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APPLICATIONS OF TAYLOR SERIES FOR CARLEMAN'S INEQUALITY THROUGH HARDY INEQUALITY

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Abstract. In this paper, we prove the discrete Hardy inequality through the continuous case for decreasing functions using elementary properties of calculus. Also, we prove the Carleman's inequality through limiting the discrete Hardy inequality with applications of Taylor series.

1. Introduction

G. H. Hardy in [4] established the following discrete inequality:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p, \tag{1}$$

where $p > 1$, $a_k \geq 0$ and the constant $\left(\frac{p}{p-1}\right)^p$ is the best possible (see also [3], p. 239). The continuous analogue of (1) is given as

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^{\infty} f^p(x)dx, \tag{2}$$

where $p > 1$, $x > 0$, f is a nonnegative measurable function on $(0,\infty)$ and the constant $\left(\frac{p}{p-1}\right)^p$ is the best possible. The inequality (2) is

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sometimes written as

$$\int_0^\infty F^p(x)dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \tag{3}$$

where $0 < F(x) = \frac{1}{x}\int_0^x f(t)dt < \infty$ and $f > 0$. These interesting results (1) to (3) are very popular in the research environment and are usually called the classical Hardy inequalities (see also [1], [2], [3], [7], [9], [10], [11], [12] and the references therein.)

Let us also consider the inequality

$$a_1 + \sqrt{a_1 a_2} + \cdots + \sqrt[n]{a_1 a_2 \cdots a_n} < e(a_1 + a_2 + \cdots), \tag{4}$$

where $a_1, a_2 \cdots, a_n$ are positive numbers and $\sum_{j=1}^{\infty} a_j$ is convergent. This inequality (4) is due to a Swedish mathematician called Torsten Carleman who discovered it in 1922 (see [3], p.249). In order to agree further with T. Carleman, other Mathematicians also proved the inequality (4) by different methods: Thus by differentiation and the variations of the Arithmetic (A_n)–Geometric (G_n) mean inequality (i.e. $G_n \le A_n$) methods (See [5], [6] [13], [14], [15] and the references therein).

The aim of this paper is first to provide a simple proof of the discrete Hardy inequality through the continuous case and then recover the Carleman's inequality (4) through limiting the discrete Hardy inequality with applications of Taylor series. This approach here for the prove of the Carleman's inequality is mainly to demonstrate the use of Taylor series in the evaluation of large expressions.

2. Results and Discussions

We begin as follows:

THEOREM 2.1. *Let $p > 1$ and a_k be a non-increasing sequence of positive real numbers, then*

$$\sum_{k=1}^{\infty} A_k^p \le \left(\frac{p}{p-1}\right)^p \sum_{k=1}^{\infty} a_k^p,$$

where $A_k = \frac{1}{k}\sum_{j=1}^{k} a_j$ and the constant $\left(\frac{p}{p-1}\right)^p$ is the best possible.

Proof. Consider the non-increasing sequence a_k. Let $f(x) = a_k$ on $[k-1, k]$. From the integral inequality (3) we have

$$\sum_{k=1}^{\infty} \int_{k-1}^{k} F^p(x)dx \leq \sum_{k=1}^{\infty} \left(\frac{p}{p-1}\right)^p \int_{k-1}^{k} f^p(x)dx \tag{5}$$

where

$$F(x) = \frac{1}{x} \int_0^x f(t)dt.$$

Since the function $F(x)$ is non-increasing, it follows that

$$F(k) \leq F(x) \leq F(k-1).$$

Thus

$$\int_{k-1}^{k} F^p(k)dx \leq \int_{k-1}^{k} F^p(x)dx \leq \int_{k-1}^{k} F^p(k-1)dx.$$

This simplifies to

$$F^p(k) \leq \int_{k-1}^{k} F^p(x)dx \leq F^p(k-1)$$

Thus

$$\sum_{k=1}^{\infty} F^p(k) \leq \sum_{k=1}^{\infty} \int_{k-1}^{k} F^p(x)dx \leq \sum_{k=1}^{\infty} F^p(k-1). \tag{6}$$

Considering the first inequality of (6) and then applying (5), we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} F^p(k) &\leq \sum_{k=1}^{\infty} \int_{k-1}^{k} F^p(x)dx \\ &\leq \sum_{k=1}^{\infty} \left(\frac{p}{p-1}\right)^p \int_{k-1}^{k} f^p(x)dx \\ &\leq \left(\frac{p}{p-1}\right)^p \sum_{k=1}^{\infty} a_k^p. \end{aligned}$$

Putting $F(k) = A_k$, we get

$$\sum_{k=1}^{\infty} A_k^p \leq \left(\frac{p}{p-1}\right)^p \sum_{k=1}^{\infty} a_k^p.$$

□

Theorem 2.1. *Let a_k be a sequence of distinct positive real numbers not necessarily non-increasing, then*

$$\sum_{k=1}^{\infty} A_k^p \leq \left(\frac{p}{p-1}\right)^p \sum_{k=1}^{\infty} a_k^p \tag{7}$$

holds for $p > 1$.

Proof. The inequality (7) is valid for non-increasing sequence $\{a_k\}$ as in Theorem 2.1, where $A_k = \frac{1}{k}\sum_{n=1}^{k} a_n$. Assume that $\sum_{k=1}^{\infty} a_k^p < \infty$ which implies $a_k \to 0$. Now let us consider re-arrangement of the sequence $\{a_k\}$ by setting $\sup\{a_k\} = \max\{a_k\}$ and considering the sub-sequence a_{k_j} such that $\{a_{k_j}\}_{j=1}^{\infty} = \{a_j^*\}_{j=1}^{\infty}$. Denote by $(\{a_k\} \setminus \{a_j^*\})$, the difference between the two sequences. Thus

$$\begin{gathered} a_1^* = \max(\{a_k\}) = a_{k_1} \\ a_2^* = \max(\{a_k\} \setminus \{a_1^*\}) = a_{k_2} \\ a_3^* = \max(\{a_k\} \setminus \{a_1^*, a_2^*\}) = a_{k_3} \\ \vdots \\ a_n^* = \max(\{a_k\} \setminus \{a_1^*, a_2^*, \ldots, a_{(n-1)}^*\} = a_{k_n} \\ \vdots \end{gathered}$$

Thus

$$a_1^* = a_{k_1} \geq a_{k_2} \geq a_{k_3} \geq \cdots \geq a_{k_n} = a_n^* \geq \ldots$$

This shows that the sequence $\{a_j^*\}_{j=1}^{\infty}$ is non-increasing. Hence

$$\sum_{k=1}^{n} a_k \leq \sum_{j=1}^{n} a_j^* \leq \sum_{j=1}^{\infty} a_j^* = \sum_{k=1}^{\infty} a_k$$

and

$$\sum_{j=1}^{\infty} (a_j^*)^p = \sum_{k=1}^{\infty} a_k^p$$

while

$$\frac{1}{n}\sum_{k=1}^{n} a_k \leq \frac{1}{n}\sum_{j=1}^{n} a_j^*$$

for $n > 0$. Thus (7) is valid for sequence a_k not necessarily non-increasing. □

Let us now establish some results for the limiting Hardy inequality. First, we present for the case of two positive numbers followed by a generalization.

THEOREM 2.2. *Let $\{a_k\}$ be a sequence of distinct positive real numbers, then*

$$\lim_{p\to+\infty}\left(\frac{1}{2}\sum_{k=1}^{2}(a_k)^{\frac{1}{p}}\right)^p = \sqrt{a_1a_2}.$$

Proof. Let

$$S_p = \left(\frac{1}{2}\sum_{k=1}^{2}(a_k)^{\frac{1}{p}}\right)^p = \left(\frac{(a_1)^{\frac{1}{p}}+(a_2)^{\frac{1}{p}}}{2}\right)^p.$$

Applying logarithm to base e to both sides and using $x = e^{\ln x}$, we obtain

$$\begin{aligned}\ln S_p &= p\ln\left(\frac{(a_1)^{\frac{1}{p}}+(a_2)^{\frac{1}{p}}}{2}\right)\\ &= p\ln\left(\frac{e^{\frac{1}{p}\ln a_1}+e^{\frac{1}{p}\ln a_2}}{2}\right).\end{aligned}$$

Now consider the Taylor expansion

$$\begin{aligned}e^x &= 1+x+x^2\left(\frac{1}{2}+\frac{x}{3!}+\frac{x^2}{4!}+\dots\right)\\ &= 1+x+x^2f(x) \qquad (8)\end{aligned}$$

where $f(x) = \left(\frac{1}{2}+\frac{x}{3!}+\frac{x^2}{4!}+\dots\right) \to \frac{1}{2}$ when $x\to 0$. Applying the Taylor series (8) to $\ln S_p$, we obtain

$$\begin{aligned}\ln S_p &= p\ln\frac{1}{2}\left[1+1+\frac{1}{p}(\ln a_1+\ln a_2)+\frac{1}{p^2}\sum_{k=1}^{2}(\ln a_k)^2f(\frac{1}{p}\ln a_k)\right]\\ &= p\ln\left[1+\frac{1}{2p}\ln(a_1a_2)+\frac{\epsilon_p}{p}\right]\\ &= p\ln\left[1+\frac{1}{p}\ln\sqrt{a_1a_2}+\frac{\epsilon_p}{p}\right]\\ &= p\ln(1+u_p)\end{aligned}$$

where $u_p = \frac{1}{p}\ln\sqrt{a_1a_2} + \frac{\epsilon_p}{p}$ and $\epsilon_p = \frac{1}{2p}\sum_{k=1}^{2}(\ln a_k)^2 f(\frac{1}{p}\ln a_k)$ for which $\epsilon_p \to 0$ as $p \to \infty$ since $\frac{1}{p} \to 0$ as $p \to \infty$.
Also consider the Taylor expansion

$$\begin{aligned} \ln(1+u) &= u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \dots \\ &= u\left(1 - \frac{u}{2} + \frac{u^2}{3} - \frac{u^3}{4} + \dots\right) \\ &= ug(u) \end{aligned} \tag{9}$$

where $g(u) = \left(1 - \frac{u}{2} + \frac{u^2}{3} - \frac{u^3}{4} + \dots\right) \to 1$ as $u \to 0$. Applying (9), we find that

$$\begin{aligned} \ln S_p &= p\ln(1+u_p) \\ &= pu_p g(u_p) \\ &= (\ln\sqrt{(a_1a_2)} + \epsilon_p).g(u_p). \end{aligned}$$

As $p \to \infty$, $\epsilon_p \to 0$ and $u_p \to 0$ implies $g(u_p) \to 1$. Hence

$$\lim_{p\to+\infty} S_p = \sqrt{a_1a_2}.$$

□

Now we consider the general case:

THEOREM 2.3. *Let $\{a_k\}$ be a sequence of distinct positive real numbers, then*

$$\lim_{p\to+\infty}\left(\frac{1}{n}\sum_{k=1}^{n}(a_k)^{\frac{1}{p}}\right)^p = (a_1a_2\dots a_n)^{\frac{1}{n}}.$$

Equivalently,

$$\lim_{p\to+\infty}\left(\frac{1}{n}\sum_{k=1}^{n}(a_k)^{\frac{1}{p}}\right)^p = \exp\left(\frac{1}{n}\sum_{k=1}^{n}\log a_k\right).$$

Proof. Let

$$V_p = \left(\frac{1}{n}\sum_{k=1}^{n}(a_k)^{\frac{1}{p}}\right)^p = \left(\frac{(a_1)^{\frac{1}{p}} + (a_2)^{\frac{1}{p}} + \dots + (a_n)^{\frac{1}{p}}}{n}\right)^p.$$

Applying $t = \exp \ln t$, we get

$$\ln V_p = p \ln \left(\frac{\exp(\frac{1}{p}\ln a_1) + \exp(\frac{1}{p}\ln a_2) + \cdots + \exp(\frac{1}{p}\ln a_n)}{n} \right)$$

Applying $e^x = 1 + x + x^2 f(x)$ where $f(x) \to \frac{1}{2}$ when $x \to 0$, we obtain

$$\begin{aligned} \ln V_p &= p \ln \frac{1}{n} \left[n + \frac{1}{p} \ln(a_1 a_2 \ldots a_n) + \frac{1}{p^2} \sum_{k=1}^{n} (\ln a_k)^2 f(\frac{1}{p} \ln a_k) \right] \\ &= p \ln \left[1 + \frac{1}{np} \ln(a_1 a_2 \ldots a_n) + \frac{h_p}{p} \right] \\ &= p \ln(1 + m_p) \end{aligned}$$

where $m_p = \frac{1}{np} \ln(a_1 a_2 \ldots a_n) + \frac{h_p}{p}$ and $h_p = \frac{1}{np} \sum_{k=1}^{n} (\ln a_k)^2 f(\frac{1}{p} \ln a_k) \to 0$ as $p \to \infty$. Applying the Taylor expansion

$$\ln(1 + m) = m \left(1 - \frac{m}{2} + \frac{m^2}{3} - \frac{m^3}{4} + \ldots \right) = m g(m)$$

where $g(m) = \left(1 - \frac{m}{2} + \frac{m^2}{3} - \frac{m^3}{4} + \ldots \right) \to 1$ as $m \to 0$. Thus

$$\begin{aligned} \ln V_p &= p m_p g(m_p) \\ &= p \left[\frac{1}{np} \ln(a_1 a_2 \ldots a_n) + \frac{h_p}{p} \right] g(m_p) \\ &= \left[\ln(a_1 a_2 \ldots a_n)^{\frac{1}{n}} + h_p \right] g(m_p) \end{aligned}$$

As $p \to \infty$, $h_p \to 0$, $m_p \to 0$ and $g(m_p) \to 1$. Hence

$$\lim_{p \to +\infty} V_p = (a_1 a_2 \ldots a_n)^{\frac{1}{n}}$$

as required. □

Remark 1. By Theorem 2.3, we have

$$\begin{aligned} \lim_{p\to+\infty}\left(\frac{1}{n}\sum_{k=1}^{n}(a_k)^{\frac{1}{p}}\right)^p &= (a_1a_2\dots a_n)^{\frac{1}{n}} \\ &= \exp\frac{1}{n}\log(\prod_{k=1}^{n}a_k) \\ &= \exp\frac{1}{n}\left(\log a_1+\log a_2+\cdots+\log a_n\right) \\ &= \exp\left(\frac{1}{n}\sum_{k=1}^{n}\log a_k\right). \qquad \square \end{aligned}$$

THEOREM 2.4. *Let* $\{a_k\}$ *be a sequence of distinct positive real numbers, then*

$$\sum_{n=1}^{\infty}\exp\left(\frac{1}{n}\sum_{k=1}^{n}\log a_k\right)\le e\sum_{n=1}^{\infty}a_n. \tag{10}$$

Proof. Replace a_n with $(a_n)^{\frac{1}{p}}$ in inequality (1). Then

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}(a_k)^{\frac{1}{p}}\right)^p\le\left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\infty}a_n$$

Take limits on both sides. Thus

$$\sum_{n=1}^{\infty}\exp\left(\frac{1}{n}\sum_{k=1}^{n}\log a_k\right)\le e\sum_{n=1}^{\infty}a_n \tag{11}$$

since

$$\lim_{p\to\infty}\left(\frac{p}{p-1}\right)^p=e$$

and by application of Remark 1. $\square$

Equivalently

$$\sum_{n=1}^{\infty}(a_1a_2\dots a_n)^{\frac{1}{n}}\le e\sum_{n=1}^{\infty}a_n \tag{12}$$

which is the well known Carleman's inequality.

3. Conclusion

We proved the discrete Hardy inequality through the integral Hardy inequality for decreasing functions. We also established the well known Carleman's inequality through limiting Discrete Hardy inequality with applications of Taylor series.

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