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SYSTEM OF GENERALIZED NONLINEAR REGULARIZED NONCONVEX VARIATIONAL INEQUALITIES

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Abstract. In this work, we suggest a new *system of generalized nonlinear regularized nonconvex variational inequalities* in a real Hilbert space and establish an equivalence relation between this system and fixed point problems. By using the equivalence relation we suggest a new perturbed projection iterative algorithms with mixed errors for finding a solution set of *system of generalized nonlinear regularized nonconvex variational inequalities*.

1. Introduction

Variational inequalities introduced by Stampacchia [16] provided us with a powerful source to study a wide class of problems arising in mechanic, physics, optimization and control theory, linear programming, economics and engineering sciences, *see* [4, 5, 7]. In recent years, several authors studied different type of systems of variational inequalities and suggested iterative algorithms to find the approximate solutions of such system (*see* [3, 6, 9, 11, 14, 19, 20]). We remark that the almost all results concerning the system of solutions of iterative scheme for solving the system of variational inequalities and related problems are being

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considered in the setting of convex sets. Consequently the techniques are based on the projections of operator over convex sets, which may not hold in general, when the sets are nonconvex. It is known that the unified prox-regular sets are nonconvex and included the convex sets as special cases, (*see* [5, 21]).
Motivated by the recent works (*see* [1,2,8,10,12,13,17,18]), in this communication, we suggest a new *system of generalized nonlinear regularized nonconvex variational inequalities* in a real Hilbert space. We establish the equivalence between the *system of generalized nonlinear regularized nonconvex variational inequalities* and some fixed point problems. By using the equivalence relation, we define a perturbed projection iterative algorithms with mixed errors for finding a solution set of the aforementioned system. Also we prove the convergence of the defined iterative algorithms under suitable assumptions.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space with a norm and an inner product denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, *respectively.* Let $\mathcal{K}$ be a nonempty convex subset of $\mathcal{H}$ and $CB(\mathcal{H})$ denote the family of all closed and bounded subsets of $\mathcal{H}$.

Definition 2.1. The proximal normal cone of $\mathcal{K}$ at a point $u \in \mathcal{H}$ is given by

$$N^P_{\mathcal{K}}(u) = \{\zeta \in \mathcal{H} : u \in P_{\mathcal{K}}(u + \alpha\zeta)\},$$

where $\alpha > 0$ is a constant and $P_{\mathcal{K}}$ the projection operator of $\mathcal{H}$ onto $\mathcal{K}$, that is,

$$P_{\mathcal{K}}(u) = \{v \in \mathcal{K} : d_{\mathcal{K}}(u) = \|u - v\|\},$$

where $d_{\mathcal{K}}(u)$ is the usual distance function to the subset $\mathcal{K}$, that is,

$$d_{\mathcal{K}}(u) = \inf_{v\in\mathcal{K}} \|u - v\|.$$

Lemma 2.2. *Let $\mathcal{K}$ be a nonempty closed subset of $\mathcal{H}$. Then $\zeta \in N^P_{\mathcal{K}}(u)$ if and only if there exists a constant $\alpha > 0$ such that*

$$\langle\zeta, v - u\rangle \leq \alpha\|v - u\|^2, \ \forall v \in \mathcal{K}.$$

Definition 2.3. The Clarke normal cone, denoted by $N^C_{\mathcal{K}}(u)$ is defined as

$$N^C_{\mathcal{K}}(u) = \overline{co}[N^P_{\mathcal{K}}(u)],$$

where $\overline{co}\mathcal{A}$ means the closure of the convex hull of $\mathcal{A}$. It is clear that $N^P_{\mathcal{K}}(x) \subseteq N^C_{\mathcal{K}}(x)$, but converse is not true in general. Note that $N^C_{\mathcal{K}}(x)$ is closed and convex, but $N^P_{\mathcal{K}}(x)$ is convex, which may be not closed (*see* [5, 17]).

Definition 2.4. For any $r \in (0, +\infty]$, a subset $\mathcal{K}_r$ of $\mathcal{H}$ is said normalized uniformly prox-regular (or uniformly r-prox-regular) if every nonzero proximal normal to $\mathcal{K}_r$ can be realized by an r-ball. This means that for all $\overline{x} \in \mathcal{K}_r$ and all $\zeta \in N^P_{\mathcal{K}_r}(\overline{x})$ with $\|\zeta\| = 1$,

$$\langle \zeta, x - \overline{x}\rangle \leq \frac{1}{2r}\|x - \overline{x}\|^2, \ x \in \mathcal{K}.$$

Lemma 2.5. [4] *A closed set $\mathcal{K} \subseteq \mathcal{H}$ is convex if and only if it is proximally smooth of radius r for every $r > 0$.*

Proposition 2.6. *Let $r > 0$ and let $\mathcal{K}_r$ be a nonempty closed and uniformly r-prox-regular subset of $\mathcal{H}$. Set*

$$\mathcal{U}(r) = \{u \in \mathcal{X} : 0 \leq d_{\mathcal{K}_r}(u) < r\}.$$

Then the following statements are hold:

(a) *for all $x \in \mathcal{U}(r), P_{\mathcal{K}_r}(x) \neq \emptyset$;*
(b) *for all $r' \in (0, r)$, $P_{\mathcal{K}_r}$ is Lipschitz continuous mapping with constant $\frac{r}{r-r'}$ on*

$$\mathcal{U}(r') = \{u \in \mathcal{H} : 0 \leq d_{\mathcal{K}_r}(u) < r'\};$$

(c) *the proximal normal cone is closed as a set-valued mapping.*

From Proposition 2.6 (c) we have $N^C_{\mathcal{K}_r}(x) = N^P_{\mathcal{K}_r}(x)$. Therefore we define $N_{\mathcal{K}_r}(x) = N^C_{\mathcal{K}_r}(x) = N^P_{\mathcal{K}_r}(x)$ for a class of sets.

Definition 2.7. A single-valued mapping $h : \mathcal{H} \longrightarrow \mathcal{H}$ is said to be

(i) monotone if

$$\langle h(x) - h(y), x - y\rangle \geq 0, \quad \forall x, y \in \mathcal{H},$$

(ii) β-strongly monotone if there exists a constant $\beta > o$ such that

$$\langle h(x) - h(y), x - y\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in \mathcal{H},$$

(iii) inversely β-strongly monotone if there exists a constant $\beta > 0$ such that

$$\langle h(x) - h(y), x - y\rangle \geq \beta\|h(x) - h(y)\|^2, \quad \forall x, y \in \mathcal{H},$$

(iv) σ-Lipschitz continuous if there exists a constant $\sigma > 0$ such that

$$\|h(x) - h(y)\| \leq \sigma \|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

Definition 2.8. Let $Q : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ be a nonlinear single-valued mapping and $T : \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ be a set-valued mapping. Then Q is said to be

(i) monotone if

$$\langle Q(u, x) - Q(v, x), x - y \rangle \geq 0, \quad \forall x, y \in \mathcal{H}, u \in T(x), v \in T(y),$$

(ii) (κ, λ)-relaxed cocoercive with respect to the first variable of Q and T if there exist constants κ and λ such that $\langle Q(u, x) - Q(v, x), x - y \rangle \geq -\kappa \|Q(u, x) - Q(v, x)\|^2 + \lambda \|x - y\|^2$, $\forall x, y \in \mathcal{H}, u \in T(x), v \in T(y)$.

(iii) ζ-Lipschitz continuous with respect to the first variable and ϱ-Lipschitz continuous with respect to the second variable if

$$Q(x_1, y_1) - Q(x_2, y_2)\| \leq \zeta \|x_1 - x_2\| + \varrho \|y_1 - y_2\|, \quad \forall x_1, x_2, y_1, y_2 \in \mathcal{H}.$$

Definition 2.9. A two-variable set-valued mapping $T : \mathcal{H} \times \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ is $\xi - \widehat{D}$-Lipschitz continuous in the first variable, if there exists a constant $\xi > o$ such that, for all $x, x' \in \mathcal{H}$,

$$\widehat{\mathcal{D}}(T(x, y), T(x', y')) \leq \xi \|x - x'\|, \quad \forall y, y' \in \mathcal{H},$$

where $\widehat{\mathcal{D}}$ is the Hausdorff pseudo-metric, that is, for any two nonempty subsets A and B of $\mathcal{H}$

$$\widehat{\mathcal{D}}(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}.$$

3. System of Generalized Nonlinear Regularized Nonconvex Variational Inequalities

In this section, we introduce a new *system of generalized nonlinear regularized nonconvex variational inequalities* in a Hilbert space and investigated their relations.

Let $T_i, F_i : \mathcal{H} \times \mathcal{H} \longrightarrow CB(\mathcal{H})$ be nonlinear set-valued mappings, $Q_i : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ single-valued mappings and let g_i , $h_i : \mathcal{H} \longrightarrow \mathcal{H}$ be nonlinear single-valued mappings such that $\mathcal{K}_r \subseteq g_i(\mathcal{H})$ $(i = 1, \ldots, N)$. For any constants $\eta_i (i = 1, \ldots, N)$, we consider a problem of finding $x_i \in \mathcal{H} (i = 1, \ldots, N)$ and $u_i \in T_i(x_{i+1}, x_i) (i = 1, \ldots, N-1)$, $u_N \in$

$T_N(x_1, x_N)$, $v_i \in F_i(x_{i+1}, x_i)(i = 1, \dots, N-1)$ and $v_N \in F_N(x_1, x_N)$ such that $h_i(x_i) \in \mathcal{K}_r$ $(i = 1, \dots, N)$ and

$$\begin{cases} \langle \eta_i Q_i(u_i, v_i) + h_i(x_i) - g_i(x_{i+1}), g_i(x) - h_i(x_i)\rangle + \frac{1}{2r}\|g_i(x) - h_i(x_i)\|^2 \\ \qquad \geq 0 \quad (i = 1, \dots, N-1), \\ \langle \eta_N Q_N(u_N, v_N) + h_N(x_N) - g_N(x_1), g_N(x) - h_N(x_N)\rangle \\ \qquad + \frac{1}{2r}\|g_N(x) - h_N(x_N)\|^2 \geq 0, \quad \forall x \in \mathcal{K}_r,\ g_1(x), \dots, g_N(x) \in \mathcal{K}_r. \end{cases} \tag{3.1}$$

The problem (3.1) is called the *system of generalized nonlinear regularized nonconvex variational inequalities.*

LEMMA 3.1. *Let $\mathcal{K}_r$ be a uniformly r-prox-regular set, then the problem (3.1) is equivalent to finding $x_i \in \mathcal{H}(i = 1, \dots, N)$ and $u_i \in T_i(x_{i+1}, x_i)$ $(i = 1, \dots, N-1)$, $u_N \in T_N(x_1, x_N)$, $v_i \in F_i(x_{i+1}, x_i)(i = 1, \dots, N-1)$ and $v_N \in F_N(x_1, x_N)$ such that*

$$\begin{cases} 0 \in \eta_i Q_i(u_i, v_i) + h_i(x_i) - g_i(x_N) + N^P_{\mathcal{K}_r}(h_i(x_i)), (i = 1, \dots, N-1) \\ 0 \in \eta_N Q_N(u_N, v_N) + h_N(x_N) - g_N(x_1) + N^P_{\mathcal{K}_r}(h_N(x_N)), \end{cases} \tag{3.2}$$

where $N^P_{\mathcal{K}_r}(s)$ denotes the P-normal cone of $\mathcal{K}_r$ at s in the sense of nonconvex analysis.

Proof. Let (x_i, u_i, v_i) with $x_i \in \mathcal{H}$, $h_i(x_i) \in \mathcal{K}_r$ $(i = 1, \dots, N)$ and $u_i \in T_i(x_{i+1}, x_i)(i = 1, \dots, N-1)$, $u_N \in T_N(x_1, x_N)$, $v_i \in F_i(x_{i+1}, x_i)(i = 1, \dots, N-1)$, $v_N \in F_N(x_1, x_N)$ be solution sets of the system (3.1). If

$$\eta_1 Q_1(u_1, v_1) + h_1(x_1) - g_1(x_2) = 0$$

because the vector zero always belongs to any normal cone, then

$$0 \in \eta_1 Q_1(u_1, v_1) + h_1(x_1) - g_1(x_2) + N^P_{\mathcal{K}_r}(h_1(x_1)).$$

If

$$\eta_1 Q_1(u_1, v_1) + h_1(x_1) - g_1(x_2) \neq 0$$

then for all $x \in \mathcal{H}$ with $g_1(x) \in \mathcal{K}_r$

$$\langle -(\eta_1 Q_1(u_1, v_1) + h_1(x_1) - g_1(x_2)), g_1(x) - h_1(x_1)\rangle \leq \frac{1}{2r}\|g_1(x) - h_1(x_1)\|^2. \tag{3.3}$$

From Lemma 2.2 we have

$$-(\eta_1 Q_1(u_1, v_1) + h_1(x_1) - g_1(x_2)) \in N^P_{\mathcal{K}_r}(h_1(x_1))$$

and

$$0 \in \eta_1 Q_1(u_1, v_1) + h_1(x_1) - g_1(x_2) + N^P_{\mathcal{K}_r}(h_1(x_1)). \tag{3.4}$$

Similarly

$$\begin{cases} 0 \in \eta_i Q_i(u_i, v_i) + h_i(x_i) - g_i(x_N) + N^P_{\mathcal{K}_r}(h_i(x_i)) \ (i = 1, \ldots, N-1), \\ 0 \in \eta_N Q_N(u_N, v_N) + h_N(x_N) - g_N(x_1) + N^P_{\mathcal{K}_r}(h_N(x_N)). \end{cases} \tag{3.5}$$

Conversely if (x_i, u_i, v_i) with $x_i \in \mathcal{H}$, $h_i(x_i) \in \mathcal{K}_r$ $(i = 1, \ldots, N)$ and $u_i \in T_i(x_{i+1}, x_i)(i = 1, \ldots, N-1)$, $u_N \in T_N(x_1, x_N)$, $v_i \in F_i(x_{i+1}, x_i)(i = 1, \ldots, N-1)$, $v_N \in F_N(x_1, x_N)$ are solution sets of the system (3.2) then from Definition 2.4, $x_i \in \mathcal{H}(i = 1, \ldots, N)$ and $u_i \in T_i(x_{i+1}, x_i)(i = 1, \ldots, N-1)$, $u_N \in T_N(x_1, x_N)$, $v_i \in F_i(x_{i+1}, x_i)(i = 1, \ldots, N-1)$, $v_N \in F_N(x_1, x_N)$ with $h_i(x_i) \in \mathcal{K}_r(i = 1, \ldots, N)$ are solution sets of the system (3.1). □

The problem (3.2) is called *system of generalized nonlinear regularized nonconvex variational inclusions.*

4. Main results

Lemma 4.1. *Let T_i, F_i, Q_i, g_i, h_i and $\eta_i(i = 1, 2, \cdots, N)$ be the same as in the system* (3.1). *Then $(x_1, \ldots, x_N, u_1, \ldots, u_N, v_1, \ldots, v_N)$ with $x_i \in \mathcal{H}$, $h_i(x_i) \in \mathcal{K}_r$ for all $i = 1, \ldots, N$ and $u_1 \in T_1(x_2, x_1)$, $\ldots$,$u_{N-1} \in T_{N-1}(x_N, x_{N-1})$, $u_N \in T_N(x_1, x_N)$, $v_1 \in F_1(x_2, x_1)$, $\ldots$,$v_{N-1} \in F_{N-1}(x_N, x_{N-1})$, $v_N \in F_N(x_1, x_N)$ are solution sets of the system* (3.1) *if and only if*

$$\begin{cases} h_i(x_i) = P_{\mathcal{K}_r}[g_i(x_N) - \eta_i Q_i(u_i, v_i)](i = 1, \ldots, N-1), \\ h_N(x_N) = P_{\mathcal{K}_r}[g_N(x_1) - \eta_N Q_N(u_N, v_N)], \end{cases} \tag{4.1}$$

where $P_{\mathcal{K}_r}$ is the projection of $\mathcal{H}$ onto the uniformly r-prox-regular set $\mathcal{K}_r$.

Proof. Let $(x_1, \ldots, x_N, u_1, \ldots, u_N, v_1, \ldots, v_N)$ with $x_i \in \mathcal{H}$, $h_i(x_i) \in \mathcal{K}_r$ for all $i = 1, \ldots, N$ and $u_1 \in T_1(x_2, x_1)$, $\ldots$,$u_{N-1} \in T_{N-1}(x_N, x_{N-1})$, $u_N \in T_N(x_1, x_N)$, $v_1 \in F_1(x_2, x_1)$, $\ldots$, $v_{N-1} \in F_{N-1}(x_N, x_{N-1})$, $v_N \in F_N(x_1, x_N)$ are solution sets of the system (3.1). Then from Lemma 3.1

we have

$$\begin{cases} 0 \in \eta_i Q_i(u_i, v_i) + h_i(x_i) - g_i(x_{i+1}) + N^P_{\mathcal{K}_r}(h_i(x_i))(i = 1, \ldots, N-1), \\ 0 \in \eta_N Q_N(u_N, v_N) + h_N(x_N) - g_N(x_1) + N^P_{\mathcal{K}_r}(h_N(x_N)), \end{cases} \tag{4.2}$$

$$\Leftrightarrow \begin{cases} g_i(x_{i+1}) - \eta_i Q_i(u_i, v_i) \in (I + N^P_{\mathcal{K}_r})(h_i(x_i))(i = 1, \ldots, N-1), \\ g_N(x_1) - \eta_N Q_N(u_N, v_N) \in (I + N^P_{\mathcal{K}_r})(h_N(x_N)),, \end{cases} \tag{4.3}$$

$$\Leftrightarrow \begin{cases} h_i(x_i) = P_{\mathcal{K}_r}[g_i(x_{i+1}) - \eta_i Q_i(u_i, v_i)](i = 1, \ldots, N-1), \\ h_N(x_N) = P_{\mathcal{K}_r}[g_N(x_1) - \eta_N Q_N(u_N, v_N)], \end{cases} \tag{4.4}$$

where I is an identity mapping and $P_{\mathcal{K}_r} = (I + N^P_{\mathcal{K}_r})^{-1}$. □

Remark 4.2. The inequality (4.1) can be written as follows

$$\begin{cases} q_i = g_i(x_{i+1}) - \eta_i Q_i(u_i, v_i), \quad h_i(x_i) = P_{\mathcal{K}_r}[q_i](i = 1, \ldots, N-1) \\ q_N = g_N(x_1) - \eta_N Q_N(u_N, v_N), \quad h_N(x_N) = P_{\mathcal{K}_r}[q_N], \end{cases} \tag{4.5}$$

where $\eta_i > 0, i = 1, \ldots, N$ are constants.

The fixed point formulation (4.5) enables us to construct the following perturbed iterative algorithms with mixed errors.

Algorithm 4.3. Let T_i, F_i, Q_i, g_i, h_i and $\eta_i(i = 1, 2, \cdots, N)$ be the same as in the system (3.1) such that $h_1, \ldots, h_N : \mathcal{H} \longrightarrow \mathcal{H}$ be onto operators. Let $e^0_1, \ldots, e^0_N$, $r^0_1, \ldots, r^0_N \in \mathcal{H}$, $\alpha_0 \in \mathbb{R}$ and $\eta_0 > 0$. For given $q^0_1, \ldots, q^0_N \in \mathcal{H}$, we let $x^0_1, \ldots, x^0_N \in \mathcal{H}$, $u_1 \in T_1(x_2, x_1)$, $u_2 \in T_2(x_3, x_2)$, $\ldots$,$u_{N-1} \in T_{N-1}(x_N, x_{N-1})$, $u_N \in T_N(x_1, x_N)$, $v_1 \in F_1(x_2, x_1)$, $v_2 \in F_2(x_3, x_2)$, $\ldots$, $v_{N-1} \in F_{N-1}(x_N, x_{N-1})$, $v_N \in F_N(x_1, x_N)$ such that

$$\begin{cases} h_i(x^0_i) = P_{\mathcal{K}_r}(q^0_i); \;\; q^1_i = (1-\alpha_0)q^0_i + \alpha_0(g_i(x^0_{i+1}) - \eta_0 Q_i(u^0_i, v^0_i) + e^0_i) \\ \qquad\qquad\qquad\qquad\qquad\qquad +r^0_i(i = 1, \ldots, N-1), \\ h_N(x^0_N) = P_{\mathcal{K}_r}(q^0_N); \;\; q^1_N = (1-\alpha_0)q^0_N + \alpha_0(g_N(x^0_1) \\ \qquad\qquad\qquad\qquad\qquad\qquad -\eta_0 Q_N(u^0_N, v^0_N) + e^0_N) + r^0_N. \end{cases} \tag{4.6}$$

We Choose $x_1^1, \ldots, x_1^N \in \mathcal{H}$ such that $h_1(x_1^1) = P_{\mathcal{K}_r}(q_1^1)$,...,$h_N(x_1^N) = P_{\mathcal{K}_r}(q_N^1)$. By Nadler Theorem [15], there exists

$$\begin{cases} u_i^1 \in T_i(x_{i+1}^0, x_i^0); \\ \|u_i^0 - u_i^1\| \leq (1 + (1+n)^{-1})\widehat{\mathcal{D}}(T_i(x_{i+1}^0, x_i^0), T_i(x_{i+1}^1, x_i^1)) \\ \qquad\qquad\qquad\qquad\qquad\qquad (i = 1, \ldots, N-1), \\ v_i^1 \in F_i(x_{i+1}^0, x_i^0); \\ \|v_i^0 - v_i^1\| \leq (1 + (1+n)^{-1})\widehat{\mathcal{D}}(F_i(x_{i+1}^0, x_i^0), F_i(x_{i+1}^1, x_i^1)) \\ \qquad\qquad\qquad\qquad\qquad\qquad (i = 1, \ldots, N-1), \\ u_N^1 \in T_N(x_1^0, x_N^0); \\ \|u_N^0 - u_N^1\| \leq (1 + (1+n)^{-1})\widehat{\mathcal{D}}(T_N(x_1^0, x_N^0), T_N(x_1^1, x_N^1)), \\ v_N^1 \in F_N(x_1^0, x_N^0); \\ \|v_N^0 - v_N^1\| \leq (1 + (1+n)^{-1})\widehat{\mathcal{D}}(F_N(x_1^0, x_N^0), F_N(x_1^1, x_N^1)). \end{cases} \tag{4.7}$$

Continuing the above process inductively, we can obtain the sequences $\{x_1^n\}_{n=0}^\infty, \ldots, \{x_N^n\}_{n=0}^\infty$, $\{u_1^n\}_{n=0}^\infty, \ldots, \{u_N^n\}_{n=0}^\infty$ by using

$$\begin{cases} h_i(x_i^n) = P_{\mathcal{K}_r}(q_i^n); \\ q_i^{n+1} = (1-\alpha_n)q_i^n + \alpha_n(g_i(x_{i+1}^n) - \eta_i Q_i(u_i^n, v_i^n) + e_i^n) + r_i^n \\ \qquad\qquad\qquad\qquad\qquad\qquad (i = 1, \ldots, N-1), \\ h_N(x_N^n) = P_{\mathcal{K}_r}(q_N^n); \\ q_N^{n+1} = (1-\alpha_n)q_N^n + \alpha_n(g_N(x_1^n) - \eta_N Q_N(u_N^n, v_N^n) + e_N^n) + r_N^n, \end{cases} \tag{4.8}$$

and

$$\begin{cases} u_i^n \in T_i(x_{i+1}^n, x_i^n); \\ \|u_i^n - u_i^{n+1}\| \leq (1 + (1+n)^{-1})\widehat{\mathcal{D}}(T_i(x_{i+1}^n, x_i^n), T_i(x_{i+1}^{n+1}, x_i^{n+1})) \\ \qquad\qquad\qquad\qquad\qquad\qquad (i = 1, \ldots, N-1), \\ v_i^n \in F_i(x_{i+1}^n, x_i^n); \\ \|v_i^n - v_i^{n+1}\| \leq (1 + (1+n)^{-1})\widehat{\mathcal{D}}(F_i(x_{i+1}^n, x_i^n), F_i(x_{i+1}^{n+1}, x_i^{n+1})) \\ \qquad\qquad\qquad\qquad\qquad\qquad (i = 1, \ldots, N-1), \\ u_N^n \in T_N(x_1^n, x_N^n); \\ \|u_N^n - u_N^{n+1}\| \leq (1 + (1+n)^{-1})\widehat{\mathcal{D}}(T_N(x_1^n, x_N^n), T_N(x_1^{n+1}, x_N^{n+1})), \\ v_N^n \in F_N(x_1^n, x_N^n); \\ \|v_N^n - v_N^{n+1}\| \leq (1 + (1+n)^{-1})\widehat{\mathcal{D}}(F_N(x_1^n, x_N^n), F_N(x_1^{n+1}, x_N^{n+1})), \end{cases} \tag{4.9}$$

where $0 \leq \alpha_n \leq 1$ is a parameter and $\{e_1^n\}_{n=0}^{\infty}, \ldots, \{e_N^n\}_{n=0}^{\infty}$, $\{r_1^n\}_{n=0}^{\infty}$, $\ldots, \{r_N^n\}_{n=0}^{\infty}$ are sequences in $\mathcal{H}$ to take into account of a possible inexact computation of the resolvent operator satisfying the following conditions:

$$\lim_{n\longrightarrow\infty} e_i^n = \lim_{n\longrightarrow\infty} r_i^n = 0;$$

$$\sum_{n=1}^{\infty} \|e_i^n - e_i^{n-1}\| < \infty, \quad \sum_{n=1}^{\infty} \|r_i^n - r_i^{n-1}\| < \infty, \tag{4.10}$$

for all $i = 1, \ldots, N$.

THEOREM 4.4. *Let $T_i, F_i, Q_i, g_i, h_i, \eta_i$, for $i = 1, \ldots, N$ be the same as in the system* (3.1) *such that, for each $i = 1, \ldots, N$,*

(i) *Q_i is ζ_i-Lipschitz continuous with respect to the first variable with a constant $\zeta_i > 0$ and ϱ_i-Lipschitz continuous with respect to the second variables with a constant $\varrho_i > 0$;*
(ii) *T_i is $\xi_i - \widehat{\mathcal{D}}$-Lipschitz continuous in the first variables with a constant $\xi_i > 0$;*
(iii) *F_i is $\rho_i - \widehat{\mathcal{D}}$-Lipschitz continuous in the first variables with a constant $\rho_i > 0$;*
(iv) *Q_i is (κ_i, λ_i)-relaxed cocoercive with respect to the first variable of Q_i and T_i with constants $\kappa_i, \lambda_i > 0$;*
(v) *h_i is β_i-strongly monotone with respect to a constant $\beta_i > 0$ and σ_i-Lipschitz continuous with a constant $\sigma_i > 0$;*
(vi) *g_i is inversely γ_i-strongly monotone with a constant $\gamma_i > 0$ and μ_i-Lipschitz continuous mapping with a constant $\mu_i > 0$;*

If the constants $\eta_i > 0$ satisfy the following conditions:

$$\begin{aligned}
\left|\eta_1 - \frac{\kappa_1}{\xi_1^2}\right| &< \frac{\sqrt{r^2\kappa_1^2 - \xi_1^2\Big(r^2\mu_1^2 - (r-r')^2(1-\pi_2)^2\Big)}}{r\xi_1^2}, \\
&\vdots \\
\left|\eta_N - \frac{\kappa_N}{\xi_N^2}\right| &< \frac{\sqrt{r^2\kappa_N^2 - \xi_N^2\Big(r^2\mu_N^2 - (r-r')^2(1-\pi_1)^2\Big)}}{r\xi_N^2},
\end{aligned} \tag{4.11}$$

$$\begin{aligned} r\kappa_1 &> \xi_1\sqrt{r^2\mu_1^2-(r-r')^2(1-\mu_2)^2}, \\ &\vdots \\ r\kappa_N &> \xi_N\sqrt{r^2\mu_N^2-(r-r')^2(1-\mu_1)^2}, \end{aligned} \tag{4.12}$$

$$r\mu_1 > (r-r')(1-\pi_2), \ldots, r\mu_N > (r-r')(1-\pi_1), \tag{4.13}$$

and

$$\pi_i = \sqrt{1-2\beta_i+\sigma_i^2}, \quad 2\pi_i < 1+\sigma_i^2, \tag{4.14}$$

for each $i = 1, \ldots, N$, *where* $r' \in (0, r)$, *then there exists* $x_1^*, \ldots, x_N^* \in \mathcal{H}$ *with* $h_1(x_1^*), \ldots, h_N(x_N^*) \in \mathcal{K}_r$ *and* $u_1^* \in T_1(x_2^*, x_1^*), \ldots, u_{N-1}^* \in T_{N-1}(x_N^*, x_{N-1}^*)$, $u_N^* \in T_N(x_1^*, x_N^*)$, $v_1^* \in F_1(x_2^*, x_1^*), \ldots, v_{N-1}^* \in F_{N-1}(x_N^*, x_{N-1}^*), v_N^* \in F_N(x_1^*, x_N^*)$ *such that* $(x_1^*, \ldots, x_N^*,\ u_1^*, \ldots, u_N^*,\ v_1^*, \ldots, v_N^*)$ *is a solution set of system* (3.1) *and sequences* $\{(x_1^n, \ldots, x_N^n, u_1^n, \ldots, u_N^n, v_1^n, \ldots, v_N^n)\}_{n=0}^{\infty}$ *suggested by Algorithm 4.3 converges strongly to* $(x_1^*, \ldots, x_N^*, u_1^*, \ldots, u_N^*, v_1^*, \ldots, v_N^*)$.

Proof. From (4.8), we have

$$\begin{aligned} &\|q_1^{n+1}-q_1^n\| \\ \le\ & (1-\alpha_n)\|q_1^n-q_1^{n-1}\| \\ &+\alpha_n\|g_1(x_2^n)-g_1(x_2^{n-1})-\eta_1(Q_1(u_1^n,v_1^n)-Q_1(u_1^{n-1},v_1^{n-1}))\| \\ &+\alpha_n\|e_1^n-e_1^{n-1}\|+\|r_1^n-r_1^{n-1}\| \\ \le\ & (1-\alpha_n)\|q_1^n-q_1^{n-1}\|+\alpha_n\big\{\|x_2^n-x_2^{n-1}-(g_1(x_2^n)-g_1(x_2^{n-1}))\| \\ &\qquad +\|x_2^n-x_2^{n-1}-\eta_1(Q_1(u_1^n,v_1^n)-Q_1(u_1^{n-1},v_1^n))\| \\ &\qquad +\eta_1\|Q_1(u_1^{n-1},v_1^n)-Q_1(u_1^{n-1},v_1^{n-1})\|\big\} \\ &+\alpha_n\|e_1^n-e_1^{n-1}\|+\|r_1^n-r_1^{n-1}\|. \end{aligned} \tag{4.15}$$

Since g_1 is inversely γ_1-strongly monotone with respect to a constant $\gamma_1 > 0$ and μ_1-Lipschitz continuous with a constant $\mu_1 > 0$, we get

$$\begin{aligned} &\|x_2^n-x_2^{n-1}-(g_1(x_2^n)-g_1(x_2^{n-1}))\|^2 = \|x_2^n-x_2^{n-1}\|^2 \\ &-2\langle g_1(x_2^n)-g_1(x_2^{n-1}), x_2^n-x_2^{n-1}\rangle+\|g_1(x_2^n)-g_1(x_2^{n-1})\|^2 \\ \le\ & \|x_2^n-x_2^{n-1}\|^2-2\gamma_1\|g_1(x_2^n)-g_1(x_2^{n-1})\|^2+\|g_1(x_2^n)-g_1(x_2^{n-1})\|^2 \\ \le\ & \|x_2^n-x_2^{n-1}\|^2-2\gamma_1\mu_1^2\|x_2^n-x_2^{n-1}\|^2+\mu_1^2\|x_2^n-x_2^{n-1}\|^2 \\ \le\ & (1+\mu_1^2(1-2\gamma_1))\|x_2^n-x_2^{n-1}\|^2. \end{aligned} \tag{4.16}$$

Since Q_1 is ζ_1-Lipschitz continuous with respect to the first variable with constant $\zeta_1 > 0$ and second variable with constant $\varrho_1 > 0$, and T_1 is $\xi_1 - \widehat{\mathcal{D}}$-Lipschitz continuous in the first variables with constant $\xi_1 > 0$, and F_1 is $\rho_1 - \widehat{\mathcal{D}}$-Lipschitz continuous in the first variables with constant $\rho_1 > 0$, we get

$$\begin{aligned}\|Q_1(u_1^n, v_1^n) - Q_1(u_1^{n-1}, v_1^n)\| &\leq \zeta_1\|u_1^n - u_1^{n-1}\| \\ &\leq \zeta_1(1+\frac{1}{n})\widehat{\mathcal{D}}(T_1(x_2^n, x_1^n), T_1(x_2^{n-1}, x_1^{n-1})) \\ &\leq \zeta_1(1+\frac{1}{n})\xi_1\|x_2^n - x_2^{n-1}\|, \qquad (4.17)\end{aligned}$$

$$\begin{aligned}\|Q_1(u_1^{n-1}, v_1^n) - Q_1(u_1^{n-1}, v_1^{n-1})\| &\leq \varrho_1\|v_1^n - v_1^{n-1}\| \\ &\leq \varrho_1(1+\frac{1}{n})\widehat{\mathcal{D}}(F_1(x_2^n, x_1^n), F_1(x_2^{n-1}, x_1^{n-1})) \\ &\leq \varrho_1(1+\frac{1}{n})\rho_1\|x_2^n - x_2^{n-1}\|. \qquad (4.18)\end{aligned}$$

Since Q_1 is (κ_1, λ_1)-relaxed cocoercive with respect to the first variable of Q_1 and T_1 with a constants $\kappa_1, \lambda_1 > 0$, respectively and ζ_1-Lipschitz continuous with respect to the first variable with a constant $\zeta_1 > 0$ and T_1 is $\xi_1 - \widehat{\mathcal{D}}$-Lipschitz continuous in the first variables with a constant $\xi_1 > 0$, we get

$$\begin{aligned}&\|x_2^n - x_2^{n-1} - \eta_1(Q_1(u_1^n, v_1^n) - Q_1(u_1^{n-1}, v_1^n))\|^2 = \|x_2^n - x_2^{n-1}\|^2 \\ &\quad -2\eta_1\langle Q_1(u_1^n, v_1^n) - Q_1(u_1^{n-1}, v_1^n), x_2^n - x_2^{n-1}\rangle \\ &\quad +\eta_1^2\|Q_1(u_1^n, v_1^n) - Q_1(u_1^{n-1}, v_1^n)\|^2 \\ \leq\ & \|x_2^n - x_2^{n-1}\|^2 - 2\eta_1(-\kappa_1\|Q_1(u_1^n, v_1^n) - Q_1(u_1^{n-1}, v_1^n)\|^2 \\ &\quad +\lambda_1\|x_2^n - x_2^{n-1}\|^2) + \eta_1^2\zeta_1^2\|u_1^n - u_1^{n-1}\| \\ \leq\ & \|x_2^n - x_2^{n-1}\|^2 - 2\eta_1(-\kappa_1\zeta_1^2(1+n^{-1})^2\xi_1^2(\widehat{\mathcal{D}}(T_1(x_2^n, x_1^n), T_1(x_2^{n-1}, x_1^{n-1})))^2 \\ &\quad +\lambda_1\|x_2^n - x_2^{n-1}\|^2) + \eta_1^2\zeta_1^2(1+n^{-1})^2\xi_1^2(\widehat{\mathcal{D}}(T_1(x_2^n, x_1^n), T_1(x_2^{n-1}, x_1^{n-1})))^2 \\ \leq\ & \|x_2^n - x_2^{n-1}\|^2 - 2\eta_1(-\kappa_1\zeta_1^2(1+n^{-1})^2\xi_1^2\|x_2^n - x_2^{n-1}\|^2 \\ &\quad +\lambda_1\|x_2^n - x_2^{n-1}\|^2) + \eta_1^2\zeta_1^2(1+n^{-1})^2\xi_1^2\|x_2^n - x_2^{n-1}\|^2 \\ \leq\ & (1 - 2\eta_1(\lambda_1^2 - \kappa_1\zeta_1^2\xi_1^2(1+\frac{1}{n})^2) + \eta_1^2\zeta_1^2\xi_1^2(1+\frac{1}{n})^2)\|x_2^n - x_2^{n-1}\|^2. \quad (4.19)\end{aligned}$$

It follows from (4.15)-(4.19), we obtain that

$$
\begin{aligned}
&\|q_i^{n+1} - q_i^n\| \\
&\leq (1-\alpha_n)\|q_i^n - q_i^{n-1}\| + \alpha_n\Big(\sqrt{1+\mu_i^2(1-2\gamma_i)} \\
&\quad + \sqrt{1 - 2\eta_i(\lambda_i^2 - \kappa_i\zeta_i^2\xi_i^2(1+\frac{1}{n})^2) + \eta_i^2\zeta_i^2\xi_i^2(1+\frac{1}{n})^2\|} \\
&\quad + \varrho_i(1+\frac{1}{n})\rho_i\Big)\|x_{i+1}^n - x_{i+1}^{n-1} + \alpha_n\|e_i^n - e_i^{n-1}\| + \|r_i^n - r_i^{n-1}\| \\
&\quad (i = 1, \ldots, N-1),
\end{aligned}
$$

$$
\begin{aligned}
&\|q_N^{n+1} - q_N^n\| \\
&\leq (1-\alpha_n)\|q_N^n - q_N^{n-1}\| + \alpha_n\Big(\sqrt{1+\mu_N^2(1-2\gamma_N)} \\
&\quad + \sqrt{1 - 2\eta_N(\lambda_N^2 - \kappa_N\zeta_N^2\xi_N^2(1+\frac{1}{n})^2) + \eta_N^2\zeta_N^2\xi_N^2(1+\frac{1}{n})^2} \\
&\quad + \varrho_N(1+\frac{1}{n})\rho_N\Big)\|x_1^n - x_1^{n-1}\| + \alpha_n\|e_N^n - e_N^{n-1}\| \\
&\quad + \|r_N^n - r_N^{n-1}\|. \qquad (4.20)
\end{aligned}
$$

By using (4.8), we get that

$$
\begin{aligned}
&\|x_1^n - x_1^{n-1}\| \\
\leq\ & \|x_1^n - x_1^{n-1} - (h_1(x_1^n) - h_1(x_1^{n-1}))\| + \|h_1(x_1^n) - h_1(x_1^{n-1})\| \\
=\ & \|x_1^n - x_1^{n-1} - (h_1(x_1^n) - h_1(x_1^{n-1}))\| + \|P_{\mathcal{K}_r}(q_1^n) - P_{\mathcal{K}_r}(q_1^{n-1})\| \\
\leq\ & \|x_1^n - x_1^{n-1} - (h_1(x_1^n) - h_1(x_1^{n-1}))\| + \frac{r}{r-r'}\|q_1^n - q_1^{n-1}\|. \quad (4.21)
\end{aligned}
$$

Since h_1 is β_1-strongly monotone with respect to the constant $\beta_1 > 0$ and σ_1-Lipschitz continuous with a constant $\sigma_1 > 0$, we have

$$
\begin{aligned}
&\|x_1^n - x_1^{n-1} - (h_1(x_1^n) - h_1(x_1^{n-1}))\|^2 \\
=\ & \|x_1^n - x_1^{n-1}\|^2 - 2\langle h_1(x_1^n) - h_1(x_1^{n-1}), x_1^n - x_1^{n-1}\rangle \\
& + \|h_1(x_1^n) - h_1(x_1^{n-1})\|^2 \\
\leq\ & \|x_1^n - x_1^{n-1}\|^2 - 2\beta_1\|x_1^n - x_1^{n-1}\|^2 + \sigma_1^2\|x_1^n - x_1^{n-1}\|^2 \\
=\ & (1 - 2\beta_1 + \sigma_1^2)\|x_1^n - x_1^{n-1}\|^2. \qquad (4.22)
\end{aligned}
$$

By (4.21) and (4.22), we obtain

$$
\|x_1^n - x_1^{n-1}\| \leq \sqrt{1 - 2\beta_1 + \sigma_1^2}\|x_1^n - x_1^{n-1}\| + \frac{r}{r-r'}\|q_1^n - q_1^{n-1}\| \quad (4.23)
$$

that is

$$\|x_1^n - x_1^{n-1}\| \leq \frac{r}{(r-r')(1-\sqrt{1-2\beta_1+\sigma_1^2})}\|q_1^n - q_1^{n-1}\|. \quad (4.24)$$

Similarly, we can prove that

$$\begin{cases} \|x_i^n - x_i^{n-1}\| \leq \frac{r}{(r-r')(1-\sqrt{1-2\beta_i+\sigma_i^2})}\|q_i^n - q_i^{n-1}\|(i=2,\ldots,N-1), \\ \|x_N^n - x_N^{n-1}\| \leq \frac{r}{(r-r')(1-\sqrt{1-2\beta_N+\sigma_N^2})}\|q_N^n - q_N^{n-1}\|. \end{cases} \quad (4.25)$$

It follows from (4.20), (4.24) and (4.25) that

$$\begin{aligned} \|q_i^{n+1} - q_i^n\| &\leq (1-\alpha_n)\|q_i^n - q_i^{n-1}\| + \alpha_n \frac{r(\vartheta_i + \Omega_i(n))}{(r-r')(1-\pi_{i+1})}\|q_{i+1}^n - q_{i+1}^{n-1}\| \\ &\quad + \alpha_n\|e_i^n - e_i^{n-1}\| + \|r_i^n - r_i^{n-1}\|(i=1,\ldots,N-1), \\ \|q_N^{n+1} - q_N^n\| &\leq (1-\alpha_n)\|q_N^n - q_N^{n-1}\| + \alpha_n \frac{r(\vartheta_N + \Omega_N(n))}{(r-r')(1-\pi_1)}\|q_1^n - q_1^{n-1}\| \\ &\quad + \alpha_n\|e_N^n - e_N^{n-1}\| + \|r_N^n - r_N^{n-1}\|, \end{aligned} \quad (4.26)$$

where $\vartheta_i = \sqrt{1+\mu_i^2(1-2\gamma_i)}$, $\pi_i = \sqrt{1-2\beta_i+\sigma_i^2}$ and

$$\Omega_i(n) = \sqrt{1-2\eta_i(\lambda_i^2 - \kappa_i\zeta_i^2\xi_i^2(1+\frac{1}{n})^2) + \eta_i^2\zeta_i^2\xi_i^2(1+\frac{1}{n})^2} + \varrho_i(1+\frac{1}{n})\rho_i,$$

for all $i = 1, 2, \ldots, N$.
Now we define $\|\cdot\|_*$ on $\underbrace{\mathcal{H}\times\ldots\times\mathcal{H}}_{N-\text{times}}$ by

$$\|(x_1,\ldots,x_N)\|_* = \|x_1\|+\ldots+\|x_N\|, \quad \text{for all } (x_1,\ldots,x_N) \in \underbrace{\mathcal{H}\times\ldots\times\mathcal{H}}_{N-\text{times}}.$$

It is obvious that $(\underbrace{\mathcal{H}\times\ldots\times\mathcal{H}}_{N-\text{times}}, \|\cdot\|_*)$ is a Hilbert space, applying (4.26) we have

$$\begin{aligned} &\|(q_1^{n+1},\ldots,q_N^{n+1}) - (q_1^n,\ldots,q_N^n)\|_* \\ &\quad \leq (1-\alpha_n)\|(q_1^n,\ldots,q_N^n) - (q_1^{n-1},\ldots,q_N^{n-1})\|_* \\ &\qquad + \alpha_n\Theta(n)\|(q_1^n,\ldots,q_N^n) - (q_1^{n-1},\ldots,q_N^{n-1})\|_* \\ &\qquad + \alpha_n\|(e_1^n,\ldots,e_N^n) - (e_1^{n-1},\ldots,e_N^{n-1})\|_* \\ &\qquad + \|(r_1^n,\ldots,r_N^n) - (r_1^{n-1},\ldots,r_N^{n-1})\|_*. \end{aligned} \quad (4.27)$$

Put

$$\Theta(n) = \max\left\{\frac{r(\vartheta_1+\Omega_1(n))}{(r-r')(1-\pi_2)},\dots,\frac{r(\vartheta_N+\Omega_N(n))}{(r-r')(1-\pi_1)}\right\}. \tag{4.28}$$

Let $\Theta(n) \longrightarrow \Theta$, as $n \longrightarrow \infty$, where

$$\Theta = \max\left\{\frac{r(\vartheta_1+\Omega_1)}{(r-r')(1-\pi_2)},\dots,\frac{r(\vartheta_N+\Omega_N)}{(r-r')(1-\pi_1)}\right\}. \tag{4.29}$$

By (4.11),we know that $0 \leq \Theta < 1$. For $\Theta = \frac{1}{2}(\Theta+1) \in (\Theta, 1)$ there exists $n_0 \geq 1$ such that $\Theta(n) = \widehat{\Theta}$ for each $n \geq n_0$. So it follows from (4.24) that, for each $n \geq n_0$,

$$\begin{aligned}
&\|(q_1^{n+1},\dots,q_N^{n+1})-(q_1^n,\dots,q_N^n)\|_*\\
\leq\ & (1-\alpha_n)\|(q_1^n,\dots,q_N^n)-(q_1^{n-1},\dots,q_N^{n-1})\|_*\\
&+\alpha_n\widehat{\Theta}\|(q_1^n,\dots,q_N^n)-(q_1^{n-1},\dots,q_N^{n-1})\|_*\\
&+\alpha_n\|(e_1^n,\dots,e_N^n)-(e_1^{n-1},\dots,e_N^{n-1})\|_*\\
&+\|(r_1^n,\dots,r_N^n)-(r_1^{n-1},\dots,r_N^{n-1})\|_*\\
=\ & (1-\alpha_n(1-\widehat{\Theta}))\|(q_1^n,\dots,q_N^n)-(q_1^{n-1},\dots,q_N^{n-1})\|_*\\
&+\alpha_n\|(e_1^n,\dots,e_N^n)-(e_1^{n-1},\dots,e_N^{n-1})\|_*\\
&+\|(r_1^n,\dots,r_N^n)-(r_1^{n-1},\dots,r_N^{n-1})\|_*\\
\leq\ & (1-\alpha_n(1-\widehat{\Theta}))\Big((1-\alpha_n(1-\widehat{\Theta}))\|(q_1^{n-1},\dots,q_N^{n-1})-(q_1^{n-2},\dots,q_N^{n-2})\|_*\\
&+\alpha_n\|(e_1^{n-1},\dots,e_N^{n-1})-(e_1^{n-2},\dots,e_N^{n-2})\|_*\\
&+\|(r_1^{n-1},\dots,r_N^{n-1})-(r_1^{n-2},\dots,r_N^{n-2})\|_*\Big)\\
&+\alpha_n\|(e_1^n,\dots,e_N^n)-(e_1^{n-1},\dots,e_N^{n-1})\|_*\\
&+\|(r_1^n,\dots,r_N^n)-(r_1^{n-1},\dots,r_N^{n-1})\|_*\\
=\ & (1-\alpha_n(1-\widehat{\Theta}))^2\|(q_1^{n-1},\dots,q_N^{n-1})-(q_1^{n-2},\dots,q_N^{n-2})\|_*\\
&+\alpha_n\Big((1-\alpha_n(1-\widehat{\Theta}))\|(e_1^{n-1},\dots,e_N^{n-1})-(e_1^{n-2},\dots,e_N^{n-2})\|_*\\
&+\|(e_1^n,\dots,e_N^n)-(e_1^{n-1},\dots,e_N^{n-1})\|_*\Big)\\
&+(1-\alpha_n(1-\widehat{\Theta}))\|(r_1^{n-1},\dots,r_N^{n-1})-(r_1^{n-2},\dots,r_N^{n-2})\|_*\\
&+\|(r_1^n,\dots,r_N^n)-(r_1^{n-1},\dots,r_N^{n-1})\|_*\\
\leq\ & \\
&\vdots
\end{aligned}$$

$$
\begin{aligned}
&\leq\ (1-\alpha_n(1-\widehat{\Theta}))^{n-n_0}\|(q_1^{n_0+1},\dots,q_N^{n_0+1})-(q_1^{n_0},\dots,q_N^{n_0})\|_*\\
&+\alpha_n\sum_{i=1}^{n-n_0}(1-\alpha_n(1-\widehat{\Theta}))^{i-1}\|(e_1^{n-(i-1)},\dots,e_N^{n-(i-1)})-(e_1^{n-i},\dots,e_N^{n-i})\|_*\\
&+\sum_{i=1}^{n-n_0}(1-\alpha_n(1-\widehat{\Theta}))^{i-1}\|(r_1^{n-(i-1)},\dots,r_N^{n-(i-1)})-(r_1^{n-i},\dots,r_N^{n-i})\|_*.
\end{aligned}
\tag{4.30}
$$

Thus, for any $m\geq n>n_0$, we get that

$$
\begin{aligned}
&\|(q_1^m,\dots,q_N^m)-(q_1^n,\dots,q_N^n)\|_*\\
\leq\ &\sum_{j=n}^{m-1}\|(q_1^{j+1},\dots,q_N^{j+1})-(q_1^j,\dots,q_N^j)\|_*\\
\leq\ &\sum_{j=n}^{m-1}(1-\alpha_n(1-\widehat{\Theta}))^{j-n_0}\|(q_1^{n_0+1},\dots,q_N^{n_0+1})-(q_1^{n_0},\dots,q_N^{n_0})\|_*\\
&+\alpha_n\sum_{j=n}^{m-1}\sum_{i=1}^{j-n_0}(1-\alpha_n(1-\widehat{\Theta}))^{i-1}\\
&\qquad\cdot\|(e_1^{n-(i-1)},\dots,e_N^{n-(i-1)})-(e_1^{n-i},\dots,e_N^{n-i})\|_*\\
&+\sum_{j=n}^{m-1}\sum_{i=1}^{j-n_0}(1-\alpha_n(1-\widehat{\Theta}))^{i-1}\\
&\qquad\cdot\|(r_1^{n-(i-1)},\dots,r_N^{n-(i-1)})-(r_1^{n-i},\dots,r_N^{n-i})\|_*.
\end{aligned}
\tag{4.31}
$$

Since $(1-\alpha_n(1-\widehat{\Theta}))\in(0,1)$, it follows from (4.10) and (4.31) that

$$\|(q_1^m,\dots,q_N^m)-(q_1^n,\dots,q_N^n)\|_*=\|q_1^m-q_1^n\|+\ldots+\|q_N^m-q_N^n\|\longrightarrow 0$$

as $n\longrightarrow\infty$. So $\{q_i^n\}(i=1,\cdots,N)$ are Cauchy sequences in $\mathcal{H}$, there exist $q_i^*(i=1,\cdots,N)\in\mathcal{H}$ such that $q_i^n\longrightarrow q_i^*(i=1,\cdots,N)$ as $n\longrightarrow\infty$. By (4.24) and (4.25), it follows that the sequences $\{x_i^n\}(i=1,\cdots,N)$ are also Cauchy sequences in $\mathcal{H}$. Hence there exist $x_i^*(i=1,\cdots,N)\in\mathcal{H}$ such that $x_i^n\longrightarrow x_i^*(i=1,\cdots,N)$ as $n\longrightarrow\infty$. Since for each $i=1,\dots,N$, T_i are ξ_i-$\widehat{\mathcal{D}}$-Lipschitz continuous in the first variable and also F_i are ρ_i-$\widehat{\mathcal{D}}$-Lipschitz continuous in the first variable, it follow from

(4.7) that

$$\begin{array}{rcl}
\|u_i^n - u_i^{n+1}\| & \leq & (1+(1+n)^{-1})\widehat{\mathcal{D}}(T_i(x_{i+1}^n, x_i^n), T_i(x_{i+1}^{n+1}, x_i^{n+1})) \\
& \leq & (1+(1+n)^{-1})\xi_i\|x_{i+1}^n - x_{i+1}^{n+1}\| \longrightarrow 0, (i=1,\dots,N-1) \\
\|v_i^n - v_i^{n+1}\| & \leq & (1+(1+n)^{-1})\widehat{\mathcal{D}}(F_i(x_{i+1}^n, x_i^n), F_i(x_{i+1}^{n+1}, x_i^{n+1})) \\
& \leq & (1+(1+n)^{-1})\rho_i\|x_{i+1}^n - x_{i+1}^{n+1}\| \longrightarrow 0, (i=1,\dots,N-1) \\
\|u_N^n - u_N^{n+1}\| & \leq & (1+(1+n)^{-1})\widehat{\mathcal{D}}(T_N(x_1^n, x_N^n), T_1(x_1^{n+1}, x_N^{n+1})) \\
& \leq & (1+(1+n)^{-1})\xi_N\|x_1^n - x_1^{n+1}\| \longrightarrow 0, \\
\|v_N^n - v_N^{n+1}\| & \leq & (1+(1+n)^{-1})\widehat{\mathcal{D}}(F_N(x_1^n, x_N^n), F_1(x_1^{n+1}, x_N^{n+1})) \\
& \leq & (1+(1+n)^{-1})\rho_N\|x_1^n - x_1^{n+1}\| \longrightarrow 0, \text{ as } n \longrightarrow \infty.
\end{array} \tag{4.32}$$

Hence $\{u_i^n\}(i=1,\cdots,N)$ are Cauchy sequences in $\mathcal{H}$ and also $\{v_i^n\}(i=1,\cdots,N)$ are Cauchy sequences in $\mathcal{H}$ and so there exist $u_i^*(i=1,\cdots,N)\in\mathcal{H}$ such that $x_i^n \longrightarrow x_i^*(i=1,\cdots,N)$ as $n\longrightarrow\infty$. Further $u_1^n \in T_1(x_2^n, x_1^n)$ we have

$$\begin{array}{rcl}
d(u_1^*, T_1(x_2^*, x_1^*)) & := & \inf\{\|u_1^* - t\| : t \in T_1(x_2^*, x_1^*)\} \\
& \leq & \|u_1^* - u_1^n\| + d(u_1^n, T_1(x_2^*, x_1^*)) \\
& \leq & \|u_1^* - u_1^n\| + \widehat{\mathcal{D}}(T_1(x_2^n, x_1^n), T_1(x_2^{n+1}, x_1^{n+1})) \\
& \leq & \|u_1^* - u_1^n\| + \|x_2^n - x_2^*\| \longrightarrow 0, \text{ as } n \longrightarrow \infty.
\end{array} \tag{4.33}$$

Hence $d(u_1^*, T_1(x_2^*, x_1^*)) = 0$ and so $u_1^n \longrightarrow u_1^* \in T_1(x_2^*, x_1^*)$.
Similarly we can show that $d(v_1^*, F_1(x_2^*, x_1^*)) = 0$ and so $v_1^n \longrightarrow v_1^* \in F_1(x_2^*, x_1^*)$.
By the same method, we can prove that

$$\begin{cases}
d(u_{i-1}^*, T_{i-1}(x_i^*, x_{i-1}^*)) \leq \|u_{i-1}^* - u_{i-1}^n\| + \|x_i^n - x_i^*\| \longrightarrow 0, (i=3,\dots,N) \\
d(v_{i-1}^*, F_{i-1}(x_i^*, x_{i-1}^*)) \leq \|v_{i-1}^* - v_{i-1}^n\| + \|x_i^n - x_i^*\| \longrightarrow 0, (i=3,\dots,N) \\
d(u_N^*, T_N(x_1^*, x_N^*)) \ \leq \|u_N^* - u_N^n\| + \|x_1^n - x_1^*\| \longrightarrow 0, \\
d(v_N^*, F_N(x_1^*, x_N^*)) \ \leq \|v_N^* - v_N^n\| + \|x_1^n - x_1^*\| \longrightarrow 0, \text{ as } n \longrightarrow \infty.
\end{cases}$$

Therefore $u_i^* \in T_i(x_{i+1}^*, x_i^*)(i=2,\dots,N-1)$,$u_N^* \in T_N(x_1^*, x_N^*)$ and also $v_i^* \in F_i(x_{i+1}^*, x_i^*)(i=2,\dots,N-1)$,$v_N^* \in F_N(x_1^*, x_N^*)$. Since g_i and

$Q_i(i = 1, \ldots, N)$ are continuous, it follows from (4.8) and (4.10) that

$$
\begin{aligned}
q_i^* &= g_i(x_{i+1}^*) - \eta_i Q_i(u_i^*, v_i^*)(i = 1, \ldots, N-1), \\
q_N^* &= g_N(x_1^*) - \eta_N Q_N(u_N^*, v_N^*).
\end{aligned} \tag{4.34}
$$

Since $h_1, \ldots, h_N$ and $P_{\mathcal{K}_r}$ are continuous mappings, it follows from (4.8) and (4.34) that

$$
\begin{cases}
h_i(x_i^*) \;\;\; = P_{\mathcal{K}_r}(q_i^*) = P_{\mathcal{K}_r}(g_i(x_{i+1}^*) - \eta_i Q_i(u_i^*, v_i))(i = 1, \ldots, N-1), \\
h_N(x_N^*) = P_{\mathcal{K}_r}(q_N^*) = P_{\mathcal{K}_r}(g_N(x_1^*) - \eta_N Q_N(u_N^*, v_N^*)).
\end{cases}
$$

Now Lemma 4.1, guarantees that $(x_1^*, \ldots, x_N^*, u_1^*, \ldots, u_N^*, v_1^*, \ldots, v_N^*)$ is a solution set of the system (3.1). □

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