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right) |z|^{n-1} \leqq \frac{\Big(2n(1+\kappa) - (\mu+\kappa)[1-(-1)^n]\Big)\,(\lambda[\varphi_n - 1] + 1)\psi_n}{2(1-\mu)},$$

which is precisely the hypothesis of Theorem 3.2. Further, the function $f_n(z)$ given by

$$f_n(z) = z + \frac{2(1-\mu)}{\Big(2n(1+\kappa) - (\mu+\kappa)[1-(-1)^n]\Big)\,(\lambda[\varphi_n - 1] + 1)\psi_n} z^n$$

satisfies the hypothesis of Theorem 3.2. Therefore, we have

$$\frac{z\Big(H_{a,b,c}\big(f_n(z)\big)\Big)'}{H_{a,b,c}(f_n(z)} - 1 = \frac{(n-1)A_n z^{n-1}}{1 + A_n z^{n-1}},$$

where

$$A_n = \frac{2(1-\mu)}{\Big(2n(1+\kappa) - (\mu+\kappa)[1-(-1)^n]\Big)\,(\lambda[\varphi_n - 1] + 1)\psi_n} \left(\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right).$$

Now, if $|z| = r_{11}$, we get

$$(k+1)\left|\frac{(n-1)A_n r_{11}^{n-1}}{1 + A_n r_{11}^{n-1}}\right| = (1-\rho),$$

which shows that the radius r_{11} for $f_n(z)$ is sharp. □

Corollary 3.2. *Assume that $f \in \mathcal{S}_s^r(\lambda, \kappa, \mu)$. Then $H_{a,b,c}(f) \in k\text{–}\mathcal{UCV}(\rho)$ in the disk $|z| < r_{12}$, where*

$$r_{12} = \inf_n \left(\frac{(1-\rho)\Big(2n(1+\kappa) - (\mu+\kappa)[1-(-1)^n]\Big)\,(\lambda[\varphi_n - 1] + 1)\psi_n}{2n[(n-1)(k+1) + 1 - \rho](1-\mu)} \cdot \left(\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right)\right)^{1/(n-1)}$$

for $k \geqq 0$, $0 \leqq \rho < 1$, $0 < b \leqq 1$ and $0 < a \leqq c$.

Proof. It is well known that $g = zf' \in k\text{–}\mathcal{US}^*(\rho) \Longleftrightarrow f \in k\text{–}\mathcal{UCV}(\rho)$. Since Theorem 3.2 is true for the class $k - \mathcal{US}^*(\rho)$, upon replacing $f(z)$ by $zf'(z)$ in Theorem 3.2, we obtain the required result. □

Example 3.2. Consider $k = 0$ in Theorem 3.2. Then $H_{a,b,c}(\mathcal{S}_s^r(\lambda, \kappa, \mu)) \in \mathcal{S}^*(\rho)$, in the disk $|z| < r_{13}$, where

$$r_{13} = \inf_n \left(\frac{(1-\rho)(n\cos\phi + \kappa(n-1) - \mu)(\lambda[\varphi_n - 1] + 1)\psi_n}{(n-\rho)(\cos\phi - \mu)} \cdot \left(\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right) \right)^{1/(n-1)}$$

for $0 \leqq \rho < 1$, $0 < b \leqq 1$ and $0 < a \leqq c$.

Remark 3.2. *Since $zf' \in \mathcal{S}^*(\rho) \iff f \in \mathcal{C}(\rho)$, if we replace $f(z)$ by $zf'(z)$ in Example 3.2, we get $H_{a,b,c}(\mathcal{S}_s^r(\lambda, \kappa, \mu)) \in \mathcal{C}(\rho)$ in the disk $|z| < r_{14}$, where*

$$r_{14} = \inf_n \left(\frac{(1-\rho)(n\cos\phi + \kappa(n-1) - \mu)(\lambda[\varphi_n - 1] + 1)\psi_n}{n(n-\rho)(\cos\phi - \mu)} \cdot \left(\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right) \right)^{1/(n-1)}$$

for $0 \leqq \rho < 1$, $0 < b \leqq 1$ and $0 < a \leqq c$.

For $a > -1$ and $b > -1$, consider $\eta(t)$ given in (3.1) with the following particular value:

$$\eta(t) = \begin{cases} (a+1)(b+1)\left(\dfrac{t^a - t^b}{b-a}\right) & (b \neq a) \\[2ex] (a+1)^2 t^a \log\left(\dfrac{1}{t}\right) & (b = a). \end{cases} \tag{3.4}$$

The integral transform $V_\eta\big(f(z)\big)$, which is defined in (3.1) with $\eta(t)$ given by (3.4), becomes the convolution operator $\mathcal{G}_{a,b}\big(f(z)\big)$, where

$$\mathcal{G}_{a,b}\big(f(z)\big) = \left(\sum_{n=1}^{\infty} \frac{(a+1)(b+1)}{(a+n)(b+n)} z^n \right) * f(z). \tag{3.5}$$

The integral transform $\mathcal{G}_{a,b}\big(f(z)\big)$ has been studied extensively using duality techniques by several authors for certain classes of analytic functions. For example, see [6].

THEOREM 3.3. *Assume that* $f \in \mathcal{S}_s^r(\lambda, \kappa, \mu)$. *Then* $\mathcal{G}_{a,b}\big(f\big) \in k\text{–}\mathcal{US}^*(\rho)$ *in the disk* $|z| < r_{21}$, *where*

$$r_{21} = \begin{cases} \left(R_n \left(\dfrac{(a+1)(b+1)}{(a+n)(b+n)}\right)\right)^{1/(n-1)} & (a \neq b) \\[2ex] \left(R_n \left(\dfrac{a+1}{a+n-1}\right)^2\right)^{1/(n-1)} & (a = b), \end{cases}$$

where

$$R_n := \frac{(1-\rho)\Big(2n(1+\kappa) - (\mu+\kappa)[1-(-1)^n]\Big)(\lambda[\varphi_n - 1] + 1)\psi_n}{2((n-1)(k+1) + 1 - \rho)(1-\mu)}$$

for $k \geqq 0$, $0 \leqq \rho < 1$, $a > -1$ *and* $b > -1$. *The result is sharp for the function given by* (4.1).

Proof. To show that $\mathcal{G}_{a,b}\big(f\big) \in k\text{–}\mathcal{US}^*(\rho)$, for $f \in \mathcal{S}_s^r(\lambda, \kappa, \mu)$, it is sufficient to obtain

$$(k+1)\left|\frac{z(\mathcal{G}_{a,b}(f(z)))'}{\mathcal{G}_{a,b}(f(z))} - 1\right| \leqq 1 - \rho \qquad (|z| < r_{21}). \tag{3.6}$$

As in our demonstration of Lemma 2.1, we consider each of the following two possibilities.
Case (i): $a \neq b$. The condition (3.6) is equivalent to

$$\sum_{n=2}^{\infty} \frac{[(n-1)(k+1) + 1 - \rho]}{1-\rho} \left(\frac{(a+1)(b+1)}{(a+n)(b+n)}\right) a_n |z|^{n-1} \leqq 1 \qquad (|z| < r_{21}).$$

Using Lemma 2.1, the above condition is satisfied if

$$\begin{aligned} &\frac{(n-1)(k+1) + 1 - \rho}{1-\rho} \left(\frac{(a+1)(b+1)}{(a+n)(b+n)}\right) |z|^{n-1} \\ &\quad \leqq \frac{\Big(2n(1+\kappa) - (\mu+\kappa)[1-(-1)^n]\Big)(\lambda[\varphi_n - 1] + 1)\psi_n}{2(1-\mu)}. \end{aligned}$$

Case (ii): $a = b$. The condition (3.6) is equivalent to

$$\sum_{n=2}^{\infty} \frac{(n-1)(k+1) + 1 - \rho}{1-\rho} \left(\frac{a+1}{a+n-1}\right)^2 a_n |z|^{n-1} \leqq 1 \qquad (|z| < r_{21}).$$

Using Lemma 2.1, the above condition is satisfied if

$$\frac{(n-1)(k+1)+1-\rho}{1-\rho}\left(\frac{a+1}{a+n-1}\right)^2|z|^{n-1}$$
$$\leqq \frac{\Big(2n(1+\kappa)-(\mu+\kappa)[1-(-1)^n]\Big)(\lambda[\varphi_n-1]+1)\psi_n}{2(1-\mu)}.$$

Both the obtained inequalities are implied by the hypothesis of Theorem 3.3. Hence the result is true.

Now the function $f_n(z)$ given by

$$f_n(z)=z+\frac{2(1-\mu)}{\Big(2n(1+\kappa)-(\mu+\kappa)[1-(-1)^n]\Big)(\lambda[\varphi_n-1]+1)\psi_n}z^n$$

satisfies the hypothesis of Theorem 3.3. Therefore, we have

$$\frac{z\Big(\mathcal{G}_{a,b}\big(f_n(z)\big)\Big)'}{\mathcal{G}_{a,b}\big(f_n(z)\big)}-1=\frac{(n-1)A_nz^{n-1}}{1+A_nz^{n-1}},$$

where
A_n :

$$=\begin{cases}\dfrac{2(1-\mu)}{(2n(1+\kappa)-(\mu+\kappa)(1-(-1)^n))(\lambda[\varphi_n-1]+1)\psi_n}\dfrac{(a+1)(b+1)}{(a+n)(b+n)} & (a\neq b)\\[2ex] \dfrac{2(1-\mu)}{\Big(2n(1+\kappa)-(\mu+\kappa)[1-(-1)^n]\Big)(\lambda[\varphi_n-1]+1)\psi_n}\left(\dfrac{a+1}{a+n-1}\right)^2 & (a=b),\end{cases}$$

which, for $|z|=r_{21}$, yields

$$(k+1)\left|\frac{(n-1)A_nr_{21}^{n-1}}{1+A_nr_{21}^{n-1}}\right|=(1-\rho)$$

leading to sharpness of the result. □

COROLLARY 3.3. *Assume that* $f(z) \in \mathcal{S}_s^r(\lambda, \kappa, \mu)$. *Then* $\mathcal{G}_{a,b}\big(f(z)\big) \in k\text{–}\mathcal{UCV}(\rho)$ *in the disk* $|z| < r_{22}$, *where*

$$r_{22} = \begin{cases} \left(S_n\left(\dfrac{(a+1)(b+1)}{(a+n)(b+n)}\right)\right)^{1/(n-1)} & (a \neq b) \\ \left(S_n\left(\dfrac{a+1}{a+n-1}\right)^2\right)^{1/(n-1)} & (a = b), \end{cases}$$

where

$$S_n := \frac{(1-\rho)\Big(2n(1+\kappa) - (\mu+\kappa)[1-(-1)^n]\Big)(\lambda[\varphi_n - 1]+1)\psi_n}{2n[(n-1)(k+1)+1-\rho](1-\mu)}$$

for $k \geqq 0$, $0 \leqq \rho < 1$, $a > -1$ *and* $b > -1$.

Proof. It is well known that

$$zf' \in k\text{–}\mathcal{US}^*(\rho) \Longleftrightarrow f \in k\text{–}\mathcal{UCV}(\rho).$$

Since Theorem 3.3 is true for the class k-$\mathcal{US}^*(\rho)$, upon replacing $f(z)$ by $zf'(z)$ in Theorem 3.3, we get the required result. $\square$

EXAMPLE 3.3. Consider $k = 0$ in Theorem 3.3. Then $\mathcal{G}_{a,b}(\mathcal{S}_s^r(\lambda, \kappa, \mu)) \in \mathcal{S}^*(\rho)$ in the disk $|z| < r_{23}$, where

$$r_{23} = \begin{cases} \left(T_n\left(\dfrac{(a+1)(b+1)}{(a+n)(b+n)}\right)\right)^{1/(n-1)} & (a \neq b) \\ \left(T_n\left(\dfrac{a+1}{a+n-1}\right)^2\right)^{1/(n-1)} & (a = b), \end{cases}$$

and

$$T_n := \frac{(1-\rho)\Big(2n(1+\kappa) - (\mu+\kappa)[1-(-1)^n]\Big)(\lambda[\varphi_n - 1]+1)\psi_n}{2(n-\rho)(1-\mu)}$$

$(a > -1;\ b > -1)$.

REMARK 3.3. *Since* $zf' \in \mathcal{S}^*(\rho) \Longleftrightarrow f \in \mathcal{C}(\rho)$, *if we replace* $f(z)$ *by* $zf'(z)$ *in Example 3.3, we find that* $\mathcal{G}_{a,b}(\mathcal{S}_s^r(\lambda,\kappa,\mu)) \in \mathcal{C}(\rho)$, *in the disk* $|z| < r_{24}$, *where*

$$r_{24} = \begin{cases} \left(U_n \left(\dfrac{(a+1)(b+1)}{(a+n)(b+n)} \right) \right)^{1/(n-1)} & (a \neq b) \\[2ex] \left(U_n \left(\dfrac{a+1}{a+n-1} \right)^2 \right)^{1/(n-1)} & (a = b) \end{cases}$$

and

$$U_n := \frac{(1-\rho)\Big(2n(1+\kappa) - (\mu+\kappa)[1-(-1)^n]\Big)(\lambda[\varphi_n - 1] + 1)\psi_n}{2n(n-\rho)(1-\mu)}$$

$(a > -1;\ b > -1)$.

We end this section by showing that the class $\mathcal{S}_s^r(\lambda,\kappa,\mu)$ is invariant under the integral operators $F_{c,\delta}\big(f(z)\big)$, $H_{a,b,c}\big(f(z)\big)$ and $\mathcal{G}_{a,b}\big(f(z)\big)$ using Lemma 2.1.

THEOREM 3.4. *The class* $\mathcal{S}_s^r(\lambda,\kappa,\mu)$ *is closed under the operators given by* (3.2) *for* $\delta \geqq 0$, (3.3) *for* $0 < b \leqq 1$ *and* $0 < a \leqq c$, *and by* (3.5).

Proof. It is sufficient to verify that the Taylor coefficients of the respective operator satisfy Lemma 2.1. The series representation of $F_{c,\delta}\big(f(z)\big)$ is given by

$$F_{c,\delta}\big(f(z)\big) = z + \sum_{n=2}^{\infty} a_n \left(\frac{c+1}{c+n} \right)^{\delta} z^n,$$

where $f \in \mathcal{S}_s^r(\lambda,\kappa,\mu)$ is of the form (1.1). Clearly, we have

$$\left(\frac{c+1}{c+n} \right)^{\delta} \leqq 1 \qquad (n \geqq 2)$$

and $\delta \geqq 0$. Hence the function $f \in \mathcal{S}_s^r(\lambda,\kappa,\mu)$ satisfies the following inequality:

$$\sum_{n=2}^{\infty} \frac{\Big(2n(1+\kappa) - (\mu+\kappa)[1-(-1)^n]\Big)(\lambda[\varphi_n - 1] + 1)\psi_n}{2(1-\mu)} a_n \left(\frac{c+1}{c+n} \right)^{\delta} \leqq 1,$$

which, by Lemma 2.1, implies that $F_{c,\delta}\big(f(z)\big) \in \mathcal{S}_s^r(\lambda,\kappa,\mu)$.

From the following series representation of $H_{a,b,c}(f(z))$:

$$H_{a,b,c}\big(f(z)\big) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n z^n$$

and the fact that

$$\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \leqq 1 \qquad (0 < b \leqq 1;\ 0 < a \leqq c;\ n \geqq 2),$$

it is easy to see that $f \in \mathcal{S}_s^r(\lambda, \kappa, \mu)$ satisfies the following inequality:

$$\sum_{n=2}^{\infty} \frac{\Big(2n(1+\kappa) - (\mu+\kappa)[1-(-1)^n]\Big)\,(\lambda[\varphi_n - 1] + 1)\psi_n}{2(1-\mu)} a_n \left(\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right) \leqq 1,$$

which, by Lemma 2.1, implies that $H_{a,b,c}\big(f\big) \in \mathcal{S}_s^r(\lambda, \kappa, \mu)$.

Now it remains to prove that the class $\mathcal{S}_s^r(\lambda, \kappa, \mu)$ is closed under the operator $\mathcal{G}_{a,b}\big(f(z)\big)$, where $f(z)$ is of the form (1.1). The proof can be divided into the following two cases.

Case (i): $a \neq b$. The series representation of $\mathcal{G}_{a,b}\big(f(z)\big)$ is given by

$$\mathcal{G}_{a,b}\big(f(z)\big) = z + \sum_{n=2}^{\infty} \frac{(a+1)(b+1)}{(a+n)(b+n)} a_n z^n.$$

Since

$$\frac{(a+1)(b+1)}{(a+n)(b+n)} \leqq 1 \qquad (n \geqq 2),$$

$f \in \mathcal{S}_s^r(\lambda, \kappa, \mu)$ gives

$$\sum_{n=2}^{\infty} \frac{\Big(2n(1+\kappa) - (\mu+\kappa)[1-(-1)^n]\Big)\,(\lambda[\varphi_n - 1] + 1)\psi_n}{2(1-\mu)} a_n \left(\frac{(a+1)(b+1)}{(a+n)(b+n)}\right)$$
$$\leqq 1,$$

which, by Lemma 2.1, provides the desired result.

Case (ii): $a = b$. The series expansion of $\mathcal{G}_{a,a}\big(f(z)\big)$ is given by

$$\mathcal{G}_{a,a}\big(f(z)\big) = z + \sum_{n=2}^{\infty} \left(\frac{a+1}{a+n-1}\right)^2 a_n z^n,$$

which implies that

$$\left(\frac{a+1}{a+n-1}\right)^2 \leqq 1 \qquad (n \geqq 2).$$

Hence $f \in \mathcal{S}_s^r(\lambda,\kappa,\mu)$ means

$$\sum_{n=2}^{\infty} \frac{\Big(2n(1+\kappa)-(\mu+\kappa)[1-(-1)^n]\Big)(\lambda[\varphi_n-1]+1)\psi_n}{2(1-\mu)} a_n \left(\frac{a+1}{a+n-1}\right)^2 \leqq 1.$$

Thus, clearly, Lemma 2.1 implies that $\mathcal{G}_{a,b}\big(f\big) \in \mathcal{S}_s^r(\lambda,\kappa,\mu)$. □

4. Partial Sums

In the section, the following lemma will be used for obtaining our results.

LEMMA 4.1. ([25]) *If the Taylor series expansion of the analytic function $w(z)$ is given by*

$$w(z) = \sum_{n=1}^{\infty} b_n z^n \qquad (z \in \mathbb{D}),$$

then

$$\Re\left(\frac{1+w(z)}{1-w(z)}\right) > 0 \qquad (z \in \mathbb{D})$$

if and only if

$$|w(z)| \leqq |z| \qquad (z \in \mathbb{D}).$$

THEOREM 4.1. *Let $f \in \mathcal{S}_s^r(\lambda,\kappa,\mu)$ be of the form* (1.1) *in $\mathbb{D}$. Then*

$$\Re\left(\frac{f(z)}{f_m(z)}\right)$$

$$\geqq \frac{\Big(2(m+1)(1+\kappa)-(\mu+\kappa)[1+(-1)^m]\Big)(\lambda[\varphi_{m+1}-1]+1)\psi_{m+1}-2(1-\mu)}{\Big(2(m+1)(1+\kappa)-(\mu+\kappa)[1+(-1)^m]\Big)(\lambda[\varphi_{m+1}-1]+1)\psi_{m+1}}.$$

The result is sharp for the function given by

$$f_n(z) = z + \frac{2(1-\mu)}{\Big(2n(1+\kappa)-(\mu+\kappa)[1-(-1)^n]\Big)(\lambda[\varphi_n-1]+1)\psi_n} z^n \quad (n \geqq 2). \tag{4.1}$$

Proof. Consider

$$A_{m+1} = \Big(2(m+1)(1+\kappa)-(\mu+\kappa)[1+(-1)^m]\Big)(\lambda[\varphi_{m+1}-1]+1)\psi_{m+1}.$$

Then we have

$$\frac{A_{m+1}}{2(1-\mu)}\left(\frac{f(z)}{f_m(z)}-\frac{A_{m+1}-2(1-\mu)}{A_{m+1}}\right)$$
$$=\frac{1+\sum_{n=2}^{m}a_nz^{n-1}+\frac{A_{m+1}}{2(1-\mu)}\sum_{n=m+1}^{\infty}a_nz^{n-1}}{1+\sum_{n=2}^{m}a_nz^{n-1}}=:\frac{1+w(z)}{1-w(z)}.$$

A simple computation gives

$$|w(z)|\leqq\frac{\frac{A_{m+1}}{2(1-\mu)}\sum_{n=m+1}^{\infty}|a_n|}{2-2\sum_{n=2}^{m}|a_n|-\frac{A_{m+1}}{2(1-\mu)}\sum_{n=m+1}^{\infty}|a_n|}$$

We now note that $|w(z)|\leqq 1$ if and only if

$$\frac{A_{m+1}}{2(1-\mu)}\sum_{n=m+1}^{\infty}|a_n|\leqq 2-2\sum_{n=2}^{m}|a_n|-\frac{A_{m+1}}{2(1-\mu)}\sum_{n=m+1}^{\infty}|a_n|,$$

which is equivalent to

$$\sum_{n=2}^{m}|a_n|+$$
$$\frac{\Big(2(m+1)(1+\kappa)-(\mu+\kappa)[1+(-1)^m]\Big)(\lambda[\varphi_{m+1}-1]+1)\psi_{m+1}}{2(1-\mu)}\sum_{n=m+1}^{\infty}|a_n|$$
$$\leqq 1. \tag{4.2}$$

The left-hand side of (4.2) is bounded above by

$$\sum_{n=2}^{\infty}\frac{\Big(2n(1+\kappa)-(\mu+\kappa)[1-(-1)^n]\Big)(\lambda[\varphi_n-1]+1)\psi_n}{2(1-\mu)}|a_n|, \qquad \text{if}$$

$$\frac{1}{2(1-\mu)}\left(\sum_{n=2}^{m}\left[\Big(2n(1+\kappa)-(\mu+\kappa)[1-(-1)^n]\Big)(\lambda[\varphi_n-1]+1)\psi_n-2(1-\mu)\right]|a_n|\right.$$
$$+\sum_{n=m+1}^{\infty}\left[\Big(2n(1+\kappa)-(\mu+\kappa)[1-(-1)^n]\Big)(\lambda[\varphi_n-1]+1)\psi_n\right.$$
$$\left.\left.-(2(m+1)(1+\kappa)-(\mu+\kappa)(1+(-1)^m))(\lambda[\varphi_{m+1}-1]+1)\psi_{m+1}\right]|a_n|\right)\geqq 0,$$

which, by Lemma 4.1, implies that the result holds true.

To verify that the function $f(z)$ given by

$$f(z)=z+\frac{2(1-\mu)}{\Big(2(m+1)(1+\kappa)-(\mu+\kappa)[1+(-1)^m]\Big)(\lambda[\varphi_{m+1}-1]+1)\psi_{m+1}}z^{m+1}$$

$(m \geqq 1)$
gives the sharp result, by considering $z = re^{i\pi/m}$, we observe that $f(z)$ is an infinite series with only two non-zero terms and the corresponding $f_m(z)$, which is obtained by the taking first m terms, gives $f_m(z) = z$. Hence we have

$$\Re\left(\frac{f(z)}{f_m(z)}\right)$$
$$=1+\Re\left(\frac{2(1-\mu)}{\Big(2(m+1)(1+\kappa)-(\mu+\kappa)[1+(-1)^m]\Big)(\lambda[\varphi_{m+1}-1]+1)\psi_{m+1}}z^{m}\right)$$
$$\longrightarrow\frac{\Big(2(m+1)(1+\kappa)-(\mu+\kappa)[1+(-1)^m]\Big)(\lambda[\varphi_{m+1}-1]+1)\psi_{m+1}-2(1-\mu)}{\Big(2(m+1)(1+\kappa)-(\mu+\kappa)[1+(-1)^m]\Big)(\lambda[\varphi_{m+1}-1]+1)\psi_{m+1}}$$

when $r \to 1-$. □

THEOREM 4.2. *Let $f \in \mathcal{S}_s^r(\lambda,\kappa,\mu)$ be of the form* (1.1) *in* $\mathbb{D}$. *Then*

$$\Re\left(\frac{f_m(z)}{f(z)}\right)$$
$$\geqq\frac{\Big(2(m+1)(1+\kappa)-(\mu+\kappa)[1+(-1)^m]\Big)(\lambda[\varphi_{m+1}-1]+1)\psi_{m+1}}{\Big(2(m+1)(1+\kappa)-(\mu+\kappa)[1+(-1)^m]\Big)(\lambda[\varphi_{m+1}-1]+1)\psi_{m+1}+2(1-\mu)}.$$

The result is sharp for the function given by (4.1).

Proof. Consider

$$A_{m+1}=\Big(2(m+1)(1+\kappa)-(\mu+\kappa)[1+(-1)^m]\Big)(\lambda[\varphi_{m+1}-1]+1)\psi_{m+1}.$$

Then

$$\frac{A_{m+1}+2(1-\mu)}{2(1-\mu)}\left(\frac{f_m(z)}{f(z)}-\frac{A_{m+1}}{A_{m+1}+2(1-\mu)}\right)$$
$$=\frac{1+\sum_{n=2}^{m}a_n z^{n-1}-\frac{A_{m+1}}{2(1-\mu)}\sum_{n=m+1}^{\infty}a_n z^{n-1}}{1+\sum_{n=2}^{\infty}a_n z^{n-1}}=:\frac{1+w(z)}{1-w(z)}.$$

A simple computation gives

$$|w(z)|\leqq\frac{\frac{A_{m+1}+2(1-\mu)}{2(1-\mu)}\sum_{n=m+1}^{\infty}|a_n|}{2-2\sum_{n=2}^{m}|a_n|-\frac{A_{m+1}-2(1-\mu)}{2(1-\mu)}\sum_{n=m+1}^{\infty}|a_n|}.$$

We note that $|w(z)|\leqq 1$ if and only if

$$\frac{A_{m+1}+2(1-\mu)}{2(1-\mu)}\sum_{n=m+1}^{\infty}|a_n|\leqq 2-2\sum_{n=2}^{m}|a_n|-\frac{A_{m+1}-2(1-\mu)}{2(1-\mu)}\sum_{n=m+1}^{\infty}|a_n|,$$

which is equivalent to

$$\sum_{n=2}^{m}|a_n|+\frac{\Big(2(m+1)(1+\kappa)-(\mu+\kappa)[1+(-1)^m]\Big)(\lambda[\varphi_{m+1}-1]+1)\psi_{m+1}}{2(1-\mu)}\sum_{n=m+1}^{\infty}|a_n| \leqq 1. \tag{4.3}$$

The left-hand side of (4.3) is bounded above by

$$\sum_{n=2}^{\infty}\frac{\Big(2n(1+\kappa)-(\mu+\kappa)[1-(-1)^n]\Big)(\lambda[\varphi_n-1]+1)\psi_n}{2(1-\mu)}|a_n|,$$

which, by Lemma 4.1, gives the required result. The argument for sharpness follows from Theorem 4.1. □

THEOREM 4.3. *Let $f\in\mathcal{S}_s^r(\lambda,\kappa,\mu)$ be of the form (1.1) in $\mathbb{D}$. Then* $\Re\left(\frac{f'(z)}{f'_m(z)}\right)\geqq$

$$\frac{\Big(2(m+1)(1+\kappa)-(\mu+\kappa)[1+(-1)^m]\Big)(\lambda[\varphi_{m+1}-1]+1)\psi_{m+1}-2(m+1)(1-\mu)}{\Big(2(m+1)(1+\kappa)-(\mu+\kappa)[1+(-1)^m]\Big)(\lambda[\varphi_{m+1}-1]+1)\psi_{m+1}}.$$

The result is sharp for the function given by (4.1).

Proof. Consider

$$A_{m+1} = \Big(2(m+1)(1+\kappa) - (\mu+\kappa)[1+(-1)^m]\Big)(\lambda[\varphi_{m+1}-1]+1)\psi_{m+1}.$$

Then we have

$$\frac{A_{m+1}}{2(m+1)(1-\mu)}\left(\frac{f'(z)}{f'_m(z)} - \frac{A_{m+1}-2(m+1)(1-\mu)}{A_{m+1}}\right)$$
$$= \frac{1+\displaystyle\sum_{n=2}^{m} na_n z^{n-1} + \frac{A_{m+1}}{2(m+1)(1-\mu)}\sum_{n=m+1}^{\infty} na_n z^{n-1}}{1+\displaystyle\sum_{n=2}^{m} na_n z^{n-1}} =: \frac{1+w(z)}{1-w(z)}.$$

A simple computation gives

$$|w(z)| \leqq \frac{\dfrac{A_{m+1}}{2(m+1)(1-\mu)}\displaystyle\sum_{n=m+1}^{\infty} n|a_n|}{2-2\displaystyle\sum_{n=2}^{m} n|a_n| - \dfrac{A_{m+1}}{2(m+1)(1-\mu)}\displaystyle\sum_{n=m+1}^{\infty} n|a_n|}.$$

We now note that $|w(z)| \leqq 1$ if and only if

$$\frac{A_{m+1}}{2(m+1)(1-\mu)}\sum_{n=m+1}^{\infty} n|a_n| \leqq 2-2\sum_{n=2}^{m} n|a_n| - \frac{A_{m+1}}{2(m+1)(1-\mu)}\sum_{n=m+1}^{\infty} n|a_n|,$$

which is equivalent to

$$\sum_{n=2}^{m} n|a_n| + \frac{(2(m+1)(1+\kappa)-(\mu+\kappa)(1+(-1)^m))(\lambda[\varphi_{m+1}-1]+1)\psi_{m+1}}{2(m+1)(1-\mu)}\sum_{n=m+1}^{\infty} n|a_n| \leqq 1. \tag{4.4}$$

Using Lemma 4.1, it is easy to see that the left-hand side of (4.4) is bounded above by

$$\sum_{n=2}^{\infty} \frac{\Big(2n(1+\kappa) - (\mu+\kappa)[1-(-1)^n]\Big)(\lambda[\varphi_n-1]+1)\psi_n}{2(1-\mu)}|a_n|,$$

which implies the required result. For sharpness, the argument for sharpness as in Theorem 4.1 gives the necessary justification. $\square$

THEOREM 4.4. *Let $f \in \mathcal{S}_s^r(\lambda, \kappa, \mu)$ be of the form* (1.1) *in* $\mathbb{D}$. *Then* $\Re\left(\frac{f_m'(z)}{f'(z)}\right) \geqq$

$$\frac{\Big(2(m+1)(1+\kappa)-(\mu+\kappa)[1+(-1)^m]\Big)(\lambda[\varphi_{m+1}-1]+1)\psi_{m+1}}{\Big(2(m+1)(1+\kappa)-(\mu+\kappa)[1+(-1)^m]\Big)(\lambda[\varphi_{m+1}-1]+1)\psi_{m+1}+2(m+1)(1-\mu)}.$$

The result is sharp for the function given in (4.1).

Proof. Consider

$$A_{m+1}=\Big(2(m+1)(1+\kappa)-(\mu+\kappa)[1+(-1)^m]\Big)(\lambda[\varphi_{m+1}-1]+1)\psi_{m+1}.$$

Then we have

$$\frac{A_{m+1}+2(m+1)(1-\mu)}{2(m+1)(1-\mu)}\left(\frac{f_m'(z)}{f'(z)}-\frac{A_{m+1}}{A_{m+1}+2(m+1)(1-\mu)}\right)$$
$$=\frac{1+\sum_{n=2}^{m} na_n z^{n-1}-\frac{A_{m+1}}{2(m+1)(1-\mu)}\sum_{n=m+1}^{\infty} na_n z^{n-1}}{1+\sum_{n=2}^{\infty} na_n z^{n-1}}$$
$$=:\frac{1+w(z)}{1-w(z)}.$$

A simple computation gives

$$|w(z)| \leqq \frac{\frac{A_{m+1}+2(m+1)(1-\mu)}{2(m+1)(1-\mu)}\sum_{n=m+1}^{\infty} n|a_n|}{2-2\sum_{n=2}^{m} n|a_n|-\frac{A_{m+1}-2(m+1)(1-\mu)}{2(m+1)(1-\mu)}\sum_{n=m+1}^{\infty} n|a_n|}.$$

We note that $|w(z)| \leqq 1$ if and only if

$$\frac{A_{m+1}+2(m+1)(1-\mu)}{2(m+1)(1-\mu)}\sum_{n=m+1}^{\infty} n|a_n|$$
$$\leqq 2-2\sum_{n=2}^{m} n|a_n|-\frac{A_{m+1}-2(m+1)(1-\mu)}{2(m+1)(1-\mu)}\sum_{n=m+1}^{\infty} n|a_n|,$$

which is equivalent to

$$\sum_{n=2}^{m} n|a_n| + \frac{\Big(2(m+1)(1+\kappa)-(\mu+\kappa)[1+(-1)^m]\Big)(\lambda[\varphi_{m+1}-1]+1)\psi_{m+1}}{2(m+1)(1-\mu)} \sum_{n=m+1}^{\infty} n|a_n| \leqq 1. \tag{4.5}$$

Clearly, the left-hand side of (4.5) is bounded above by

$$\sum_{n=2}^{\infty} \frac{\Big(2n(1+\kappa)-(\mu+\kappa)[1-(-1)^n]\Big)(\lambda[\varphi_n-1]+1)\psi_n}{2(1-\mu)}|a_n|.$$

This, together with Lemma 4.1, guarantees the required result. Sharpness of the given function is obvious if we follow an argument similar to that in the proof of Theorem 4.1. □

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