

A RESOLVENT APPROACH FOR SOLVING A SET-VALUED VARIATIONAL INCLUSION PROBLEM USING WEAK-RRD SET-VALUED MAPPING

IQBAL AHMAD, RAIS AHMAD, AND JAVID IQBAL

ABSTRACT. The resolvent operator approach of [2] is applied to solve a set-valued variational inclusion problem in ordered Hilbert spaces. The resolvent operator under consideration is called relaxed resolvent operator and we demonstrate some of its properties. To obtain the solution of a set-valued variational inclusion problem, an iterative algorithm is developed and weak-RRD set-valued mapping is used. The problem as well as main result of this paper are more general than many previous problems and results available in the literature.

1. Introduction

The variational inclusion is an important generalization of the variational inequality and is applicable to solve many problems related to optimization and control, economic and transportation equilibrium, engineering and basic sciences. A lot of work concerned with the ordered variational inequalities and ordered equations is done by H-G Li and his co-authors, see [10, 12, 13, 16, 17].

Most of the problems related to variational inclusions are solved by maximal monotone operators and their generalizations such as H -monotonicity [6], H -accretivity [5] and many more, see e.g., [3, 7, 8] and reference therein. Almost all the splitting methods are based on the resolvent

Received April 5, 2016. Revised May 16, 2016. Accepted May 17, 2016.

2010 Mathematics Subject Classification: 49J40, 47H05.

Key words and phrases: Algorithm, Inclusion, Mapping, Resolvent; Space.

© The Kangwon-Kyungki Mathematical Society, 2016.

This is an Open Access article distributed under the terms of the Creative Commons Attribution Non-Commercial License (<http://creativecommons.org/licenses/by-nc/3.0/>) which permits unrestricted non-commercial use, distribution and reproduction in any medium, provided the original work is properly cited.

operator of the form $[I + \lambda M]^{-1}$, where M is a set-valued monotone mapping, λ is a positive constant and I is the identity mapping.

In this paper, we consider a resolvent operator $[(I - R) + \lambda M]^{-1}$, where R is a single-valued mapping and is called relaxed resolvent operator, see [2]. We prove that the relaxed resolvent operator is a comparison mapping as well as Lipschitz continuous with respect to operator \oplus . Finally, a set-valued variational inclusion problem is solved by using weak-RRD set-valued mapping.

2. Preliminaries

Let X is a real ordered Hilbert space equipped with a norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$, d be the metric induced by the norm $\|\cdot\|$, 2^X (respectively, $CB(X)$) is the family of nonempty (respectively, closed and bounded) subsets of X , and $D(\cdot, \cdot)$ is the Hausdörff metric on $CB(X)$ defined by

$$D(P, Q) = \max \left\{ \sup_{x \in P} d(x, Q), \sup_{y \in Q} d(P, y) \right\},$$

where $P, Q \in CB(X)$, $d(x, Q) = \inf_{y \in Q} d(x, y)$ and $d(P, y) = \inf_{x \in P} d(x, y)$.

Let us recall some known concepts and results.

DEFINITION 2.1. A nonempty closed convex subset C of X is said to be a cone if,

- (i) for any $x \in C$ and any $\lambda > 0$, $\lambda x \in C$;
- (ii) if $x \in C$ and $-x \in C$, then $x = 0$.

DEFINITION 2.2. [4] Let C be the cone of X . C is said to be normal if and only if there exists a constant $\lambda_{C_N} > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq \lambda_{C_N} \|y\|$, where λ_{C_N} is called the normal constant of C .

DEFINITION 2.3. Let C be the cone in X . For arbitrary elements $x, y \in X$, $x \leq y$ if and only if $x - y \in C$, then the relation \leq in X is a partial ordered relation in X . The Hilbert space X equipped with the ordered relation \leq defined by the cone C is called ordered Hilbert space.

DEFINITION 2.4. [19] For arbitrary elements $x, y \in X$, if $x \leq y$ (or $y \leq x$) holds, then x and y are said to be comparable to each other (denoted by $x \propto y$).

DEFINITION 2.5. [19] For arbitrary elements $x, y \in X$, $\text{lub}\{x, y\}$ and $\text{glb}\{x, y\}$ mean least upper bound and greatest upper bound of the set $\{x, y\}$. Suppose $\text{lub}\{x, y\}$ and $\text{glb}\{x, y\}$ exist, some binary operations are defined as follows:

- (i) $x \vee y = \text{lub}\{x, y\}$,
- (ii) $x \wedge y = \text{glb}\{x, y\}$,
- (iii) $x \oplus y = (x - y) \vee (y - x)$,
- (iv) $x \odot y = (x - y) \wedge (y - x)$.

The operations \vee, \wedge, \oplus and \odot are called OR, AND, XOR and XNOR operations, respectively.

PROPOSITION 2.1. [4] If $x \propto y$, then $\text{lub}\{x, y\}$ and $\text{glb}\{x, y\}$ exist, $x - y \propto y - x$, and $0 \leq (x - y) \vee (y - x)$.

PROPOSITION 2.2. [4] For any positive integer n , if $x \propto y_n$ and $y_n \rightarrow y^*$ ($n \rightarrow \infty$), then $x \propto y^*$.

PROPOSITION 2.3. [4, 12] Let \oplus be an XOR operation and \odot be an XNOR operation. Then the following relation hold:

- (i) $x \odot x = 0$, $x \odot y = y \odot x = -(x \oplus y) = -(y \oplus x)$,
- (ii) $(\lambda x) \oplus (\lambda y) = |\lambda|(x \oplus y)$,
- (iii) $x \odot 0 \leq 0$, if $x \propto 0$,
- (iv) $0 \leq x \oplus y$, if $x \propto y$,
- (v) if $x \propto y$, then $x \oplus y = 0$ if and only if $x = y$,
- (vi) $(x + y) \odot (u + v) \geq (x \odot u) + (y \odot v)$,
- (vii) $(x + y) \odot (u + v) \geq (x \odot v) + (y \odot u)$,
- (viii) $\alpha x \oplus \beta x = |\alpha - \beta|x = (\alpha \oplus \beta)x$, if $x \propto 0$, $\forall x, y, u, v \in X$ and $\alpha, \beta, \lambda \in \mathbb{R}$.

PROPOSITION 2.4. [4] Let C be a normal cone in X with normal constant λ_{C_N} , then for each $x, y \in X$, the following relations hold:

- (i) $\|0 \oplus 0\| = \|0\| = 0$,
- (ii) $\|x \vee y\| \leq \|x\| \vee \|y\| \leq \|x\| + \|y\|$,
- (iii) $\|x \oplus y\| \leq \|x - y\| \leq \lambda_{C_N} \|x \oplus y\|$,
- (iv) if $x \propto y$, then $\|x \oplus y\| = \|x - y\|$.

DEFINITION 2.6. [12] Let $A : X \rightarrow X$ be a single-valued mapping.

- (i) A is said to be comparison mapping, if for each $x, y \in X$, $x \propto y$ then $A(x) \propto A(y)$, $x \propto A(x)$ and $y \propto A(y)$.
- (ii) A is said to be strongly comparison mapping, if A is a comparison mapping and $A(x) \propto A(y)$ if and only if $x \propto y$, for all $x, y \in X$.

DEFINITION 2.7. A mapping $A : X \rightarrow X$ is said to be β -ordered compression mapping, if A is a comparison mapping and

$$A(x) \oplus A(y) \leq \beta(x \oplus y), \text{ for } 0 < \beta < 1.$$

DEFINITION 2.8. [11, 14] Let $R : X \rightarrow X$ be a strong comparison and β -ordered compression mapping and $M : X \rightarrow CB(X)$ be a set-valued mapping.

Then

- (i) M is said to be a comparison mapping, if for any $v_x \in M(x)$, $x \propto v_x$, and if $x \propto y$, then for any $v_x \in M(x)$ and any $v_y \in M(y)$, $v_x \propto v_y$, for all $x, y \in X$;
- (ii) a comparison mapping M is said to be ordered rectangular, if for each $x, y \in X$, $v_x \in M(x)$ and $v_y \in M(y)$ such that

$$\langle v_x \odot v_y, -(x \oplus y) \rangle = 0;$$

- (iii) a comparison mapping M is said to be γ_R -ordered rectangular with respect to R , if there exists a constant $\gamma_R > 0$, for any $x, y \in X$, there exist $v_x \in M(R(x))$ and $v_y \in M(R(y))$ such that

$$\langle v_x \odot v_y, -(R(x) \oplus R(y)) \rangle \geq \gamma_R \|R(x) \oplus R(y)\|^2,$$

holds, where v_x and v_y are said to be γ_R -elements, respectively;

- (iv) M is said to be a weak comparison mapping with respect to R , if for any $x, y \in X$, $x \propto y$, then there exist $v_x \in M(R(x))$ and $v_y \in M(R(y))$ such that $x \propto v_x$, $y \propto v_y$ and $v_x \propto v_y$, where v_x and v_y are said to be weak comparison elements, respectively.
- (v) M is said to be a λ -weak ordered different comparison mapping with respect to R , if there exists a constant $\lambda > 0$ such that for any $x, y \in X$, there exist $v_x \in M(R(x))$, $v_y \in M(R(y))$, $\lambda(v_x - v_y) \propto (x - y)$ holds, where v_x and v_y are said to be λ -elements, respectively;
- (vi) a weak comparison mapping M is said to be a (γ_R, λ) -weak-RRD mapping with respect to R , if M is a γ_R -ordered rectangular and λ -weak ordered different comparison mapping with respect to R and $(R + \lambda M)(X) = X$, for $\lambda > 0$ and there exist $v_x \in M(R(x))$ and $v_y \in M(R(y))$ such that v_x and v_y are (γ_R, λ) -elements, respectively.

REMARK 2.1. Let X be a real ordered Hilbert space. Let $R : X \rightarrow X$ be a single-valued mapping and $M : X \rightarrow CB(X)$ be a set-valued

mapping, then the following relation hold, details of which can be found in [14].

- (i) Every λ -ordered monotone mapping is a λ -weak ordered different comparison mapping.
- (ii) If $R = I$ (identity mapping), then a γ_I -ordered rectangular mapping is an ordered rectangular mapping.
- (iii) An ordered RME mapping is λ -weak-RRD mapping.

DEFINITION 2.9. A set-valued mapping $A : X \rightarrow CB(X)$ is said to be D -Lipschitz continuous, if for each $x, y \in X$, $x \propto y$, there exists a constant δ_A such that

$$D(A(x), A(y)) \leq \delta_A \|x \oplus y\|, \forall x, y \in X.$$

DEFINITION 2.10. Let $M : X \rightarrow CB(X)$ be a set-valued mapping, $R : X \rightarrow X$ be single-valued mapping and $I : X \rightarrow X$ be an identity mapping. Then a weak comparison mapping M is said to be a (γ', λ) -weak-RRD mapping with respect to $(I - R)$, if M is a γ' -ordered rectangular and λ -weak ordered different comparison mapping with respect to $(I - R)$ and $[(I - R) + \lambda M](X) = X$, for $\lambda > 0$ and there exist $v_x \in M((I - R)(x))$ and $v_y \in M((I - R)(y))$ such that v_x and v_y are (γ', λ) -elements, respectively.

EXAMPLE 2.1. Let $X = \mathbb{R}$ with usual inner product. Let $R : X \rightarrow X$ be a mapping defined by

$$R(x) = \frac{x}{2}, \forall x \in X.$$

and the set-valued mapping $M : X \rightarrow CB(X)$ is defined by

$$M(x) = \begin{cases} \{\frac{x}{3}\}, & \text{if } x \neq 0, \\ \{1\}, & \text{if } x = 0. \end{cases}$$

Then, it is easy to check that R is 1-ordered compression and M is $(\frac{1}{7}, 1)$ -weak-RRD mapping with respect to R .

3. Formulation of the problem and some basic properties

Let X be an ordered Hilbert space and $A, B, C : X \rightarrow CB(X)$ be the set-valued mappings and $f, p : X \rightarrow X$ are the single-valued mappings. Suppose that $M : X \rightarrow CB(X)$ be a set-valued mapping. We consider the following problem:

For some $\rho \in X$ and any $\tau > 0$, find $u \in X, w \in A(u), v \in B(u), z \in C(u)$ such that

$$\rho \in f(w) - p(v) + \tau M(z). \quad (3.1)$$

This problem is called an ordered variational inclusion problem involving weak-RRD set-valued mapping.

Below are some special cases of problem (3.1).

- (i) If $\rho = 0$ and $\tau = 1$, then problem (3.1) reduces to the problem of finding $u \in X, w \in A(u), y \in B(u)$ and $z \in C(u)$ such that

$$0 \in f(w) - p(y) + M(z). \quad (3.2)$$

Problem (3.2) was introduced and studied by [9].

- (ii) If $\rho = 0, \tau = 1, A = B = I$ (identity mapping) and C is a single-valued mapping, then problem (3.1) reduces to the problem of finding $u \in X$ such that

$$0 \in f(u) - p(u) + M(C(u)). \quad (3.3)$$

Problem (3.3) was introduced and studied by [1].

- (iii) If $f = p = 0, A = B = 0$ and $C = I$, then problem (3.1) reduces to the problem of finding $u \in X$ such that

$$\rho \in \tau M(u). \quad (3.4)$$

Problem (3.4) was introduced and studied by [11].

- (iv) If $f = p = B = 0$ and $A = C = I$, then problem (3.1) reduces to the problem of finding $u \in X$ such that

$$\rho \in f(u) + \tau M(u). \quad (3.5)$$

Problem (3.5) was introduced and studied by [15].

DEFINITION 3.1. Let C be a normal cone with normal constant λ_{C_N} and $M : X \rightarrow CB(X)$ be weak-RRD set-valued mapping. Let $I : X \rightarrow X$ be the identity mapping and $R : X \rightarrow X$ be a single-valued mapping. The relaxed resolvent operator $J_{\lambda, M}^{I-R} : X \rightarrow X$ associated with I, R and M is defined by

$$J_{\lambda, M}^{I-R}(x) = [(I - R) + \lambda M]^{-1}(x), \text{ for all } x \in X \text{ and } \lambda > 0. \quad (3.6)$$

Now, we show that the relaxed resolvent operator defined by (3.6) is single-valued, a comparison mapping as well as Lipschitz continuous.

PROPOSITION 3.1. [2] Let $R : X \rightarrow X$ be a β -ordered compression mapping and $M : X \rightarrow CB(X)$ be the set-valued ordered rectangular mapping. Then the operator $J_{\lambda, M}^{I-R} : X \rightarrow X$ is a single-valued, for all $\lambda > 0$.

PROPOSITION 3.2. Let $M : X \rightarrow CB(X)$ be a (γ_R, λ) -weak-RRD set-valued mapping with respect to $J_{\lambda, M}^{(I-R)}$. Let $R : X \rightarrow X$ be a strongly comparison mapping with respect to $J_{\lambda, M}^{(I-R)}$ and $I : X \rightarrow X$ be the identity mapping. Then the resolvent operator $J_{\lambda, M}^{(I-R)} : X \rightarrow X$ is a comparison mapping.

Proof. Let M be a (γ_R, λ) -weak-RRD set-valued mapping with respect to $J_{\lambda, M}^{(I-R)}$. That is, M is γ_R -ordered rectangular and λ -weak-ordered different comparison mapping with respect to $J_{\lambda, M}^{(I-R)}$ so that $x \propto J_{\lambda, M}^{(I-R)}(x)$. For any $x, y \in X$, let $x \propto y$, and let

$$v_x = \frac{1}{\lambda}(x - (I - R)(J_{\lambda, M}^{(I-R)}(x))) \in M(J_{\lambda, M}^{(I-R)}(x)) \quad (3.7)$$

and

$$v_y = \frac{1}{\lambda}(y - (I - R)(J_{\lambda, M}^{(I-R)}(y))) \in M(J_{\lambda, M}^{(I-R)}(y)). \quad (3.8)$$

Using (3.7) and (3.8), we have

$$\begin{aligned} v_x - v_y &= \left(\frac{1}{\lambda}(x - (I - R)(J_{\lambda, M}^{(I-R)}(x))) \right) \\ &\quad - \left(\frac{1}{\lambda}(y - (I - R)(J_{\lambda, M}^{(I-R)}(y))) \right) \\ &= \frac{1}{\lambda} \left(x - y + (I - R)(J_{\lambda, M}^{(I-R)}(y)) - (I - R)(J_{\lambda, M}^{(I-R)}(x)) \right). \end{aligned}$$

Since M is λ -weak-ordered different comparison mapping with respect to $J_{\lambda, M}^{(I-R)}$, we have

$$\begin{aligned} \lambda(v_x - v_y) - (x - y) &= (x - y) + (I - R)(J_{\lambda, M}^{(I-R)}(y)) \\ &\quad - (I - R)(J_{\lambda, M}^{(I-R)}(x)) - (x - y) \\ &= (I - R)(J_{\lambda, M}^{(I-R)}(y)) - (I - R)(J_{\lambda, M}^{(I-R)}(x)). \end{aligned}$$

Since R is strongly comparison mapping with respect to $J_{\lambda, M}^{(I-R)}$, $(I - R)$ is also strongly comparison mapping with respect to $J_{\lambda, M}^{(I-R)}$. Therefore, $J_{\lambda, M}^{(I-R)}(x) \propto J_{\lambda, M}^{(I-R)}(y)$. The proof is completed. \square

PROPOSITION 3.3. *Let $M : X \rightarrow CB(X)$ be a (γ_R, λ) -weak-RRD set-valued mapping with respect to $J_{\lambda, M}^{(I-R)}$. Let $R : X \rightarrow X$ be a comparison and β -ordered compression mapping with respect to $J_{\lambda, M}^{(I-R)}$ with condition $\lambda\gamma_R > \beta + 1$. Then the following condition holds:*

$$\left\| J_{\lambda, M}^{(I-R)}(x) \oplus J_{\lambda, M}^{(I-R)}(y) \right\| \leq \frac{1}{\lambda\gamma_R - \beta - 1} \|x \oplus y\|.$$

Proof. Let M be a (γ_R, λ) -weak-RRD set-valued mapping with respect to $J_{\lambda, M}^{(I-R)}$. That is, M is γ_R -ordered rectangular and λ -weak-ordered different comparison mapping with respect to $J_{\lambda, M}^{(I-R)}$. Then for any $x, y \in X$, set $u_x = J_{\lambda, M}^{(I-R)}(x)$, $u_y = J_{\lambda, M}^{(I-R)}(y)$, and let

$$\begin{aligned} v_x &= \frac{1}{\lambda} (x - (I - R)(J_{\lambda, M}^{(I-R)}(x))) \in M(J_{\lambda, M}^{(I-R)}(x)) \\ \text{and} \\ v_y &= \frac{1}{\lambda} (y - (I - R)(J_{\lambda, M}^{(I-R)}(y))) \in M(J_{\lambda, M}^{(I-R)}(y)). \end{aligned}$$

Since R is β -ordered compression mapping and using Proposition 2.3, we have

$$\begin{aligned} v_x \oplus v_y &= \frac{1}{\lambda} [(x - (I - R)(u_x)) \oplus (y - (I - R)(u_y))] \\ &\leq \frac{1}{\lambda} [x \oplus y + (I - R)(u_x) \oplus (I - R)(u_y)] \\ &\leq \frac{1}{\lambda} [x \oplus y + u_x \oplus u_y + R(u_x) \oplus R(u_y)] \\ &\leq \frac{1}{\lambda} [x \oplus y + u_x \oplus u_y + \beta(u_x \oplus u_y)] \\ &= \frac{1}{\lambda} [x \oplus y + (1 + \beta)(u_x \oplus u_y)]. \end{aligned}$$

Since M is γ_R -ordered rectangular mapping with respect to $J_{\lambda, M}^{(I-R)}$, we have

$$\begin{aligned} \gamma_R \|u_x \oplus u_y\|^2 &\leq \langle v_x \odot v_y, -(u_x \oplus u_y) \rangle \\ &= \langle v_x \oplus v_y, u_x \oplus u_y \rangle \\ &\leq \langle \frac{1}{\lambda} [x \oplus y + (1 + \beta)(u_x \oplus u_y)], (u_x \oplus u_y) \rangle \\ &= \frac{1}{\lambda} \langle x \oplus y, (u_x \oplus u_y) \rangle + \frac{(1 + \beta)}{\lambda} \langle (u_x \oplus u_y), (u_x \oplus u_y) \rangle \\ &\leq \frac{1}{\lambda} \|x \oplus y\| \|u_x \oplus u_y\| + \frac{(1 + \beta)}{\lambda} \|u_x \oplus u_y\|^2. \end{aligned}$$

It follows that $(\gamma_R - \frac{(1+\beta)}{\lambda}) \|u_x \oplus u_y\| \leq \frac{1}{\lambda} \|x \oplus y\|$ and consequently, we have

$$\|J_{\lambda, M}^{(I-R)}(x) \oplus J_{\lambda, M}^{(I-R)}(y)\| \leq \frac{1}{\lambda \gamma_R - \beta - 1} \|x \oplus y\|.$$

The proof is completed. \square

4. An iterative algorithm and existence result

In this section, we define an iterative algorithm to obtain the solution of ordered variational inclusion problem involving weak-RRD set-valued mapping (3.1).

ITERATIVE ALGORITHM 4.1. Let $A, B, C : X \rightarrow CB(X)$ be the set-valued mappings, $R, f, p : X \rightarrow X$ be the single-valued mappings and $I : X \rightarrow X$ be the identity mapping. Suppose that $M : X \rightarrow CB(X)$ is a set-valued mapping.

For any given initial $u_0 \in X$, $w_0 \in A(u_0)$, $v_0 \in B(u_0)$, $z_0 \in C(u_0)$, let

$$u_1 = u_0 - z_0 + J_{\lambda, M}^{(I-R)}[(I - R)(z_0) + \frac{\lambda}{\tau}(\rho - (f(w_0) - p(v_0)))].$$

Since $w_0 \in A(u_0) \in CB(X)$, $v_0 \in B(u_0) \in CB(X)$ and $z_0 \in C(u_0) \in CB(X)$, by Nadler's theorem [18], there exist $w_1 \in A(u_1)$, $v_1 \in B(u_1)$, $z_1 \in C(u_1)$ and suppose that $u_0 \times u_1$, $w_0 \times w_1$, $v_0 \times v_1$ and $z_0 \times z_1$

such that

$$\begin{aligned}\|w_1 \oplus w_0\| &= \|w_1 - w_0\| \leq D(A(u_1), A(u_0)), \\ \|v_1 \oplus v_0\| &= \|v_1 - v_0\| \leq D(B(u_1), B(u_0)), \\ \|z_1 \oplus z_0\| &= \|z_1 - z_0\| \leq D(C(u_1), C(u_0)).\end{aligned}$$

Continuing the above process inductively, we can define the iterative sequences $\{u_n\}$, $\{w_n\}$, $\{v_n\}$ and $\{z_n\}$ with the supposition that $u_n \propto u_{n+1}$, $w_n \propto w_{n+1}$, $v_n \propto v_{n+1}$ and $z_n \propto z_{n+1}$, for all $n \in \mathbb{N}$. We define the following iterative algorithm schemes:

$$u_{n+1} = u_n - z_n + J_{\lambda, M}^{(I-R)} \left[(I - R)(z_n) + \frac{\lambda}{\tau} (\rho - (f(w_n) - p(v_n))) \right], \quad (4.1)$$

$$u_{n+1} \in A(u_{n+1}), \|w_{n+1} \oplus w_n\| = \|w_{n+1} - w_n\| \leq D(A(u_{n+1}), A(u_n)), \quad (4.2)$$

$$v_{n+1} \in B(u_{n+1}), \|v_{n+1} \oplus v_n\| = \|v_{n+1} - v_n\| \leq D(B(u_{n+1}), B(u_n)), \quad (4.3)$$

$$z_{n+1} \in C(u_{n+1}), \|z_{n+1} \oplus z_n\| = \|z_{n+1} - z_n\| \leq D(C(u_{n+1}), C(u_n)), \quad (4.4)$$

where $\lambda, \rho, \tau > 0$ are constants.

The fixed point formulation of problem (3.1) is as follows.

LEMMA 4.1. *Let $u \in X$, $w \in A(u)$, $v \in B(u)$ and $z \in C(u)$ be a solution of ordered variational inclusion problem involving weak-RRD set-valued mapping (3.1) if and only if (u, w, v, z) satisfies the following relation:*

$$u = u - z + J_{\lambda, M}^{(I-R)} \left[(I - R)(z) + \frac{\lambda}{\tau} (\rho - (f(w) - p(v))) \right].$$

Where

$$J_{\lambda, M}^{(I-R)} = [(I - R) + \lambda M]^{-1},$$

and $\lambda, \tau, \rho > 0$ are constants.

Proof. The proof directly follows from the definition of the relaxed resolvent operator $J_{\lambda, M}^{(I-R)}$. \square

THEOREM 4.1. *Let X be a real ordered Hilbert space and C be a normal cone with normal constant λ_{C_N} . Let $R, f, P : X \rightarrow X$ be the*

single-valued mappings such that R is comparison, β -ordered compression, f is comparison, λ_f -ordered compression and P is comparison, λ_p -ordered compression mappings and $I : X \rightarrow X$ is an identity mapping. Let $A, B, C : X \rightarrow CB(X)$ be the set-valued mappings such that A, B and C are D -Lipschitz continuous mappings with constants δ_A, δ_B and δ_C , respectively. Suppose that $M : X \rightarrow CB(X)$ is a (γ_R, λ) -weak-RRD set-valued mapping such that the following condition is satisfied

$$\lambda_{C_N}[\tau\lambda\gamma_R(\delta_C + 1) + \lambda\lambda_f\delta_A + \lambda\lambda_p\delta_B] < \tau\lambda\gamma_R + (\lambda_{C_N} - 1)\tau(\beta + 1). \quad (4.5)$$

Then, the iterative sequences $\{u_n\}$, $\{w_n\}$, $\{v_n\}$ and $\{z_n\}$ generated by Algorithm 4.1 converge strongly to u, w, v and z , respectively and (u, w, v, z) is a solution of ordered variational inclusion problem involving weak-RRD set-valued mapping (3.1), where $u \in X$, $w \in A(u)$, $v \in B(u)$ and $z \in C(u)$.

Proof. Let us introduce the term $h(u_n) = [(I - R)(z_n) + \frac{\lambda}{\tau}(\rho - (f(w_n) - p(v_n)))]$. Using Algorithm 4.1 and Proposition 2.3, we obtain

$$\begin{aligned} 0 &\leq u_{n+1} \oplus u_n \\ &= \left(u_n - z_n + J_{\lambda, M}^{(I-R)}(h(u_n)) \right) \oplus \left(u_{n-1} - z_{n-1} + J_{\lambda, M}^{(I-R)}(h(u_{n-1})) \right) \\ &\leq u_n \oplus u_{n-1} + z_n \oplus z_{n-1} + \left(J_{\lambda, M}^{(I-R)}(h(u_n)) \oplus J_{\lambda, M}^{(I-R)}(h(u_{n-1})) \right). \end{aligned} \quad (4.6)$$

Using Definition 2.2, Proposition 3.3 and from (4.6), we have

$$\begin{aligned} \|u_{n+1} \oplus u_n\| &\leq \lambda_{C_N} \left\| u_n \oplus u_{n-1} + z_n \oplus z_{n-1} \right. \\ &\quad \left. + \left(J_{\lambda, M}^{(I-R)}(h(u_n)) \oplus J_{\lambda, M}^{(I-R)}(h(u_{n-1})) \right) \right\| \\ &\leq \lambda_{C_N} \left[\|u_n \oplus u_{n-1}\| + \|z_n \oplus z_{n-1}\| \right. \\ &\quad \left. + \left\| J_{\lambda, M}^{(I-R)}(h(u_n)) \oplus J_{\lambda, M}^{(I-R)}(h(u_{n-1})) \right\| \right] \\ &\leq \lambda_{C_N} \left[\|u_n \oplus u_{n-1}\| + D(C(u_n), C(u_{n-1})) \right] \end{aligned}$$

$$\begin{aligned}
& + \left\| J_{\lambda, M}^{(I-R)}(h(u_n)) \oplus J_{\lambda, M}^{(I-R)}(h(u_{n-1})) \right\| \\
\leq & \lambda_{C_N} \left[\|u_n \oplus u_{n-1}\| + \delta_C \|u_n \oplus u_{n-1}\| \right. \\
& \left. + \frac{1}{\lambda\gamma_R - \beta - 1} \|h(u_n) \oplus h(u_{n-1})\| \right] \quad (4.7)
\end{aligned}$$

Since R is β -ordered compression mapping, f is λ_f -ordered compression mapping, p is λ_p -ordered compression mapping and D -Lipschitz continuity of A, B and C with constants δ_A, δ_B and δ_C , respectively, we have

$$\begin{aligned}
& \|h(u_n) \oplus h(u_{n-1})\| \\
= & \left\| [(I - R)(z_n) + \frac{\lambda}{\tau}(\rho - (f(w_n) - p(v_n)))] \right. \\
& \left. \oplus [(I - R)(z_{n-1}) + \frac{\lambda}{\tau}(\rho - (f(w_{n-1}) - p(v_{n-1})))] \right\| \\
\leq & \|(I - R)(z_n) \oplus (I - R)(z_{n-1})\| + \frac{\lambda}{\tau} \|(\rho - (f(w_n) - p(v_n))) \\
& \oplus (\rho - (f(w_{n-1}) - p(v_{n-1})))\| \\
\leq & \|z_n \oplus z_{n-1}\| + \|R(z_n) \oplus R(z_{n-1})\| + \frac{\lambda}{\tau} [\|f(w_n) \oplus f(w_{n-1})\| \\
& + \|p(v_n) \oplus p(v_{n-1})\|] \\
\leq & D(C(u_n), C(u_{n-1})) + \|R(z_n) \oplus R(z_{n-1})\| + \frac{\lambda}{\tau} [\|f(w_n) \\
& \oplus f(w_{n-1})\| + \|p(v_n) \oplus p(v_{n-1})\|] \\
\leq & \delta_C \|u_n \oplus u_{n-1}\| + \beta D(C(u_n), C(u_{n-1})) \\
& + \frac{\lambda}{\tau} [\lambda_f D(A(u_n), A(u_{n-1})) + \lambda_p D(B(u_n), B(u_{n-1}))] \\
\leq & \delta_C \|u_n \oplus u_{n-1}\| + \beta \delta_C \|u_n \oplus u_{n-1}\| + \frac{\lambda}{\tau} [\lambda_f \delta_A \|u_n \oplus u_{n-1}\| \\
& + \lambda_p \delta_B \|u_n \oplus u_{n-1}\|] \\
\leq & \left[\delta_C + \beta \delta_C + \frac{\lambda}{\tau} (\lambda_f \delta_A + \lambda_p \delta_B) \right] \|u_n \oplus u_{n-1}\|,
\end{aligned}$$

which implies that

$$\|h(u_n) \oplus h(u_{n-1})\| \leq \left[\delta_C + \beta \delta_C + \frac{\lambda}{\tau} (\lambda_f \delta_A + \lambda_p \delta_B) \right] \|u_n \oplus u_{n-1}\|. \quad (4.8)$$

Using (4.8), (4.7) becomes

$$\begin{aligned} \|u_{n+1} \oplus u_n\| &\leq \lambda_{C_N} \left[1 + \delta_C + \frac{1}{\lambda\gamma_R - \beta - 1} (\delta_C + \beta\delta_C \right. \\ &\quad \left. + \frac{\lambda}{\tau} (\lambda_f\delta_A + \lambda_p\delta_B)) \right] \|u_n \oplus u_{n-1}\| \end{aligned} \quad (4.9)$$

By Proposition 2.4, we have

$$\begin{aligned} \|u_{n+1} - u_n\| = \|u_{n+1} \oplus u_n\| &\leq \lambda_{C_N} \left[1 + \delta_C + \frac{1}{\lambda\gamma_R - \beta - 1} (\delta_C + \beta\delta_C \right. \\ &\quad \left. + \frac{\lambda}{\tau} (\lambda_f\delta_A + \lambda_p\delta_B)) \right] \|u_n - u_{n-1}\|. \end{aligned}$$

i.e.,

$$\|u_{n+1} - u_n\| \leq \Theta \|u_n - u_{n-1}\|,$$

where

$$\Theta = \lambda_{C_N} \left[1 + \delta_C + \frac{1}{\lambda\gamma_R - \beta - 1} (\delta_C + \beta\delta_C + \frac{\lambda}{\tau} (\lambda_f\delta_A + \lambda_p\delta_B)) \right].$$

By condition (4.5), we have $0 < \Theta < 1$, thus $\{u_n\}$ is a Cauchy sequence in X and since X is a complete space, there exists $u \in X$ such that $u_n \rightarrow u$ as $n \rightarrow \infty$. From (4.2), (4.3) and (4.4) of Algorithm 4.1 and D -Lipschitz continuity of A, B and C , we have

$$\|w_{n+1} - w_n\| \leq D(A(u_{n+1}), A(u_n)) \leq \delta_A \|u_{n+1} - u_n\|, \quad (4.10)$$

$$\|v_{n+1} - v_n\| \leq D(B(u_{n+1}), B(u_n)) \leq \delta_B \|u_{n+1} - u_n\|, \quad (4.11)$$

$$\|z_{n+1} - z_n\| \leq D(C(u_{n+1}), C(u_n)) \leq \delta_C \|u_{n+1} - u_n\|. \quad (4.12)$$

It is clear from (4.10), (4.11) and (4.12) that $\{w_n\}$, $\{v_n\}$ and $\{z_n\}$ are also Cauchy sequences in X and so there exist w, v and z in X such that $w_n \rightarrow w, v_n \rightarrow v$ and $z_n \rightarrow z$ as $n \rightarrow \infty$. By using the continuity of the operators $A, B, C, J_{\lambda, M}^{I-R}$ and iterative Algorithm 4.1, we have

$$u = u - z + J_{\lambda, M}^{I-R} \left[(I - R)z + \frac{\lambda}{\tau} (\rho - (f(w) - p(v))) \right].$$

By Lemma 4.1, we conclude that (u, w, v, z) is a solution of problem (3.1). It remains to show that $w \in A(u), v \in B(u)$ and $z \in C(u)$. In fact

$$\begin{aligned} d(w, A(u)) &\leq \|w - w_n\| + d(w_n, A(u)) \\ &\leq \|w - w_n\| + D(A(u_n), A(u)) \\ &\leq \|w - w_n\| + \delta_A \|u_n - u\| \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence $w \in A(u)$. Similarly, we can show that $v \in B(u)$ and $z \in C(u)$. This completes the proof. \square

References

- [1] S. Adly, *Perturbed algorithm and sensitivity analysis for a general class of variational inclusions*, J. Math. Anal. Appl. **201** (1996), 609–630.
- [2] I. Ahmad and R. Ahmad, *The relaxed resolvent operator for solving fuzzy variational inclusion problem involving ordered RME set-valued mapping*, Caspian J. Math. Sci., (To appear).
- [3] S. Chang and N. Huang, *Generalized strongly nonlinear quasi-complementarity problems in Hilbert spaces*, J. Math. Anal. Appl. **158** (1991), 194–202.
- [4] Y.H. Du, *Fixed points of increasing operators in ordered Banach spaces and applications*, Appl. Anal. **38** (1990), 1–20.
- [5] Y.P. Fang and N.-J. Huang, *H-accretive operator and resolvent operator technique for variational inclusions in Banach spaces*, Appl. Math. Lett. **17** (6) (2004), 647–653.
- [6] Y.P. Fang and N.-J. Huang, *H-monotone operator and resolvent operator technique for variational inclusions*, Appl. Math. Comput. **145** (2-3) (2003), 795–803.
- [7] Y.P. Fang and N.-J. Huang, *Approximate solutions for non-linear variational inclusions with (H, η) -monotone operator*, Research Report, Sichuan University (2003).
- [8] Y.P. Fang, Y.-J. Chu and J.K. Kim, *(H, η) -accretive operator and approximating solutions for systems of variational inclusions in Banach spaces*, to appear in Appl. Math. Lett.
- [9] N.J. Huang, *A new completely general class of variational inclusions with non-compact valued mappings*, Comput. Math. Appl. **35** (10) (1998), 9–14.
- [10] H.G. Li, L.P. Li, J.M. Zheng and M.M. Jin, *Sensitivity analysis for generalized set-valued parametric ordered variational inclusion with (α, λ) -NODSM mappings in ordered Banach spaces*, Fixed Point Theory Appl. **2014** (2014): 122.
- [11] H.G. Li, X.B. Pan, Z.Y. Deng and C.Y. Wang, *Solving GNOVI frameworks involving (γ_G, λ) -weak-GRD set-valued mappings in positive Hilbert spaces*, Fixed Point Theory Appl. **2014** (2014): 146.
- [12] H.G. Li, *A nonlinear inclusion problem involving (α, λ) -NODM set-valued mappings in ordered Hilbert space*, Appl. Math. Lett. **25** (2012) 1384–1388.
- [13] H.G. Li, D. Qiu and Y. Zou, *Characterizations of weak-ANODD set-valued mappings with applications to approximate solution of GNMOQV inclusions involving \oplus operator in ordered Banach spaces*, Fixed Point Theory Appl. **2013** (2013), doi: 10.1186/1687-1812-2013-241.
- [14] H.G. Li, *Nonlinear inclusion problems for ordered RME set-valued mappings in ordered Hilbert spaces*, Nonlinear Funct. Anal. Appl. **16** (1) (2001), 1–8.

- [15] H.G. Li, L.P. Li, M.M. Jin, *A class of nonlinear mixed ordered inclusion problems for ordered (α_A, λ) -ANODM set-valued mappings with strong comparison mapping A*, Fixed Point Theory Appl. **2014** (2014): 79.
- [16] H.G. Li, *Approximation solution for general nonlinear ordered variational inequalities and ordered equations in ordered Banach space*, Nonlinear Anal. Forum **13** (2) (2008), 205–214.
- [17] H.G. Li, *Approximation solution for a new class general nonlinear ordered variational inequalities and ordered equations in ordered Banach space*, Nonlinear Anal. Forum **14** (2009), 89–97.
- [18] S.B. Nadler, *Multi-valued contraction mappings*, Pacific J. Math. **30** (1969), 475–488.
- [19] H.H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, Berlin Heidelberg (1974), doi:10.1007/978-3-642-65970-6.

Iqbal Ahmad
Department of Mathematics
Aligarh Muslim University
Aligarh 202002, India
E-mail: iqbalahmad.120@gmail.com

Rais Ahmad
Department of Mathematics
Aligarh Muslim University
Aligarh 202002, India
E-mail: raisain_123@rediffmail.com

Javid Iqbal
Department of Mathematical Sciences
Baba Ghulam Shah Badshah University
Rajouri, Jammu and Kashmir, India
E-mail: javid2iqbal@Yahoo.co.in