

GENERALIZED WAVELETS AND THE GENERALIZED WAVELET TRANSFORM ON \mathbb{R}^d FOR THE HECKMAN-OPDAM THEORY

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ABSTRACT. By using the Heckman-Opdam theory on \mathbb{R}^d given in [20], we define and study in this paper, the generalized wavelets on \mathbb{R}^d and the generalized wavelet transform on \mathbb{R}^d , and we establish their properties. Next, we prove for the generalized wavelet transform Plancherel and inversion formulas.

1. Introduction

Fourier analysis is one of the most important tools used by mathematicians and physicists. Besides, in the nineteenth century, Fourier analysis was the only technique for the decomposition of a signal and its reconstruction without loss of information. Unfortunately, it provides a frequency analysis but does not allow the temporal localization of abrupt changes.

A procedure for analyzing a frequency that depends on the time, called continuous wavelet transform, was discovered by the Physicist George Zweig in 1975 while studying the reaction of the ear to sound. Notable contribution to the continuous wavelet transform studies can be attributed to Pierre Goupillaud, Grossmann and Morlet's formulation of

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what this transform is now known (see [5][6]). The basic idea is to replace in usual Fourier transform, the function analyzed by the product of this function by a regular function, called a wavelet (see [11]). If we denote by g this wavelet on \mathbb{R}^d of L^2 -norm with respect to the Lebesgue measure, equal to 1, for a scale $a > 0$ and position $b \in \mathbb{R}^d$, the continuous wavelet transform for a function f , is expressed by the following integral (see [11]):

$$\Phi_g(f)(a, b) = \int_{\mathbb{R}^d} f(x) \overline{g_{a,b}(x)} dx, \quad (a, b) \in]0, +\infty[\times \mathbb{R}^d,$$

where $g_{a,b}$ is the wavelet defined by

$$g_{a,b}(x) = \mathcal{T}_b g_a(x), \quad x \in \mathbb{R}^d,$$

with g_a the function given by

$$g_a(x) = \frac{1}{a^d} g\left(\frac{x}{a}\right).$$

It satisfies

$$\mathcal{F}(g_a)(\lambda) = \mathcal{F}(g)(a\lambda), \quad \lambda \in \mathbb{R}^d,$$

where \mathcal{F} is the classical Fourier transform on \mathbb{R}^d and \mathcal{T}_b , $b \in \mathbb{R}^d$, the classical translation operator defined by

$$\mathcal{T}_b g(x) = g(b - x), \quad x \in \mathbb{R}^d.$$

To recover the original signal $f(x)$, the inverse of the continuous wavelet transform Φ_g can be exploited:

$$f(x) = \frac{1}{C_g} \int_0^{+\infty} \left(\int_{\mathbb{R}^d} \Phi_g(f)(a, b) g_{a,b}(x) dx \right) \frac{da}{a}, \quad x \in \mathbb{R}^d,$$

where C_g is a constant given for almost all $\lambda \in \mathbb{R}^d$, by

$$C_g = \int_0^{+\infty} |\mathcal{F}(g)(a\lambda)|^2 \frac{da}{a},$$

and satisfies

$$0 < C_g < +\infty.$$

One of the aims of the continuous wavelet transform, is to provide an easily interpretable visual representation of signal. Moreover, this transform can be applied to wide scientific research areas ranging from signal analysis in geophysics and acoustics, to quantum theory and pure Mathematics (see [1][11]).

Next, the theory of wavelets and continuous wavelet transform has

been extended to the harmonic analysis associated with the Dunkl operators on \mathbb{R}^d (see [7][9][15]) and on hypergroups, in particular to the Chébli-Trimèche hypergroups (see [2][14]).

As nowadays the harmonic analysis associated to the Cherednik operators and the Heckman-Opdam theory has known remarkable development, it is naturel question to ask whether there exists the equivalent of the theory of wavelets and continuous wavelet transform relating to this harmonic analysis.

In this paper, we study generalized wavelets and generalized continuous wavelet transform associated to the Heckman-Opdam theory on W -invariant functions on \mathbb{R}^d . To achieve this, we consider the Cherednik operators $T_j, j = 1, 2, \dots, d$, on \mathbb{R}^d associated to a root system \mathcal{R} , a reflection group W and a non negative multiplicity function k . Thanks to these operators, Heckman and Opdam have developed a theory generalizing the harmonic analysis on symmetric spaces (see [8][12]). Next, we introduce the Heckman-Opdam hypergeometric function $F_\lambda, \lambda \in \mathbb{C}^d$, given by

$$\forall x \in \mathbb{R}^d, \quad F_\lambda(x) = \frac{1}{|W|} \sum_{w \in W} G_\lambda(wx),$$

where $G_\lambda, \lambda \in \mathbb{C}^d$, is the unique solution of the differential-difference system

$$\begin{cases} T_j G_\lambda(x) = i\lambda_j G_\lambda(x), & j = 1, 2, \dots, d, x \in \mathbb{R}^d, \\ G_\lambda(0) = 1. \end{cases}$$

By using the function F_λ , we define the hypergeometric Fourier transform \mathcal{H}^W for regular W -invariant function f on \mathbb{R}^d by

$$\mathcal{H}^W(f)(\lambda) = \int_{\mathbb{R}^d} f(x) F_{-\lambda}(x) \mathcal{A}_k(x) dx, \quad \lambda \in \mathbb{R}^d,$$

where \mathcal{A}_k is a weight function, and the hypergeometric translation operator $\mathcal{T}_x^W, x \in \mathbb{R}^d$, by

$$\mathcal{H}^W(\mathcal{T}_x^W(f))(\lambda) = F_\lambda(x) \mathcal{H}^W(f)(\lambda), \quad \lambda \in \mathbb{R}^d.$$

We recall the main results of the harmonic analysis associated to the Heckman-Opdam theory on W -invariant functions (see [20]). With the aid of these results, we define and study the generalized wavelet transform $\Phi_g(f)$ given for a regular W -invariant function f on \mathbb{R}^d by

$$\Phi_g(f)(a, b) = \int_{\mathbb{R}^d} f(x) \overline{g_{a,b}(x)} \mathcal{A}_k(x) dx, \quad (a, b) \in]0, +\infty[\times \mathbb{R}^d,$$

where $g_{a,b}$ is the generalized wavelet defined, for $a > 0$ and $b \in \mathbb{R}^d$, by

$$g_{a,b}(x) = \mathcal{T}_b^W g_a(x), \quad x \in \mathbb{R}^d,$$

with g_a the function given by

$$\mathcal{H}^W(g_a)(\lambda) = \mathcal{H}^W(g)(a\lambda), \quad \lambda \in \mathbb{R}^d.$$

Next, we prove for the transform Φ_g Plancherel and inversion formulas.

2. The Cherednik operators and their eigenfunctions (see [12][13])

We consider \mathbb{R}^d with the standard basis $\{e_i, i = 1, 2, \dots, d\}$ and the inner product $\langle \cdot, \cdot \rangle$ for which this basis is orthonormal. We extend this inner product to a complex bilinear form on \mathbb{C}^d .

2.1. The root system, the multiplicity function and the Cherednik operators.

Let $\alpha \in \mathbb{R}^d \setminus \{0\}$ and $\check{\alpha} = \frac{2}{\|\alpha\|^2} \alpha$. We denote by

$$r_\alpha(x) = x - \langle \check{\alpha}, x \rangle \alpha, \quad x \in \mathbb{R}^d,$$

the reflection in the hyperplan $H_\alpha \subset \mathbb{R}^d$ orthogonal to α .

A finite set $\mathcal{R} \subset \mathbb{R}^d \setminus \{0\}$ is called a root system if $r_\alpha \mathcal{R} = \mathcal{R}$, for all $\alpha \in \mathcal{R}$. For a given root system \mathcal{R} the reflections $r_\alpha, \alpha \in \mathcal{R}$, generate a finite group $W \subset O(d)$, called the reflection group associated with \mathcal{R} . For a given $\beta \in \mathbb{R}^d \setminus \cup_{\alpha \in \mathcal{R}} H_\alpha$, we fix the positive subsystem $\mathcal{R}_+ = \{\alpha \in \mathcal{R}, \langle \alpha, \beta \rangle > 0\}$, then for each $\alpha \in \mathcal{R}$ either $\alpha \in \mathcal{R}_+$ or $-\alpha \in \mathcal{R}_+$. We denote by \mathcal{R}_+^0 the set of positive indivisible roots. Let

$$\mathfrak{a}^+ = \{x \in \mathbb{R}^d, \forall \alpha \in \mathcal{R}, \langle \alpha, x \rangle > 0\}$$

be the positive Weyl chamber. We denote by $\overline{\mathfrak{a}^+}$ its closure. Let also $\mathbb{R}_{reg}^d = \mathbb{R}^d \setminus \cup_{\alpha \in \mathcal{R}} H_\alpha$ be the set of regular elements in \mathbb{R}^d .

A function $k : \mathcal{R} \rightarrow [0, +\infty[$ on the root system \mathcal{R} is called a multiplicity function if it is invariant under the action of the reflection group W . We introduce the index

$$\gamma = \gamma(\mathcal{R}) = \sum_{\alpha \in \mathcal{R}_+} k(\alpha). \tag{2.1}$$

Moreover, let \mathcal{A}_k be the weight function

$$\forall x \in \mathbb{R}^d, \mathcal{A}_k(x) = \prod_{\alpha \in \mathcal{R}_+} |2 \sinh \langle \frac{\alpha}{2}, x \rangle|^{2k(\alpha)},$$

which is W -invariant.

The Cherednik operators $T_j, j = 1, 2, \dots, d$, on \mathbb{R}^d associated with the reflection group W and the multiplicity function k , are defined for f of class C^1 on \mathbb{R}^d and $x \in \mathbb{R}_{reg}^d$ by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \mathcal{R}_+} \frac{k(\alpha)\alpha^j}{1 - e^{-\langle \alpha, x \rangle}} \{f(x) - f(r_\alpha x)\} - \rho_j f(x),$$

where

$$\rho_j = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\alpha^j, \text{ and } \alpha^j = \langle \alpha, e_j \rangle.$$

In the case $k(\alpha) = 0$, for all $\alpha \in \mathcal{R}_+$, the operators $T_j, j = 1, 2, \dots, d$, reduce to the corresponding partial derivatives. We suppose in the following that $k \neq 0$.

The Cherednik operators form a commutative system of differential-difference operators.

For f of class C^1 on \mathbb{R}^d with compact support and g of class C^1 on \mathbb{R}^d , we have for $j = 1, 2, \dots, d$:

$$\int_{\mathbb{R}^d} T_j f(x)g(x)\mathcal{A}_k(x)dx = - \int_{\mathbb{R}^d} f(x)(T_j + S_j)g(x)\mathcal{A}_k(x)dx,$$

with

$$\forall x \in \mathbb{R}^d, S_j g(x) = \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\alpha^j g(r_\alpha x).$$

(See [16] p.302-303).

REMARK 2.1. The Dunkl operators $T_j, j = 1, 2, \dots, d$, associated to the root system \mathcal{R} , the reflection group W and the multiplicity function k are defined, for f of class C^1 on \mathbb{R}^d and $x \in \mathbb{R}_{reg}^d$, by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \mathcal{R}_+} \frac{k(\alpha)\alpha^j}{\langle \alpha, x \rangle} \{f(x) - f(r_\alpha x)\}.$$

(See [7][15]).

2.2. The Opdam-Cherednik kernel and the Heckman-Opdam hypergeometric function (see [12][13][16][17][20]).

We denote by $G_\lambda, \lambda \in \mathbb{C}^d$, the eigenfunction of the operators $T_j, j = 1, 2, \dots, d$. It is the unique analytic function on \mathbb{R}^d which satisfies the differential-difference system

$$\begin{cases} T_j G_\lambda(x) = i\lambda_j G_\lambda(x), & j = 1, 2, \dots, d, x \in \mathbb{R}^d, \\ G_\lambda(0) = 1. \end{cases} \quad (2.2)$$

It is called the Opdam-Cherednik kernel.

We consider the function F_λ defined by

$$\forall x \in \mathbb{R}^d, \quad F_\lambda(x) = \frac{1}{|W|} \sum_{w \in W} G_\lambda(wx). \quad (2.3)$$

This function is the unique analytic function on \mathbb{R}^d , which satisfies the differential system

$$\begin{cases} p(T)F_\lambda(x) = p(i\lambda)F_\lambda(x), & x \in \mathbb{R}^d, \\ F_\lambda(0) = 1, \end{cases}$$

for all W -invariant polynomials p on \mathbb{C}^d and $p(T) = p(T_1, T_2, \dots, T_d)$.

The function $F_\lambda(x)$ called the Heckman-Opdam hypergeometric function, it is W -invariant both in λ and x .

The functions G_λ and F_λ possess the following properties

- i) For all $\lambda \in \mathbb{C}^d$, the functions $x \rightarrow G_\lambda(x)$ and $x \rightarrow F_\lambda(x)$ are of class C^∞ on \mathbb{R}^d .
- ii) For all $x \in \mathbb{R}^d$, the functions $\lambda \rightarrow G_\lambda(x)$ and $\lambda \rightarrow F_\lambda(x)$ are entire on \mathbb{C}^d .
- iii) For all $x \in \mathbb{R}^d$ and $\lambda \in \mathbb{C}^d$, we have

$$\overline{G_\lambda(x)} = G_{-\bar{\lambda}}(x) \quad \text{and} \quad \overline{F_\lambda(x)} = F_{-\bar{\lambda}}(x). \quad (2.4)$$

- iv) For all $x \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}^d$, we have

$$|G_\lambda(x)| \leq |W|^{1/2} \quad \text{and} \quad |F_\lambda(x)| \leq |W|^{1/2}. \quad (2.5)$$

- v) Let p and q be polynomials of degree m and n . Then, there exists a positive constant M such that for all $\lambda \in \mathbb{C}^d$ and $x \in \mathbb{R}^d$, we have

$$\left| p\left(\frac{\partial}{\partial \lambda}\right) q\left(\frac{\partial}{\partial x}\right) G_\lambda(x) \right| \leq M(1 + \|x\|)^m (1 + \|\lambda\|)^n F_0(x) e^{-\max_{w \in W} \operatorname{Im}\langle w\lambda, x \rangle}. \quad (2.6)$$

The same inequality is also true for the function $F_\lambda(x)$.

vi) The function $F_0(x)$ satisfies the estimate

$$\forall x \in \bar{\mathfrak{a}}_+, F_0(x) \asymp e^{-\langle \rho, x \rangle} \prod_{\alpha \in \mathcal{R}_+^0} (1 + \langle \alpha, x \rangle).$$

vii) The function $G_\lambda(x), \lambda \in \mathbb{C}^d$, admits the following Laplace type representation

$$\forall x \in \mathbb{R}^d, G_\lambda(x) = \langle K_x, e^{i\langle \lambda, \cdot \rangle} \rangle, \tag{2.7}$$

where K_x is a some distribution on \mathbb{R}^d with support in $\Gamma = \text{conv}\{wx, w \in W\}$ (the convex hull for the orbit of x under W). (See [16] p.306).

viii) From (2.3), (2.6) we deduce that the function $F_\lambda(x), \lambda \in \mathbb{C}^d$, possesses the Laplace type representation

$$\forall x \in \mathbb{R}^d, F_\lambda(x) = \langle K_x^W, e^{i\langle \lambda, \cdot \rangle} \rangle, \tag{2.8}$$

where K_x^W is the distribution on \mathbb{R}^d with support in Γ , given by

$$K_x^W = \frac{1}{|W|} \sum_{w \in \mathcal{R}_+} K_{wx}. \tag{2.9}$$

REMARK 2.2. The functions $G_\lambda(x)$ and $F_\lambda(x)$ corresponding to the Dunkl operators $T_j, j = 1, 2, \dots, d$, are denoted respectively $K(x, \lambda)$ and $J_W(x, \lambda)$ and called respectively the Dunkl kernel and the generalized Bessel function. (See [7]).

EXAMPLE 2.1. For $d = 1$ and $W = \mathbb{Z}_2$, the root system is $\mathcal{R} = \{-2\alpha, -\alpha, \alpha, 2\alpha\}$ with $\alpha = 2$. Here $\mathcal{R}_+ = \{\alpha, 2\alpha\}$. We consider the multiplicity function k . We put $k_1 = k(\alpha) + k(2\alpha)$, $k_2 = k(2\alpha)$, and $\rho = k(\alpha) + 2k(2\alpha) = k_1 + 2k_2$.

The Cherednik operator is the following

$$T_1 f(x) = \frac{d}{dx} f(x) + \left(\frac{2k(\alpha)}{1 - e^{-2x}} + \frac{4k(2\alpha)}{1 - e^{-4x}} \right) (f(x) - f(-x)) - \rho f(x),$$

which can also be written in the form

$$T_1 f(x) = \frac{d}{dx} f(x) + (k_1 \coth(x) + k_2 \tanh(x)) (f(x) - f(-x)) - \rho f(-x).$$

The Opdam-Cherednik kernel is given by

$$\forall x \in \mathbb{R}, \forall \lambda \in \mathbb{C}, G_\lambda(x) = \varphi_\lambda^{(a,b)}(x) + \frac{1}{i\lambda - \rho} \frac{d}{dx} \varphi_\lambda^{(a,b)}(x),$$

where $\varphi_\lambda^{(a,b)}(x)$ is the Jacobi function (see[10]), with $a = k_1 - \frac{1}{2}$ and $b = k_2 - \frac{1}{2}$.

The Heckman-Opdam hypergeometric function has the form

$$\forall x \in \mathbb{R}, \forall \lambda \in \mathbb{C}, F_\lambda(x) = \varphi_\lambda^{(a,b)}(x).$$

(See [4] p.164-165 and 167.)

EXAMPLE 2.2.

1. The root system of type B_2 on \mathbb{R}^2 can be identified with the set \mathcal{R} given by

$$\mathcal{R} = \{\pm e_1, \pm e_2\} \cup \{\pm e_1 \pm e_2\},$$

where $\{e_1, e_2\}$ is the standard basis of \mathbb{R}^2 .

The root system \mathcal{R} can also be written in the form

$$\mathcal{R} = \{\pm \alpha_1, \pm \alpha_2, \pm \alpha_3, \pm \alpha_4\},$$

with,

$$\alpha_1 = e_1, \quad \alpha_2 = e_2, \quad \alpha_3 = (e_1 - e_2), \quad \alpha_4 = (e_1 + e_2).$$

We denote by \mathcal{R}_+ the set of positive roots

$$\mathcal{R}_+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}.$$

The Weyl group W is isomorphic to the hyperoctahedral group which is generated by permutations and sign changes of the $e_i, i = 1, 2$.

The multiplicity function $k : \mathcal{R} \rightarrow [0, +\infty[$ can be written in the form $k = (k_1, k_2)$, where k_1 is the value on the roots α_1, α_2 , and k_2 is the value on the roots α_3, α_4 .

The Cherednik operators $T_j, j = 1, 2$, associated with the Weyl group W and the multiplicity function k , can be written for f of class \mathcal{C}^1 on \mathbb{R}^2 and $x \in \mathbb{R}_{reg}^2$ in the following form

$$\begin{aligned} T_1 f(x) &= \frac{\partial}{\partial x_1} f(x) + k_1 \frac{\{f(x) - f(r_{\alpha_1} x)\}}{1 - e^{-\langle \alpha_1, x \rangle}} \\ &\quad + k_2 \left[\frac{f(x) - f(r_{\alpha_3} x)}{1 - e^{-\langle \alpha_3, x \rangle}} + \frac{f(x) - f(r_{\alpha_4} x)}{1 - e^{-\langle \alpha_4, x \rangle}} \right] - \left(\frac{1}{2} k_1 + k_2 \right) f(x), \\ T_2 f(x) &= \frac{\partial}{\partial x_2} f(x) + k_1 \frac{\{f(x) - f(r_{\alpha_2} x)\}}{1 - e^{-\langle \alpha_2, x \rangle}} \\ &\quad + k_2 \left[-\frac{f(x) - f(r_{\alpha_3} x)}{1 - e^{-\langle \alpha_3, x \rangle}} + \frac{f(x) - f(r_{\alpha_4} x)}{1 - e^{-\langle \alpha_4, x \rangle}} \right] - \frac{1}{2} k_1 f(x). \end{aligned}$$

(See [18]).

2. The root system of type C_2 on \mathbb{R}^2 can be identified with the set \mathcal{R} given by

$$\mathcal{R} = \{\pm 2e_1, \pm 2e_2\} \cup \{\pm e_1 \pm e_2\},$$

which can also be written in the form

$$\mathcal{R} = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm\alpha_4\},$$

with,

$$\alpha_1 = 2e_1, \quad \alpha_2 = 2e_2, \quad \alpha_3 = (e_1 - e_2), \quad \alpha_4 = (e_1 + e_2).$$

The set of positive roots is the following

$$\mathcal{R}_+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}.$$

If we denote by $W(C_2)$ the Weyl group associated with the root system \mathcal{R} of type C_2 , then we have

$$W(C_2) = W(B_2).$$

We denote by $k = (k_1, k_2)$ the multiplicity function of the root system \mathcal{R} of C_2 , where k_1 is the value on the roots α_1, α_2 , and k_2 is the value on the roots α_3, α_4 . (See [18]).

EXAMPLE 2.3. We consider the root system \mathcal{R} on \mathbb{R}^d given by

$$\mathcal{R} = \{\pm\alpha_i, \pm 2\alpha_i, i = 1, 2, \dots, d\}.$$

We denote by \mathcal{R}_+ the set of positive roots

$$\mathcal{R} = \{\alpha_i, 2\alpha_i, i = 1, 2, \dots, d\}.$$

The Cherednik operators $T_j, j = 1, 2, \dots, d$, associated to the Weyl group W and the multiplicity function k are defined, for f of class \mathcal{C}^1 on \mathbb{R}^d and $x \in \mathbb{R}_{reg}^d$, by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{i=1}^d \left[\frac{k(\alpha_i)}{1 - e^{-\langle x, \alpha_i \rangle}} + \frac{2k(2\alpha_i)}{1 - e^{-2\langle x, \alpha_i \rangle}} \right] \alpha_i^j \{f(x) - f(r_{\alpha_i} x)\} - \frac{1}{2} \left(\sum_{i=1}^d (k(\alpha_i) + 2k(2\alpha_i)) \right) f(x),$$

with $\alpha_i^j = \langle \alpha_i, e_j \rangle$. (See [19]).

3. The harmonic analysis associated to the Heckman-Opdam theory on \mathbb{R}^d

In this section, we give the harmonic analysis associated to the Heckman-Opdam theory (the hypergeometric Fourier transform, the hypergeometric translation operator and the hypergeometric convolution product). We shall precise these notions needed in the following subsections.

3.1. The harmonic analysis associated to the Heckman-Opdam theory on the space of W -invariant C^∞ -functions.

Notations. We denote by

- $\mathcal{E}(\mathbb{R}^d)^W$ the space of C^∞ -functions on \mathbb{R}^d , which are W -invariant.
- $\mathcal{D}(\mathbb{R}^d)^W$ the space of C^∞ -functions on \mathbb{R}^d , with compact support and W -invariant.
- $\mathcal{S}(\mathbb{R}^d)^W$ the space of W -invariant functions from the classical Schwartz space $\mathcal{S}(\mathbb{R}^d)$.
- $\mathcal{S}_2(\mathbb{R}^d)^W$ the space of C^∞ -functions on \mathbb{R}^d , which are W -invariant, and such that for all $\ell, n \in \mathbb{N}$,

$$p_{\ell,n}(f) = \sup_{\substack{|\mu| \leq n \\ x \in \mathbb{R}^d}} (1 + \|x\|)^\ell (F_0(x))^{-1} |D^\mu f(x)| < +\infty,$$

where

$$D^\mu = \frac{\partial^{|\mu|}}{\partial x_1^{\mu_1} \dots \partial x_d^{\mu_d}}, \quad \mu = (\mu_1, \dots, \mu_d) \in \mathbb{N}^d, \quad |\mu| = \sum_{i=1}^d \mu_i.$$

Its topology is defined by the semi-norms $p_{\ell,n}$, $\ell, n \in \mathbb{N}$.

- $PW_a(\mathbb{C}^d)^W$, $a > 0$, the space of entire functions g on \mathbb{C}^d , which are W -invariant and satisfying

$$\forall m \in \mathbb{N}, q_m(g) = \sup_{\lambda \in \mathbb{C}^d} (1 + \|\lambda\|)^m e^{-a\|Im\lambda\|} |g(\lambda)| < +\infty.$$

The topology of $PW_a(\mathbb{C}^d)$ is defined by the semi-norms q_m , $m \in \mathbb{N}$.

We set

$$PW(\mathbb{C}^d)^W = \cup_{a>0} PW_a(\mathbb{C}^d)^W.$$

This space is called the Paley-Wiener space. It is equipped with the inductive limit topology.

3.1.1. *The hypergeometric Fourier transform.*

The hypergeometric Fourier transform \mathcal{H}^W has been defined and studied first by E.M.Opdam in [12] on the space of W -invariant C^∞ -functions on \mathbb{R}^d .

DEFINITION 3.1. The hypergeometric Fourier transform \mathcal{H}^W is defined for f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) by

$$\forall \lambda \in \mathbb{C}^d, \mathcal{H}^W(f)(\lambda) = \int_{\mathbb{R}^d} f(x)F_{-\lambda}(x)\mathcal{A}_k(x)dx. \tag{3.1}$$

REMARK 3.1. We have also the relation

$$\forall \lambda \in \mathbb{C}^d, \mathcal{H}^W(f)(\lambda) = \int_{\mathbb{R}^d} f(x)F_\lambda(-x)\mathcal{A}_k(x)dx. \tag{3.2}$$

PROPOSITION 3.1. For all f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) we have the following relations

$$\forall \lambda \in \mathbb{R}^d, \mathcal{H}^W(\bar{f})(\lambda) = \overline{\mathcal{H}^W(\check{f})(\lambda)}, \tag{3.3}$$

$$\forall \lambda \in \mathbb{R}^d, \mathcal{H}^W(f)(\lambda) = \mathcal{H}^W(\check{f})(-\lambda), \tag{3.4}$$

where \check{f} is the function defined by

$$\forall x \in \mathbb{R}^d, \check{f}(x) = f(-x).$$

Proof. We deduce these results from relations (2.4),(3.1),(3.2). □

THEOREM 3.1. (See [12][13]).

i) The hypergeometric Fourier transform \mathcal{H}^W is a topological isomorphism from

- $\mathcal{D}(\mathbb{R}^d)^W$ onto $PW(\mathbb{C}^d)^W$.
- $\mathcal{S}_2(\mathbb{R}^d)^W$ onto $\mathcal{S}(\mathbb{R}^d)^W$.

ii) Let f be in $\mathcal{D}(\mathbb{R}^d)^W$. Then $\text{supp}f \subset B(0, a)$, the closed ball of center 0 and radius $a > 0$, if and only if its hypergeometric Fourier transform $\mathcal{H}^W(f)$ belongs to $PW_a(\mathbb{C}^d)^W$.

iii) The inverse transform $(\mathcal{H}^W)^{-1}$ is given by

$$\forall x \in \mathbb{R}^d, (\mathcal{H}^W)^{-1}(h)(x) = \int_{\mathbb{R}^d} h(\lambda)F_\lambda(x)\mathcal{C}_k^W(\lambda)d\lambda,$$

where

$$\mathcal{C}_k^W(\lambda) = c|c_k(\lambda)|^{-2}, \tag{3.5}$$

with c a positive constant chosen in such a way that $\mathcal{C}_k^W(-\rho) = 1$, and

$$c_k(\lambda) = \prod_{\alpha \in \mathcal{R}_+} \frac{\Gamma(\langle i\lambda, \check{\alpha} \rangle + \frac{1}{2}k(\frac{\alpha}{2}))}{\Gamma(\langle i\lambda, \check{\alpha} \rangle + k(\alpha) + \frac{1}{2}k(\frac{\alpha}{2}))}, \tag{3.6}$$

with the convention that $k(\frac{\alpha}{2}) = 0$ if $\frac{\alpha}{2} \notin \mathcal{R}$.

REMARK 3.2. The function \mathcal{C}_k^W is continuous on \mathbb{R}^d and satisfies the estimate

$$\forall \lambda \in \mathbb{R}^d, |\mathcal{C}_k^W(\lambda)| \leq \text{const.}(1 + \|\lambda\|)^s, \tag{3.7}$$

for some $s > 0$.

3.1.2. *The hypergeometric transmutation operators V_k^W and ${}^tV_k^W$.*

K.Trimèche has introduced in [16][17][20] the hypergeometric transmutation operators V_k^W and ${}^tV_k^W$. These operators are useful to define and study, in the following subsection, the hypergeometric translation operator.

By using the distribution K_x^W given by (2.9) we define the hypergeometric transmutation operator V_k^W on $\mathcal{E}(\mathbb{R}^d)^W$ by

$$\forall x \in \mathbb{R}^d, V_k^W(g)(x) = \langle K_x^W, g \rangle. \tag{3.8}$$

This operator is called also the trigonometric Dunkl intertwining operator. It satisfies the relation

$$\forall x \in \mathbb{R}^d, \forall \lambda \in \mathbb{C}^d, V_k^W(e^{i\langle \lambda, \cdot \rangle}) = F_\lambda(x). \tag{3.9}$$

The operator V_k^W is the unique linear topological isomorphism from $\mathcal{E}(\mathbb{R}^d)^W$ onto itself satisfying the transmutation relations

$$\forall x \in \mathbb{R}^d, p(T)V_k^W(g)(x) = V_k^W(p(D)g)(x), g \in \mathcal{E}(\mathbb{R}^d)^W,$$

for all W -invariant polynomials p on \mathbb{C}^d , $p(T) = p(T_1, T_2, \dots, T_d)$ and $p(D) = p(D_1, D_2, \dots, D_d)$ with $D_j = \frac{\partial}{\partial x_j}$, $j = 1, 2, \dots, d$, and the condition

$$V_k^W(g)(0) = g(0). \tag{3.10}$$

The dual ${}^tV_k^W$ of the operator V_k^W is defined by the following duality relation

$$\int_{\mathbb{R}^d} {}^tV_k^W(f)(y)g(y)dy = \int_{\mathbb{R}^d} V_k^W(g)(x)f(x)\mathcal{A}_k(x)dx, \tag{3.11}$$

with f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) and g in $\mathcal{E}(\mathbb{R}^d)^W$.

The operator ${}^tV_k^W$ is a linear topological isomorphism from $\mathcal{D}(\mathbb{R}^d)^W$ onto itself

- $\mathcal{S}_2(\mathbb{R}^d)^W$ onto $\mathcal{S}(\mathbb{R}^d)^W$, satisfying the transmutation relations

$$\forall y \in \mathbb{R}^d, {}^tV_k^W(p(T)f)(y) = p(D_\rho){}^tV_k^W(f)(y), f \in \mathcal{D}(\mathbb{R}^d)^W \text{ (resp. } \mathcal{S}_2(\mathbb{R}^d)^W),$$

for all W -invariant polynomials p on $\mathbb{C}^d, p(T) = p(T_1, T_2, \dots, T_d)$, and $p(D_\rho) = p(D_{1,\rho_1}, D_{2,\rho_2}, \dots, D_{d,\rho_d})$ with $D_{j,\rho_j} = \frac{\partial}{\partial x_j} - 2\rho_j, j = 1, 2, \dots, d$.

REMARK 3.3. By applying the relation (3.11) with the function $g(y) = e^{-i\langle \lambda, y \rangle}, \lambda \in \mathbb{R}^d$, we deduce from the relations (3.9),(3.1) that the operator ${}^tV_k^W$ satisfies for f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$), the following relation

$$\forall \lambda \in \mathbb{R}^d, \mathcal{F}o {}^tV_k(f)(\lambda) = \mathcal{H}^W(f)(\lambda), \tag{3.12}$$

where \mathcal{F} is the classical Fourier transform on \mathbb{R}^d .

3.1.3. *The hypergeometric translation operator \mathcal{T}_x^W and its dual ${}^t\mathcal{T}_x^W$ on the space of W -invariant C^∞ -functions.*

By using the hypergeometric transmutation operators V_k^W and ${}^tV_k^W$, K.Trimèche has defined and studied in [17][20], the hypergeometric translation operator $\mathcal{T}_x^W, x \in \mathbb{R}^d$ and its dual ${}^t\mathcal{T}_x^W$. We give in this subsection the properties of these operators on the space of W -invariant C^∞ -functions.

DEFINITION 3.2. We define the hypergeometric translation operator $\mathcal{T}_x^W, x \in \mathbb{R}^d$, on $\mathcal{E}(\mathbb{R}^d)^W$ by

$$\forall y \in \mathbb{R}^d, \mathcal{T}_x^W(f)(y) = (V_k^W)_x(V_k^W)_y[(V_k^W)^{-1}(f)(x + y)]. \tag{3.13}$$

PROPOSITION 3.2. *The operator $\mathcal{T}_x^W, x \in \mathbb{R}^d$, satisfies the following properties*

- i) For all $x \in \mathbb{R}^d$, the operator \mathcal{T}_x^W is continuous from $\mathcal{E}(\mathbb{R}^d)^W$ into itself.
- ii) For all f in $\mathcal{E}(\mathbb{R}^d)^W$ and $x, y \in \mathbb{R}^d$, we have

$$\mathcal{T}_x^W(f)(0) = f(x) \text{ and } \mathcal{T}_x^W(f)(y) = \mathcal{T}_y^W(f)(x). \tag{3.14}$$

- iii) For all $x, y \in \mathbb{R}^d$ and $\lambda \in \mathbb{C}^d$, we have the product formula

$$\mathcal{T}_x^W(F_\lambda)(y) = F_\lambda(x).F_\lambda(y), \tag{3.15}$$

where F_λ is the Heckman-Opdam hypergeometric function given by (2.3).

- Proof.* i) We deduce the result from (3.13) and the continuity of the operator V_k^W from $\mathcal{E}(\mathbb{R}^d)^W$ into itself.
 ii) The relations (3.13),(3.10) give the results.
 iii) We deduce formula (3.15) from the relations (3.13),(3.9). \square

DEFINITION 3.3. We define the hypergeometric translation operator dual ${}^t\mathcal{T}_x^W$, $x \in \mathbb{R}^d$, on $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $S_2(\mathbb{R}^d)^W$) by

$$\forall y \in \mathbb{R}^d, \quad {}^t\mathcal{T}_x^W(f)(y) = (V_k^W)_x ({}^tV_k^W)_y^{-1} [({}^tV_k^W(f))(y-x)]. \quad (3.16)$$

PROPOSITION 3.3. The operator ${}^t\mathcal{T}_x^W$, $x \in \mathbb{R}^d$ possesses the following properties

- i) For all $x \in \mathbb{R}^d$, the operator ${}^t\mathcal{T}_x^W$ is continuous from $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $S_2(\mathbb{R}^d)^W$) into itself.
 ii) For all f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $S_2(\mathbb{R}^d)^W$) and $x, y \in \mathbb{R}^d$, we have

$${}^t\mathcal{T}_x^W(f)(y) = {}^t\mathcal{T}_{-y}^W(f)(-x). \quad (3.17)$$

- iii) For all f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $S_2(\mathbb{R}^d)^W$) and h in $\mathcal{E}(\mathbb{R}^d)^W$, we have

$$\int_{\mathbb{R}^d} {}^t\mathcal{T}_x^W(f)(y)h(y)\mathcal{A}_k(y)dy = \int_{\mathbb{R}^d} f(y)\mathcal{T}_x^W(h)(y)\mathcal{A}_k(y)dy. \quad (3.18)$$

- iv) For all f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $S_2(\mathbb{R}^d)^W$) and $x \in \mathbb{R}^d$, we have

$$\forall \lambda \in \mathbb{C}^d, \quad \mathcal{H}^W({}^t\mathcal{T}_x^W(f))(\lambda) = F_{-\lambda}(x)\mathcal{H}^W(f)(\lambda). \quad (3.19)$$

- v) For all f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $S_2(\mathbb{R}^d)^W$) and $x, y \in \mathbb{R}^d$, we have

$${}^t\mathcal{T}_x^W(f)(y) = \int_{\mathbb{R}^d} F_{-\lambda}(x)F_{\lambda}(y)\mathcal{H}^W(f)(\lambda)\mathcal{C}_k^W(\lambda)d\lambda. \quad (3.20)$$

- vi) For all f in $\mathcal{D}(\mathbb{R}^d)^W$ with support in the closed ball $B(0, a)$ of center 0 and radius $a > 0$, we have

$$\text{supp } {}^t\mathcal{T}_x^W(f) \subset B(0, a + \|x\|). \quad (3.21)$$

Proof. i) We deduce the result from (3.16) and the fact that the operator ${}^tV_k^W$ is a topological isomorphism from $\mathcal{D}(\mathbb{R}^d)^W$ into itself (resp. from $S_2(\mathbb{R}^d)^W$ into $S(\mathbb{R}^d)^W$).

- ii) The relation (3.16) give the result.

- iii) We deduce (3.18) from the relations (3.16),(3.13) and Proposition

3.5 of [17].

iv) From the relations (3.1),(3.18),(3.15), we have

$$\begin{aligned} \forall \lambda \in \mathbb{C}^d, \mathcal{H}^W({}^t\mathcal{T}_x^W(f))(\lambda) &= \int_{\mathbb{R}^d} {}^t\mathcal{T}_x^W(f)(y)F_{-\lambda}(y)\mathcal{A}_k(y)dy, \\ &= \int_{\mathbb{R}^d} f(y)\mathcal{T}_x^W(F_{-\lambda})(y)\mathcal{A}_k(y)dy, \\ &= F_{-\lambda}(x) \int_{\mathbb{R}^d} f(y)F_{-\lambda}(y)\mathcal{A}_k(y)dy, \end{aligned}$$

thus,

$$\forall \lambda \in \mathbb{C}^d, \mathcal{H}^W({}^t\mathcal{T}_x^W(f))(\lambda) = F_{-\lambda}(x)\mathcal{H}^W(f)(\lambda).$$

v) We deduce (3.20) from (3.19) and Theorem 3.1 iii).

vi) We obtain (3.21) from the relations (3.19),(2.5),(2.6) and Theorem 3.1 ii). □

3.1.4. The hypergeometric convolution product.

In this subsection, we define the hypergeometric convolution product by using the hypergeometric translation operator $\mathcal{T}_x^W, x \in \mathbb{R}^d$, and we study its properties on the space of W -invariant C^∞ -functions (see [17][20]).

DEFINITION 3.4. The hypergeometric convolution product $f *_{\mathcal{H}^W} g$ of the functions f, g in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) is defined by

$$\forall x \in \mathbb{R}^d, f *_{\mathcal{H}^W} g(x) = \int_{\mathbb{R}^d} \mathcal{T}_x^W(f)(-y)g(y)\mathcal{A}_k(y)dy. \tag{3.22}$$

REMARK 3.4. We have

$$\forall x \in \mathbb{R}^d, f *_{\mathcal{H}^W} g(x) = \int_{\mathbb{R}^d} \mathcal{T}_x^W(f)(y)\check{g}(y)\mathcal{A}_k(y)dy. \tag{3.23}$$

where \check{g} is the function defined by

$$\forall y \in \mathbb{R}^d, \check{g}(y) = g(-y),$$

then, by applying the relation (3.18), the relation (3.23) can also be written in the form

$$\forall x \in \mathbb{R}^d, f *_{\mathcal{H}^W} g(x) = \int_{\mathbb{R}^d} f(y) {}^t\mathcal{T}_x^W(\check{g})(y)\mathcal{A}_k(y)dy. \tag{3.24}$$

PROPOSITION 3.4.

i) For all f, g in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$), the function $f *_{\mathcal{H}^W} g$ belongs to $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$).

ii) For all f, g in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$), we have

$$\forall \lambda \in \mathbb{R}^d, \quad \mathcal{H}^W(f *_{\mathcal{H}^W} g)(\lambda) = \mathcal{H}^W(f)(\lambda) \cdot \mathcal{H}^W(g)(\lambda). \quad (3.25)$$

Proof. i) We deduce the result from the relation (3.24) and the properties of the function ${}^t\mathcal{T}_x(\check{g})(y)$.

ii) From the relation (3.1) we have

$$\forall \lambda \in \mathbb{R}^d, \quad \mathcal{H}^W(f *_{\mathcal{H}^W} g)(\lambda) = \int_{\mathbb{R}^d} f *_{\mathcal{H}^W} g(x) F_{-\lambda}(x) \mathcal{A}_k(x) dx.$$

By using the relations (3.24), (3.17) and Fubini's theorem, we obtain

$$\forall \lambda \in \mathbb{R}^d, \quad \mathcal{H}^W(f *_{\mathcal{H}^W} g)(\lambda) = \int_{\mathbb{R}^d} f(y) \left[\int_{\mathbb{R}^d} {}^t\mathcal{T}_{-y}(g)(-x) F_{-\lambda}(x) \mathcal{A}_k(x) dx \right] \mathcal{A}_k(y) dy. \quad (3.26)$$

But from (3.2), (3.1), (3.19), (3.4), we get

$$\begin{aligned} \int_{\mathbb{R}^d} {}^t\mathcal{T}_{-y}(\check{g})(-x) F_{-\lambda}(x) \mathcal{A}_k(x) dx &= \int_{\mathbb{R}^d} {}^t\mathcal{T}_{-y}(\check{g})(-x) F_{\lambda}(-x) \mathcal{A}_k(x) dx, \\ &= \int_{\mathbb{R}^d} {}^t\mathcal{T}_{-y}(\check{g})(x) F_{\lambda}(x) \mathcal{A}_k(x) dx, \\ &= \mathcal{H}^W({}^t\mathcal{T}_{-y}(\check{g}))(-\lambda), \\ &= F_{\lambda}(-y) \mathcal{H}^W(\check{g})(-\lambda), \end{aligned}$$

thus,

$$\int_{\mathbb{R}^d} {}^t\mathcal{T}_{-y}(\check{g})(-x) F_{-\lambda}(x) \mathcal{A}_k(x) dx = F_{\lambda}(-y) \mathcal{H}^W(g)(\lambda).$$

We put this relation in (3.26) and we obtain

$$\forall \lambda \in \mathbb{R}^d, \quad \mathcal{H}^W(f *_{\mathcal{H}^W} g)(\lambda) = \mathcal{H}^W(g)(\lambda) \int_{\mathbb{R}^d} f(y) F_{\lambda}(-y) \mathcal{A}_k(y) dy,$$

we deduce (3.25) by applying (3.2). \square

COROLLARY 3.1.

i) The hypergeometric convolution product is commutative and associative on $\mathcal{D}(\mathbb{R}^d)^W$ and $\mathcal{S}_2(\mathbb{R}^d)^W$.

ii) For all f, g in $\mathcal{D}(\mathbb{R}^d)^W$, the function $f *_{\mathcal{H}^W} g$ belongs to $\mathcal{D}(\mathbb{R}^d)^W$, and if $\text{supp } f \subset B(0, a)$, $a > 0$, and $\text{supp } g \subset B(0, b)$, $b > 0$, we have

$$\text{supp } (f *_{\mathcal{H}^W} g) \subset B(0, a + b), \quad (3.27)$$

where $B(0, c)$, is the closed ball of center 0 and radius $c > 0$.

Proof. **i)** We deduce the result from Proposition 3.4 ii) and Theorem 3.1 i). **ii)** Proposition 3.4 ii) and Theorem 3.1 ii) imply the relation (3.27). \square

COROLLARY 3.2. For all f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$), we have

$$\forall x, y \in \mathbb{R}^d, \mathcal{T}_x^W(f)(y) = {}^t\mathcal{T}_x^W(\check{f})(-y), \quad (3.28)$$

where \check{f} is the function defined by

$$\forall z \in \mathbb{R}^d, \check{f}(z) = f(-z).$$

Proof. From Corollary 3.1 i), the hypergeometric convolution product is commutative, then we have

$$\forall x \in \mathbb{R}^d, \int_{\mathbb{R}^d} \mathcal{T}_x^W(g)(-y) f(y) \mathcal{A}_k(y) dy = \int_{\mathbb{R}^d} \mathcal{T}_x^W(f)(-y) g(y) \mathcal{A}_k(y) dy.$$

On the other hand, from the relation (3.24) we have

$$\forall x \in \mathbb{R}^d, \int_{\mathbb{R}^d} \mathcal{T}_x^W(g)(-y) f(y) \mathcal{A}_k(y) dy = \int_{\mathbb{R}^d} {}^t\mathcal{T}_x^W(\check{f})(y) g(y) \mathcal{A}_k(y) dy.$$

Thus, for all $x \in \mathbb{R}^d$ and g in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$), we have

$$\int_{\mathbb{R}^d} [\mathcal{T}_x^W(f)(-y) - {}^t\mathcal{T}_x^W(\check{f})(y)] g(y) \mathcal{A}_k(y) dy = 0.$$

This relation implies (3.28). \square

PROPOSITION 3.5.

i) For all f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) and $x \in \mathbb{R}^d$, we have

$$\forall \lambda \in \mathbb{C}^d, \mathcal{H}^W(\mathcal{T}_x^W(f))(\lambda) = F_\lambda(x) \mathcal{H}^W(f)(\lambda). \quad (3.29)$$

ii) For all f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) and $x, y \in \mathbb{R}^d$, we have

$$\mathcal{T}_x^W(f)(y) = \int_{\mathbb{R}^d} F_\lambda(x) F_\lambda(y) \mathcal{H}^W(f)(\lambda) \mathcal{C}_k^W(\lambda) d\lambda. \quad (3.30)$$

Proof. i) From the relations (3.2),(3.18), for all f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $S_2(\mathbb{R}^d)^W$) and $x \in \mathbb{R}^d$, we have

$$\begin{aligned} \forall \lambda \in \mathbb{C}^d, \mathcal{H}^W(\mathcal{T}_x^W(f))(\lambda) &= \int_{\mathbb{R}^d} \mathcal{T}_x^W(f)(y) \check{F}_\lambda(y) \mathcal{A}_k(y) dy, \\ &= \int_{\mathbb{R}^d} f(y) {}^t\mathcal{T}_x^W(\check{F}_\lambda)(y) \mathcal{A}_k(y) dy, \\ &= \int_{\mathbb{R}^d} \check{f}(y) {}^t\mathcal{T}_x^W(\check{F}_\lambda)(-y) \mathcal{A}_k(y) dy. \end{aligned}$$

By using the relations (3.28),(3.15), we obtain

$$\begin{aligned} \forall \lambda \in \mathbb{C}^d, \mathcal{H}^W(\mathcal{T}_x^W(f))(\lambda) &= \int_{\mathbb{R}^d} \check{f}(y) \mathcal{T}_x^W(F_\lambda)(y) \mathcal{A}_k(y) dy, \\ &= F_\lambda(x) \int_{\mathbb{R}^d} \check{f}(y) F_\lambda(y) \mathcal{A}_k(y) dy, \\ &= F_\lambda(x) \int_{\mathbb{R}^d} f(y) F_\lambda(-y) \mathcal{A}_k(y) dy. \end{aligned}$$

The relation (3.2) implies (3.29).

ii) We deduce (3.30) from the relation (3.29) and Theorem 3.1 iii). \square

3.2. The harmonic analysis associated to the Heckman-Opdam theory on the $L^p_{\mathcal{A}_k}(\mathbb{R}^d)^W$, $p = 1, 2$, spaces.

3.2.1. The hypergeometric Fourier transform.

The hypergeometric Fourier transform \mathcal{H}^W has been studied by K.Trimèche in [20] on the space $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ of W -invariant square integrable functions on \mathbb{R}^d , which has permit to prove formulas and a theorem of Plancherel.

Notations. We denote by $L^p_{\mathcal{A}_k}(\mathbb{R}^d)^W$, $1 \leq p \leq +\infty$, the space of measurable functions f on \mathbb{R}^d which are W -invariant and satisfying

$$\begin{aligned} \|f\|_{\mathcal{A}_k,p} &= \left(\int_{\mathbb{R}^d} |f(x)|^p \mathcal{A}_k(x) dx \right)^{1/p} < +\infty, \quad 1 \leq p < +\infty, \\ \|f\|_{\mathcal{A}_k,\infty} &= \text{ess sup}_{x \in \mathbb{R}^d} |f(x)| < +\infty. \end{aligned}$$

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- $L^p_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$, $1 \leq p \leq +\infty$, the space of measurable functions f on \mathbb{R}^d , which are W -invariant and satisfying

$$\|f\|_{\mathcal{C}_k^W, p} = \left(\int_{\mathbb{R}^d} |f(\lambda)|^p \mathcal{C}_k^W(\lambda) d\lambda \right)^{1/p} < +\infty, \quad 1 \leq p < +\infty,$$

$$\|f\|_{\mathcal{C}_k^W, \infty} = \text{ess sup}_{\lambda \in \mathbb{R}^d} |f(\lambda)| < +\infty.$$

REMARK 3.5.

- i) The space $\mathcal{D}(\mathbb{R}^d)^W$ is dense in the space $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$.
- ii) $S_2(\mathbb{R}^d)^W \subset L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$.

We give first the following relations relating to the hypergeometric Fourier transform on $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $S_2(\mathbb{R}^d)^W$).

PROPOSITION 3.6. For all f, g in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $S_2(\mathbb{R}^d)^W$), we have

$$\int_{\mathbb{R}^d} f(y) \overline{g(y)} \mathcal{A}_k(y) dy = \int_{\mathbb{R}^d} \mathcal{H}^W(f)(\lambda) \overline{\mathcal{H}^W(g)(\lambda)} \mathcal{C}_k^W(\lambda) d\lambda, \quad (3.31)$$

and

$$\|f\|_{\mathcal{A}_k, 2} = \|\mathcal{H}^W(f)\|_{\mathcal{C}_k^W, 2}. \quad (3.32)$$

Proof. From the relation (3.25) and Theorem 3.1 iii), we have

$$\forall x \in \mathbb{R}^d, f *_{\mathcal{H}^W} \overline{g(x)} = \int_{\mathbb{R}^d} F_\lambda(x) \mathcal{H}^W(f)(\lambda) \overline{\mathcal{H}^W(g)(\lambda)} \mathcal{C}_k^W(\lambda) d\lambda.$$

The relations (3.22),(3.3) permit to write this relation in the following form

$$\forall x \in \mathbb{R}^d, \int_{\mathbb{R}^d} \mathcal{T}_x^W(f)(y) \overline{\check{g}(y)} \mathcal{A}_k(y) dy = \int_{\mathbb{R}^d} F_\lambda(x) \mathcal{H}^W(f)(\lambda) \overline{\mathcal{H}^W(\check{g})(\lambda)} \mathcal{C}_k^W(\lambda) d\lambda.$$

We obtain (3.31) by changing \check{g} by g in the two members, by taking $x = 0$, and by using the relations

$$\forall y \in \mathbb{R}^d, \mathcal{T}_0^W(f)(y) = f(y), \quad \text{and} \quad \forall \lambda \in \mathbb{R}^d, F_\lambda(0) = 1.$$

□

DEFINITION 3.5. The hypergeometric Fourier transform \mathcal{H}^W is defined for f in $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$ by

$$\forall \lambda \in \mathbb{R}^d, \mathcal{H}^W(f)(\lambda) = \int_{\mathbb{R}^d} f(x) F_{-\lambda}(x) \mathcal{A}_k(x) dx.$$

LEMMA 3.1. *Let H be a Hilbert space, V a subspace of H dense in H , and $U: V \mapsto H$ a linear continuous application when we equip V with the norm induced by H . Then U extends to a linear continuous application from H into itself. If U is an isometry it extends to an isometry from H into itself. By taking $H = L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, $V = S_2(\mathbb{R}^d)^W$ and $U = \mathcal{H}^W$ and by using Corollary 3.3 and Lemma 3.1, we obtain the following Theorem:*

THEOREM 3.2.

i) (Plancherel formulas). *For all f, g in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ we have*

$$\int_{\mathbb{R}^d} f(x)\overline{g(x)}\mathcal{A}_k(x)dx = \int_{\mathbb{R}^d} \mathcal{H}^W(f)(\lambda)\overline{\mathcal{H}^W(g)(\lambda)}\mathcal{C}_k^W(\lambda)d\lambda. \quad (3.33)$$

and

$$\|f\|_{\mathcal{A}_k,2} = \|\mathcal{H}^W(f)\|_{\mathcal{C}_k^W,2}. \quad (3.34)$$

ii) (Plancherel theorem). *The hypergeometric Fourier transform \mathcal{H}^W extends uniquely to an isometric isomorphism from $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ onto $L^2_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$.*

COROLLARY 3.3. *For all f in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ such that $\mathcal{H}^W(f)$ belongs to $L^1_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$, we have the following inversion formula*

$$f(x) = \int_{\mathbb{R}^d} \mathcal{H}^W(f)(\lambda)F_\lambda(x)\mathcal{C}_k^W(\lambda)d\lambda, \text{ a.e. } x \in \mathbb{R}^d. \quad (3.35)$$

REMARK 3.6. The inversion formula (3.35) is also true for all function f in $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$ bounded such that $\mathcal{H}^W(f)$ belongs to $L^1_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$.

3.2.2. The hypergeometric translation operator.

DEFINITION 3.6. The hypergeometric translation operator $\mathcal{T}_x^W, x \in \mathbb{R}^d$, is defined on $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ by

$$\mathcal{H}^W(\mathcal{T}_x^W(f))(\lambda) = F_\lambda(x)\mathcal{H}^W(f)(\lambda), \quad \lambda \in \mathbb{R}^d. \quad (3.36)$$

REMARK 3.7. Note that this definition makes sense because the hypergeometric Fourier transform is, from Theorem 3.2 ii), an isomorphism from $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ onto $L^2_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$, and from (2.5), for all $\lambda \in \mathbb{R}^d$, the function $F_\lambda(x)$ is bounded.

PROPOSITION 3.7.

i) *For all f in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, we have*

$$\|\mathcal{T}_x^W(f)\|_{\mathcal{A}_k,2} \leq |W|^{1/2}\|f\|_{\mathcal{A}_k,2}. \quad (3.37)$$

ii) For all f in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, we have

$$\mathcal{T}_x^W(f)(y) = \lim_{n \rightarrow +\infty} \int_{B(0,n)} F_\lambda(x)F_\lambda(y)\mathcal{H}^W(f)(\lambda)\mathcal{C}_k^W(\lambda)d\lambda, \quad (3.38)$$

where $B(0, n)$ is the closed ball of center 0 and radius n . The limit is in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$.

iii) For all f in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ such that $\mathcal{H}^W(f)$ belongs to $L^1_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$ and $x \in \mathbb{R}^d$, we have

$$\mathcal{T}_x^W(f)(y) = \int_{\mathbb{R}^d} F_\lambda(x)F_\lambda(y)\mathcal{H}^W(f)(\lambda)\mathcal{C}_k^W(\lambda)d\lambda, \quad a.e. \quad y \in \mathbb{R}^d. \quad (3.39)$$

iv) For all f in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, we have

$$\overline{\mathcal{T}_x^W(f)(y)} = \mathcal{T}_x^W(\bar{f})(y), \quad x, y \in \mathbb{R}^d, \quad (3.40)$$

and

$$\mathcal{T}_x^W(f)(y) = \mathcal{T}_y^W(f)(x), \quad x, y \in \mathbb{R}^d. \quad (3.41)$$

Proof. i) We obtain (3.37) from (3.36), Plancherel formula (3.34) and (2.5).

ii) We deduce the result from (3.36) and Theorem 3.2 ii).

iii) The relation (3.36) and the inversion formula (3.35) imply the result.

iv) For the functions f of $S_2(\mathbb{R}^d)^W$ the relations (3.30),(3.3),(3.4) imply the relations (3.40),(3.41), we deduce these relations for the functions of $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ from the density of $S_2(\mathbb{R}^d)^W$ in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and the relation (3.38). \square

PROPOSITION 3.8. For all f in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, the mapping $x \rightarrow \mathcal{T}_x^W(f)$ is continuous from \mathbb{R}^d into $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$.

Proof. Let $x_0 \in \mathbb{R}^d$. By using Plancherel formula (3.34) and the relation (3.36), we obtain

$$\begin{aligned} \|\mathcal{T}_x^W(f) - \mathcal{T}_{x_0}^W(f)\|_{\mathcal{A}_k,2}^2 &= \|\mathcal{H}^W(\mathcal{T}_x^W(f)) - \mathcal{H}^W(\mathcal{T}_{x_0}^W(f))\|_{\mathcal{C}_k^W}^2, \\ &= \int_{\mathbb{R}^d} |F_\lambda(x) - F_\lambda(x_0)|^2 |\mathcal{H}^W(f)(\lambda)|^2 \mathcal{C}_k^W(\lambda) d\lambda. \end{aligned}$$

From the relation (2.5) and the fact that for all $\lambda \in \mathbb{R}^d$, the function $x \rightarrow F_\lambda(x)$ is continuous on \mathbb{R}^d , the dominated convergence theorem implies

$$\lim_{x \rightarrow x_0} \|\mathcal{T}_x^W(f) - \mathcal{T}_{x_0}^W(f)\|_{\mathcal{A}_k,2} = 0.$$

□

3.2.3. The hypergeometric convolution product.

In this subsection, we define the hypergeometric convolution product by using the hypergeometric translation operator $\mathcal{T}_x^W, x \in \mathbb{R}^d$, and we study its properties on the space $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ (see [17][20]).

PROPOSITION 3.9. *Let f be in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and g in $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$, then the function $f *_{\mathcal{H}^W} g$ defined all most everywhere on \mathbb{R}^d by*

$$f *_{\mathcal{H}^W} g(x) = \int_{\mathbb{R}^d} \mathcal{T}_x^W(f)(-y)g(y)\mathcal{A}_k(y)dy, \tag{3.42}$$

belongs to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, and we have

$$\|f *_{\mathcal{H}^W} g\|_{\mathcal{A}_k,2} \leq |W|^{1/2}\|f\|_{\mathcal{A}_k,2}\|g\|_{\mathcal{A}_k,1}, \tag{3.43}$$

and

$$\mathcal{H}^W(f *_{\mathcal{H}^W} g) = \mathcal{H}^W(f).\mathcal{H}^W(g). \tag{3.44}$$

Proof. Let f, g, φ in $\mathcal{D}(\mathbb{R}^d)^W$. From (3.14) and Fubini’s theorem, we have

$$\begin{aligned} & \int_{\mathbb{R}^d} f *_{\mathcal{H}^W} g(x)\overline{\varphi(x)}\mathcal{A}_k(x)dx \\ &= \int_{\mathbb{R}^d} g(y) \left(\int_{\mathbb{R}^d} \mathcal{T}_x^W(f)(-y)\overline{\varphi(x)}\mathcal{A}_k(x)dx \right) \mathcal{A}_k(y)dy, \\ &= \int_{\mathbb{R}^d} \check{g}(y) \left(\int_{\mathbb{R}^d} \mathcal{T}_y^W(f)(x)\overline{\varphi(x)}\mathcal{A}_k(x)dx \right) \mathcal{A}_k(y)dy. \end{aligned}$$

By using Hölder’s inequality and (3.37), we obtain

$$\left| \int_{\mathbb{R}^d} f *_{\mathcal{H}^W} g(x)\overline{\varphi(x)}\mathcal{A}_k(x)dx \right| \leq |W|^{1/2}\|f\|_{\mathcal{A}_k,2}\|g\|_{\mathcal{A}_k,1}\|\varphi\|_{\mathcal{A}_k,2}. \tag{3.45}$$

As the relation (3.45) remain true for all functions g in $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and f, φ in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, thus we obtain (3.43). □

PROPOSITION 3.10. *Let f and g be in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$. Then the function $f *_{\mathcal{H}^W} g$ defined on \mathbb{R}^d by*

$$f *_{\mathcal{H}^W} g(x) = \int_{\mathbb{R}^d} \mathcal{T}_x^W(f)(-y)g(y)\mathcal{A}_k(y)dy,$$

is continuous on \mathbb{R}^d , tends to zero at the infinity and we have

$$\sup_{x \in \mathbb{R}^d} |f *_{\mathcal{H}^W} g(x)| \leq |W|^{1/2} \|f\|_{\mathcal{A}_k, 2} \|g\|_{\mathcal{A}_k, 2}. \tag{3.46}$$

Proof. Let $\{f_n\}_{n \in \mathbb{N}}$ and $\{g_n\}_{n \in \mathbb{N}}$ be two sequences in $\mathcal{D}(\mathbb{R}^d)^W$ which converge respectively to f and g in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$. By using the fact that the operator $\mathcal{T}_x^W, x \in \mathbb{R}^d$ is continuous from $\mathcal{D}(\mathbb{R}^d)^W$ into itself, we deduce that the sequence $\{f_n *_{\mathcal{H}^W} g_n\}_{n \in \mathbb{N}}$ which belongs to $\mathcal{D}(\mathbb{R}^d)^W$, converges to $\{f *_{\mathcal{H}^W} g\}$ uniformly on \mathbb{R}^d . Then, the function $f *_{\mathcal{H}^W} g$ is continuous on \mathbb{R}^d and tends to zero at the infinity. The Hölder's inequality and (3.37) imply the relation (3.46). \square

PROPOSITION 3.11. *Let f and g be in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, then the function $f *_{\mathcal{H}^W} g$ belongs to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ if and only if the function $\mathcal{H}^W(f) \cdot \mathcal{H}^W(g)$ is in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, and we have $\mathcal{H}^W(f *_{\mathcal{H}^W} g) = \mathcal{H}^W(f) \cdot \mathcal{H}^W(g)$, in the L^2 -case.*

To prove this Proposition, we need the following Lemma.

LEMMA 3.2. *For all f, g in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and all ψ in $S_2(\mathbb{R}^d)^W$, we have the following identity*

$$\int_{\mathbb{R}^d} f *_{\mathcal{H}^W} g(x) (\mathcal{H}^W)^{-1}(\psi)(x) \mathcal{A}_k(x) dx = \int_{\mathbb{R}^d} \mathcal{H}^W(f)(\lambda) \mathcal{H}^W(g)(\lambda) \psi(\lambda) \mathcal{C}_k^W(\lambda) d\lambda. \tag{3.47}$$

Proof. We denote by $Z_1(f)$ and $Z_2(f)$ respectively the first and the second member of the relation (3.47). From Theorem 3.2 and Proposition 3.9, we see that $Z_1(f) = Z_2(f)$ for all f in $(L^1_{\mathcal{A}_k} \cap L^2_{\mathcal{A}_k})(\mathbb{R}^d)^W$.

On the other hand, let f_n be in $(L^1_{\mathcal{A}_k} \cap L^2_{\mathcal{A}_k})(\mathbb{R}^d)^W$ such that

$$\lim_{n \rightarrow +\infty} \|f_n - f\|_{\mathcal{A}_k, 2} = 0. \tag{3.48}$$

By using Hölder's inequality, Theorem 3.2 and Proposition 3.8 we obtain

$$|Z_1(f_n) - Z_1(f)| \leq |W|^{\frac{1}{2}} \|f_n - f\|_{\mathcal{A}_k, 2} \|g\|_{\mathcal{A}_k, 2} \|(\mathcal{H}^W)^{-1}(\psi)\|_{\mathcal{A}_k, \infty},$$

and

$$|Z_2(f_n) - Z_2(f)| \leq \|f_n - f\|_{\mathcal{A}_k, 2} \|g\|_{\mathcal{A}_k, 2} \|\psi\|_{\mathcal{A}_k, \infty}.$$

Then, from (3.48) we get

$$\lim_{n \rightarrow +\infty} Z_1(f_n) = Z_1(f),$$

and

$$\lim_{n \rightarrow +\infty} Z_2(f_n) = Z_2(f).$$

We deduce the result from the density of $(L^1_{\mathcal{A}_k} \cap L^2_{\mathcal{A}_k})(\mathbb{R}^d)^W$ in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$.
□

Proof of Proposition 3.11

Suppose that the function $f *_{\mathcal{H}^W} g$ is in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$. By Lemma 3.2 and Theorem 3.2, for all ψ in $S_2(\mathbb{R}^d)^W$, we have

$$\int_{\mathbb{R}^d} \mathcal{H}^W(f)(\lambda) \mathcal{H}^W(g)(\lambda) \psi(\lambda) \mathcal{C}_k^W(\lambda) d\lambda = \int_{\mathbb{R}^d} \mathcal{H}^W(f *_{\mathcal{H}^W} g)(\lambda) \psi(\lambda) \mathcal{C}_k^W(\lambda) d\lambda,$$

which shows that

$$\mathcal{H}^W(f)(\lambda) \mathcal{H}^W(g)(\lambda) = \mathcal{H}^W(f *_{\mathcal{H}^W} g)(\lambda), \quad \lambda \in \mathbb{R}^d.$$

Conversely, if $\mathcal{H}^W(f) \cdot \mathcal{H}^W(g)$ belongs to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$. By Lemma 3.2 and Theorem 3.2, for all ψ in $S_2(\mathbb{R}^d)^W$, we have

$$\begin{aligned} & \int_{\mathbb{R}^d} f *_{\mathcal{H}^W} g(x) (\mathcal{H}^W)^{-1}(\psi)(x) \mathcal{A}_k(x) dx \\ &= \int_{\mathbb{R}^d} (\mathcal{H}^W)^{-1}(\mathcal{H}^W(f) \cdot \mathcal{H}^W(g))(x) (\mathcal{H}^W)^{-1}(\psi)(x) \mathcal{A}_k(x) dx, \end{aligned}$$

which implies that

$$f *_{\mathcal{H}^W} g(x) = (\mathcal{H}^W)^{-1}(\mathcal{H}^W(f) \cdot \mathcal{H}^W(g))(x), \quad x \in \mathbb{R}^d.$$

This achieves the proof of Proposition 3.11.

COROLLARY 3.4. For all f, g in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, we have

$$\int_{\mathbb{R}^d} |f *_{\mathcal{H}^W} g|^2 \mathcal{A}_k(x) dx = \int_{\mathbb{R}^d} |\mathcal{H}^W(f)(\lambda)|^2 |\mathcal{H}^W(g)(\lambda)|^2 \mathcal{C}_k^W(\lambda) d\lambda, \quad (3.49)$$

where both sides are finite or infinite.

Proof.

- When $f *_{\mathcal{H}^W} g$ is in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, we deduce (3.49) from Proposition 3.11 and the Plancherel formula (3.34).
- For the other case the sides of (3.49) are infinite. □

4. Generalized wavelets and the generalized wavelet transform on \mathbb{R}^d

4.1. Generalized wavelets on \mathbb{R}^d .

By using the harmonic analysis associated to the Heckman-Opdam theory given in the previous section, we define in this subsection, the generalized wavelets on \mathbb{R}^d and we study their properties on the space of W -invariant C^∞ -functions and on the space $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$.

DEFINITION 4.1. We say that a function g in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ is a generalized wavelet on \mathbb{R}^d , if there exists a constant C_g such that

- i) $0 < C_g < +\infty$.
- ii) For almost all $\lambda \in \mathbb{R}^d$, we have

$$C_g = \int_0^{+\infty} |\mathcal{H}^W(g)(a\lambda)|^2 \frac{da}{a}. \tag{4.1}$$

EXAMPLE 4.1. Let $t > 0$. We consider the function g defined by

$$\forall x \in \mathbb{R}^d, \quad g(x) = -\mathcal{L}_k^W E_t^W(x),$$

where \mathcal{L}_k^W is the Heckman-Opdam Laplacian defined, for a function f on \mathbb{R}^d of class C^2 and W -invariant, by

$$\mathcal{L}_k^W f = \sum_{j=1}^d T_j^2 f. \tag{4.2}$$

It has the following form : For $x \in \mathbb{R}_{reg}^d$,

$$\mathcal{L}_k^W f(x) = \Delta f(x) + \sum_{\alpha \in \mathcal{R}_+} k(\alpha) \coth\left(\frac{\langle \alpha, x \rangle}{2}\right) \langle \nabla f(x), \alpha \rangle + \|\rho\|^2 f(x),$$

where Δ and ∇ are respectively the Laplacian and the gradient on \mathbb{R}^d , and E_t^W , $t > 0$, the heat kernel given by

$$\forall x \in \mathbb{R}^d, \quad E_t^W(x) = \int_{\mathbb{R}^d} e^{-t(\|\lambda\|^2 + \|\rho\|^2)} F_\lambda(x) \mathcal{C}_k^W(\lambda) d\lambda. \tag{4.3}$$

By using (2.2),(2.3),(4.2),(4.3) we obtain

$$\forall x \in \mathbb{R}^d, \quad g(x) = \int_{\mathbb{R}^d} \|\lambda\|^2 e^{-t(\|\lambda\|^2 + \|\rho\|^2)} F_\lambda(x) \mathcal{C}_k^W(\lambda) d\lambda.$$

The function g belongs to $S_2(\mathbb{R}^d)^W$, and we have

$$\forall \lambda \in \mathbb{R}^d, \quad \mathcal{H}^W(g)(\lambda) = \|\lambda\|^2 e^{-t(\|\lambda\|^2 + \|\rho\|^2)}.$$

For $\lambda \in \mathbb{R}^d \setminus \{0\}$, we have

$$\begin{aligned} C_g &= \int_0^{+\infty} |\mathcal{H}^W(g)(a\lambda)|^2 \frac{da}{a} \\ &= e^{-2t\|\rho\|^2} \int_0^{+\infty} \|\lambda\|^4 e^{-2ta^2\|\lambda\|^2} a^3 da, \end{aligned}$$

By change of variables we obtain, for almost all $\lambda \in \mathbb{R}^d$:

$$C_g = \frac{e^{-2t\|\rho\|^2}}{8t^2}.$$

DEFINITION 4.2. We define the function l_k on $]0, +\infty[$ by

$$l_k(a) = \sup_{\lambda \in \mathbb{R}^d \setminus \{0\}} \frac{|\mathcal{C}_k^W(\frac{\lambda}{a})|}{|\mathcal{C}_k^W(\lambda)|} = \sup_{\lambda \in \mathbb{R}^d \setminus \{0\}} \frac{|c_k(\lambda)|^2}{|c_k(\frac{\lambda}{a})|^2}, \tag{4.4}$$

where \mathcal{C}_k^W and c_k the functions given by the relations (3.5),(3.6).

REMARK 4.1. When $k(\alpha) \in \mathbb{N}$, for all $\alpha \in \mathcal{R}$, the function l_k has the following form

$$l_k(a) = \sup_{\lambda \in \mathbb{R}^d \setminus \{0\}} \prod_{\alpha \in \mathcal{R}_+} \prod_{n=1}^{k(\alpha)} \frac{(\langle \lambda, \check{\alpha} \rangle)^2 + (\frac{1}{2}k(\frac{\alpha}{2}) + k(\alpha) - n)^2}{(\frac{1}{a}\langle \lambda, \check{\alpha} \rangle)^2 + (\frac{1}{2}k(\frac{\alpha}{2}) + k(\alpha) - n)^2}.$$

It satisfies the estimates

i) If $a \in [1, +\infty[$

$$0 < l_k(a) \leq a^{-2\gamma},$$

with γ defined by the relation (2.1).

ii) If $a \in]0, 1[$

$$0 < l_k(a) \leq \prod_{\alpha \in \mathcal{R}_+} k(\alpha).$$

THEOREM 4.1. Let $a > 0$ and g a generalized wavelet on \mathbb{R}^d in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$. Then,

i) The function $\lambda \rightarrow \mathcal{H}^W(g)(a\lambda)$ belongs to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, and we have

$$\int_{\mathbb{R}^d} |\mathcal{H}^W(g)(a\lambda)|^2 \mathcal{C}_k^W(\lambda) d\lambda \leq \frac{l_k(a)}{a^d} \|g\|_{\mathcal{A}_k, 2}^2, \tag{4.5}$$

where l_k is the function given by the relation (4.4).

ii) There exists a function g_a in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ such that

$$\mathcal{H}^W(g_a)(\lambda) = \mathcal{H}^W(g)(a\lambda), \quad \lambda \in \mathbb{R}^d, \quad (4.6)$$

and we have

$$\|g_a\|_{\mathcal{A}_k,2}^2 \leq \frac{l_k(a)}{a^d} \|g\|_{\mathcal{A}_k,2}^2. \quad (4.7)$$

Proof.

i) By change of variables and from the relation (4.4) we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} |\mathcal{H}^W(g)(a\lambda)|^2 \mathcal{C}_k^W(\lambda) d\lambda &= \frac{1}{a^d} \int_{\mathbb{R}^d} |\mathcal{H}^W(g)(\lambda)|^2 \mathcal{C}_k^W\left(\frac{\lambda}{a}\right) d\lambda \\ &\leq \frac{l_k(a)}{a^d} \int_{\mathbb{R}^d} |\mathcal{H}^W(g)(\lambda)|^2 \mathcal{C}_k^W(\lambda) d\lambda. \end{aligned}$$

We deduce (4.5) from this relation and the Plancherel formula (3.34).

ii) The relation (4.5), Theorem 3.2.ii) and the Plancherel formula (3.34) give the results. \square

Notation. We denote by H_a the dilatation operator defined on $S_2(\mathbb{R}^d)^W$ by

$$\forall x \in \mathbb{R}^d, \quad H_a(f)(x) = f(ax). \quad (4.8)$$

PROPOSITION 4.1.

i) Let g be in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $S_2(\mathbb{R}^d)^W$). Then, for $a > 0$, the function g_a belongs to $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $S_2(\mathbb{R}^d)^W$) and we have

$$\forall x \in \mathbb{R}^d, \quad g_a(x) = \frac{1}{a^d} ({}^tV_k^W)^{-1} \circ H_{a^{-1}} \circ {}^tV_k^W(g)(x). \quad (4.9)$$

ii) Let g be in $\mathcal{D}(\mathbb{R}^d)^W$ with support in the closed ball $B(0, R)$, of center 0 and radius R . Then for $a > 0$, the function g_a belongs to $\mathcal{D}(\mathbb{R}^d)^W$ with support in $B(0, aR)$.

Proof. i) From the relations (3.12),(4.6) we obtain

$$\forall x \in \mathbb{R}^d, \quad g_a(x) = ({}^tV_k^W)^{-1} \circ \mathcal{F}^{-1} \circ H_a \circ \mathcal{F} \circ {}^tV_k^W(g)(x), \quad (4.10)$$

where \mathcal{F} is the classical Fourier transform on \mathbb{R}^d .

On the other hand, by using (4.8) and by making changes of variables, we obtain

$$\forall x \in \mathbb{R}^d, \quad \mathcal{F}^{-1} \circ H_a \circ \mathcal{F}(f)(x) = \frac{1}{a^d} H_{a^{-1}}(f)(x). \quad (4.11)$$

We deduce (4.9) from (4.10),(4.11).

ii) Let g be in $\mathcal{D}(\mathbb{R}^d)^W$ with support in $B(0, R)$. From Theorem 3.1.ii), the function $\mathcal{H}^W(g)$ belongs to $PW_R(\mathbb{C}^d)^W$. Then

$$\forall m \in \mathbb{N}, \quad \sup_{\lambda \in \mathbb{C}^d} (1 + \|\lambda\|)^m e^{-R\|Im\lambda\|} |\mathcal{H}^W(g)(\lambda)| < +\infty.$$

Thus, from this relation and (4.6) we obtain

$$\forall m \in \mathbb{N}, \quad \sup_{\lambda \in \mathbb{C}^d} (1 + \|\lambda\|)^m e^{-aR\|Im\lambda\|} |\mathcal{H}^W(g)(\lambda)| < +\infty.$$

Then, we deduce the result from this relation and Theorem 3.1.ii). \square

PROPOSITION 4.2. *Let g be a generalized wavelet on \mathbb{R}^d in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$. Then, for $a > 0$ and $b \in \mathbb{R}^d$, the function*

$$g_{a,b}(x) = \mathcal{T}_b^W g_a(x), \quad x \in \mathbb{R}^d, \quad (4.12)$$

is in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, and we have

$$C_{g_{a,b}} \leq |W|C_g. \quad (4.13)$$

Proof. As the function g is in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, then from the relation (4.12) and Proposition 3.7.i), the function $b \rightarrow g_{a,b}$ is in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$. Thus, the relations (3.36),(4.6) imply

$$\mathcal{H}^W(g_{a,b})(\lambda) = F_\lambda(b)\mathcal{H}^W(g)(a\lambda), \quad \lambda \in \mathbb{R}^d. \quad (4.14)$$

From (4.14) and Definition 4.1, we have for almost $\lambda \in \mathbb{R}^d$:

$$C_{g_{a,b}} = |F_\lambda(b)|^2 \int_0^{+\infty} |\mathcal{H}^W(g)(a_0\lambda)|^2 \frac{da_0}{a_0}. \quad (4.15)$$

We deduce (4.13) from the relations (4.15),(4.1),(2.5). \square

PROPOSITION 4.3. *Let g be in $\mathcal{D}(\mathbb{R}^d)^W$ with support in the closed ball $B(0, R)$, of center 0 and radius $R > 0$. Then, for $a > 0$ and $b \in \mathbb{R}^d$, the function $g_{a,b}$ belongs to $\mathcal{D}(\mathbb{R}^d)^W$ with support in $B(0, aR + \|b\|)$.*

Proof. As the function g belongs to $\mathcal{D}(\mathbb{R}^d)^W$ then, from (4.12),(4.14), we obtain

$$\forall \lambda \in \mathbb{C}^d, \quad \mathcal{H}^W(g_{a,b})(\lambda) = F_\lambda(b)\mathcal{H}^W(g_a)(\lambda). \quad (4.16)$$

But from (2.4),(2.5) there exists a positive constant M such that for all $\lambda \in \mathbb{C}^d$ and $b \in \mathbb{R}^d$ we have

$$|F_\lambda(b)| \leq MF_0(b)e^{\|b\|\|Im\lambda\|}, \quad (4.17)$$

and

$$|F_0(b)| \leq |W|^{\frac{1}{2}}. \tag{4.18}$$

As from Proposition 4.1.ii), the function g_a is in $\mathcal{D}(\mathbb{R}^d)^W$, then, from Theorem 3.1.i) and the relations (4.16),(4.17),(4.18), the function $\mathcal{H}^W(g_{a,b})$ belongs to $PW(\mathbb{C}^d)^W$ and then we have

$$\forall m \in \mathbb{N}, \sup_{\lambda \in \mathbb{C}^d} (1 + \|\lambda\|)^m e^{-(aR + \|b\|)\|Im\lambda\|} |\mathcal{H}^W(g_{a,b})(\lambda)| < +\infty.$$

This relation and Theorem 3.1.ii) imply that the function $g_{a,b}$ is in $\mathcal{D}(\mathbb{R}^d)^W$, and

$$supp\ g_{a,b} \subset B(0, aR + \|b\|).$$

□

COROLLARY 4.1. *Let g be a generalized wavelet on \mathbb{R}^d in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $S_2(\mathbb{R}^d)^W$). Then, for $a > 0$ and $b \in \mathbb{R}^d$, the function $g_{a,b}$ belongs to $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $S_2(\mathbb{R}^d)^W$).*

Proof. We deduce the result from Propositions 4.1 and 4.2. □

PROPOSITION 4.4. *Let g be a generalized wavelet on \mathbb{R}^d in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ such that $\mathcal{H}^W(g)$ belongs to $L^1_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$. Then, the mapping $(a, b) \longrightarrow g_{a,b}$ is continuous from $]0, +\infty[\times \mathbb{R}^d$ into $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$.*

Proof. From the density of $\mathcal{D}(\mathbb{R}^d)^W$ in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, it suffices to consider the case where g is in $\mathcal{D}(\mathbb{R}^d)^W$ with support in the closed ball $B(0, R)$, of center 0 and radius $R > 0$. Let $(a_0, b_0) \in]0, +\infty[\times \mathbb{R}^d$, from Proposition 4.3, there exists $R_0 > 0$ such that for $0 < a_0 < a$ and $b_0, b \in \mathbb{R}^d$ such that

$$|a - a_0| < 1, \quad \|b - b_0\| < 1, \quad \text{we have } supp\ g_{a,b} \subset B(0, R_0).$$

Then,

$$\|g_{a,b} - g_{a_0,b_0}\|_{\mathcal{A}_k,2}^2 \leq \left(\int_{B(0,R_0)} \mathcal{A}_k(x) dx \right) esssup_{x \in B(0,R_0)} |g_{a,b}(x) - g_{a_0,b_0}(x)|^2. \tag{4.19}$$

On the other hand, from (3.20),(3.28) we have

$$g_{a,b}(x) = \int_{\mathbb{R}^d} F_\lambda(x) F_\lambda(b) \mathcal{H}^W(g)(a\lambda) \mathcal{C}_k^W(\lambda) d\lambda, \quad a.e. \ x \in \mathbb{R}^d.$$

By using this relation and (2.5), we obtain for $x \in \mathbb{R}^d$:

$$|g_{a,b}(x) - g_{a_0,b_0}(x)| \leq |W|^{\frac{1}{2}} \int_{\mathbb{R}^d} |F_\lambda(b)\mathcal{H}^W(g)(a\lambda) - F_\lambda(b_0)\mathcal{H}^W(g)(a_0\lambda)| \mathcal{C}_k^W(\lambda) d\lambda. \tag{4.20}$$

As g is in $\mathcal{D}(\mathbb{R}^d)^W$ and the function $\mathcal{H}^W(g)(\lambda)$ is in $PW(\mathbb{C}^d)^W$ then, there exists a positive constant c_0 such that

$$\forall \Lambda \in \mathbb{R}^d, \quad |\mathcal{H}^W(g)(\Lambda)| \leq \frac{c_0}{(1 + \|\Lambda\|)^{s+d+1}}, \tag{4.21}$$

with s the constant given by (3.7).

From the relation (4.21), we deduce that

$$\forall \lambda \in \mathbb{R}^d, \quad |\mathcal{H}^W(g)(a\lambda)| \leq \frac{c_0}{(1 + \|a\lambda\|)^{s+d+1}},$$

and

$$\forall \lambda \in \mathbb{R}^d, \quad |\mathcal{H}^W(g)(a_0\lambda)| \leq \frac{c_0}{(1 + \|a_0\lambda\|)^{s+d+1}}.$$

As for $0 < a_0 < a$ we have

$$\forall \lambda \in \mathbb{R}^d, \quad \frac{1}{(1 + \|a\lambda\|)^{s+d+1}} < \frac{1}{(1 + \|a_0\lambda\|)^{s+d+1}}. \tag{4.22}$$

Then, from (2.5),(4.21),(4.22),(3.7), there exists a positive constant M such that

$$|F_\lambda(b)\mathcal{H}^W(g)(a\lambda) - F_\lambda(b_0)\mathcal{H}^W(g)(a_0\lambda)| \mathcal{C}_k^W(\lambda) \leq \frac{2M|W|}{(1 + a_0\|\lambda\|)^{d+1}},$$

with

$$\int_{\mathbb{R}^d} \frac{1}{(1 + a_0\|\lambda\|)^{d+1}} d\lambda < +\infty.$$

Thus, from the dominated convergence theorem we obtain

$$\lim_{(a,b) \rightarrow (a_0,b_0)} \int_{\mathbb{R}^d} |F_\lambda(b)\mathcal{H}^W(g)(a\lambda) - F_\lambda(b_0)\mathcal{H}^W(g)(a_0\lambda)| \mathcal{C}_k^W(\lambda) d\lambda = 0. \tag{4.23}$$

On the other hand, from (4.20) we have

$$\begin{aligned} & \text{esssup}_{x \in B(0,R_0)} |g_{a,b}(x) - g_{a_0,b_0}(x)|^2 \\ & \leq |W| \left(\int_{\mathbb{R}^d} |F_\lambda(b)\mathcal{H}^W(g)(a\lambda) - F_\lambda(b_0)\mathcal{H}^W(g)(a_0\lambda)| \mathcal{C}_k^W(\lambda) d\lambda \right)^2. \end{aligned}$$

Thus, the relation (4.23) implies that

$$\lim_{(a,b) \rightarrow (a_0,b_0)} \operatorname{ess\,sup}_{x \in B(0,R_0)} |g_{a,b}(x) - g_{a_0,b_0}(x)|^2 = 0.$$

Then, from this relation and (4.19) we obtain

$$\lim_{(a,b) \rightarrow (a_0,b_0)} \|g_{a,b} - g_{a_0,b_0}\|_{\mathcal{A}_k,2} = 0.$$

□

4.2. Generalized wavelet transform on \mathbb{R}^d .

With the aid of the results of the previous section, we define and study the generalized wavelet transform, we give some of its properties and we prove for it, Plancherel and inversion formulas.

DEFINITION 4.3. The generalized wavelet transform Φ_g on \mathbb{R}^d is defined for f in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ by

$$\Phi_g(f)(a, b) = \int_{\mathbb{R}^d} f(x) \overline{g_{a,b}(x)} \mathcal{A}_k(x) dx, \quad (a, b) \in]0, +\infty[\times \mathbb{R}^d.$$

We can also write it in the form

$$\Phi_g(f)(a, b) = \check{f} *_{\mathcal{H}^W} \overline{g_a(b)}, \tag{4.24}$$

where \check{f} is the function defined by

$$\check{f}(x) = f(-x), \quad x \in \mathbb{R}^d.$$

PROPOSITION 4.5.

i) For f in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, the function $b \rightarrow \Phi_g(f)(a, b)$ is continuous on \mathbb{R}^d , tends to zero at infinity and we have

$$\sup_{b \in \mathbb{R}^d} |\Phi_g(f)(a, b)| \leq \left(\frac{l_k(a)|W|}{a^d} \right)^{\frac{1}{2}} \|f\|_{\mathcal{A}_k,2} \|g\|_{\mathcal{A}_k,2}.$$

ii) For f in $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$, the function $b \rightarrow \Phi_g(f)(a, b)$ is defined almost everywhere on \mathbb{R}^d , belongs to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and we have

$$\|\Phi_g(f)(a, \cdot)\|_{\mathcal{A}_k,2} \leq \left(\frac{l_k(a)|W|}{a^d} \right)^{\frac{1}{2}} \|f\|_{\mathcal{A}_k,1} \|g\|_{\mathcal{A}_k,2}. \tag{4.25}$$

iii) If g is in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $S_2(\mathbb{R}^d)^W$), then for f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $S_2(\mathbb{R}^d)^W$), the function $b \rightarrow \Phi_g(a, b)$ belongs to $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $S_2(\mathbb{R}^d)^W$).

Proof. i) As f and g are in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, then from Proposition 3.10, the mapping $b \rightarrow \Phi_g(f)(a, b) = f *_{\mathcal{H}^W} \overline{g_a}(b)$ is continuous on \mathbb{R}^d , tends to zero at infinity and we have

$$\sup_{b \in \mathbb{R}^d} |\Phi_g(f)(a, b)| \leq |W|^{\frac{1}{2}} \|f\|_{\mathcal{A}_k, 2} \|g_a\|_{\mathcal{A}_k, 2}.$$

This relation and (4.7) give the relation (4.25).

ii) As f is in $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and g is in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, we obtain the results from the relation (4.24), Proposition 3.9 and the relation (4.7).

iii) We deduce the result from Proposition 4.1.ii) and Corollary 3.1 ii). \square

PROPOSITION 4.6. *Let f be in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and g in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ such that $\mathcal{H}^W(g)$ belongs to $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$. Then, the mapping $(a, b) \rightarrow \Phi_g(f)(a, b)$ is continuous on $]0, +\infty[\times \mathbb{R}^d$.*

Proof. Let $(a_0, b_0) \in]0, +\infty[\times \mathbb{R}^d$. From Definition 4.3, for $(a, b) \in]0, +\infty[\times \mathbb{R}^d$, we have

$$|\Phi_g(f)(a, b) - \Phi_g(f)(a_0, b_0)| \leq \int_{\mathbb{R}^d} |f(x)| |\overline{g_{a,b}(x)} - \overline{g_{a_0, b_0}(x)}| \mathcal{A}_k(x) dx.$$

By using Hölder inequality, we obtain

$$|\Phi_g(f)(a, b) - \Phi_g(f)(a_0, b_0)| \leq \|f\|_{\mathcal{A}_k, 2} \|g_{a,b} - g_{a_0, b_0}\|_{\mathcal{A}_k, 2}.$$

We deduce the result from Proposition 4.4. \square

THEOREM 4.2. (Plancherel formulas) *Let g be a generalized wavelet on \mathbb{R}^d in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$.*

i) *For f in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, we have*

$$\|f\|_{\mathcal{A}_k, 2}^2 = \frac{1}{C_g} \int_{\mathbb{R}^d} \int_0^{+\infty} |\Phi_g(f)(a, b)|^2 \frac{da}{a} \mathcal{A}_k(b) db. \quad (4.26)$$

ii) *For all f_1, f_2 in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, we have*

$$\int_{\mathbb{R}^d} f_1(x) \overline{f_2(x)} \mathcal{A}_k(x) dx = \frac{1}{C_g} \int_{\mathbb{R}^d} \int_0^{+\infty} \Phi_g(f_1)(a, b) \overline{\Phi_g(f_2)(a, b)} \frac{da}{a} \mathcal{A}_k(b) db. \quad (4.27)$$

Proof. i) From Definition 4.3 and Fubini-Tonelli's theorem, we have

$$\begin{aligned} & \frac{1}{C_g} \int_{\mathbb{R}^d} \int_0^{+\infty} |\Phi_g(f)(a, b)|^2 \frac{da}{a} \mathcal{A}_k(b) db \\ &= \frac{1}{C_g} \int_0^{+\infty} \left(\int_{\mathbb{R}^d} |\check{f} *_{\mathcal{H}^W} \overline{g_a(b)}|^2 \mathcal{A}_k(b) db \right) \frac{da}{a}. \end{aligned}$$

From Corollary 3.4, we deduce that

$$\begin{aligned} & \frac{1}{C_g} \int_{\mathbb{R}^d} \int_0^{+\infty} |\Phi_g(f)(a, b)|^2 \frac{da}{a} \mathcal{A}_k(b) db \\ &= \frac{1}{C_g} \int_0^{+\infty} \left(\int_{\mathbb{R}^d} |\mathcal{H}^W(\check{f})(\lambda)|^2 |\mathcal{H}^W(\overline{g_a})(\lambda)|^2 \mathcal{C}_k^W(\lambda) d\lambda \right) \frac{da}{a}. \end{aligned}$$

Then, from the relation (3.4) and Fubini-Tonelli's theorem, we get

$$\begin{aligned} & \frac{1}{C_g} \int_{\mathbb{R}^d} \int_0^{+\infty} |\Phi_g(f)(a, b)|^2 \frac{da}{a} \mathcal{A}_k(b) db \\ &= \int_{\mathbb{R}^d} |\mathcal{H}^W(f)(\lambda)|^2 \left(\frac{1}{C_g} \int_0^{+\infty} |\mathcal{H}^W(\overline{g_a})(-\lambda)|^2 \frac{da}{a} \right) \mathcal{C}_k^W(\lambda) d\lambda. \end{aligned} \tag{4.28}$$

On the other hand, by using the fact that

$$\forall x \in \mathbb{R}^d, \forall \lambda \in \mathbb{R}^d, \quad \overline{F_{-\lambda a}(x)} = F_{\lambda a}(x),$$

and by applying (3.3) we obtain for almost all $\lambda \in \mathbb{R}^d$:

$$\int_{\mathbb{R}^d} |\mathcal{H}^W(\overline{g_a})(-\lambda)|^2 \frac{da}{a} = \int_{\mathbb{R}^d} |\overline{\mathcal{H}^W(g)(a\lambda)}|^2 \frac{da}{a}.$$

Thus, from the relation (4.1), we have

$$\int_{\mathbb{R}^d} |\mathcal{H}^W(\overline{g_a})(-\lambda)|^2 \frac{da}{a} = \int_{\mathbb{R}^d} |\mathcal{H}^W(g)(a\lambda)|^2 \frac{da}{a} = C_g. \tag{4.29}$$

Then, the relation (4.26) follows from the relations (4.28), (4.29) and Plancherel formula (3.34).

ii) We deduce the result from the i). □

THEOREM 4.3. (Inversion formula) *Let g be a generalized wavelet on \mathbb{R}^d in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$. For f in $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W \cap L^\infty_{\mathcal{A}_k}(\mathbb{R}^d)^W$ continuous and*

such that $\mathcal{H}^W(f)$ belongs to $L^1_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$, we have the following inversion formula for the transform Φ_g :

$$f(x) = \frac{1}{C_g} \int_0^{+\infty} \left(\int_{\mathbb{R}^d} \Phi_g(f)(a, b) g_{a,b}(x) \mathcal{A}_k(b) db \right) \frac{da}{a}, \quad x \in \mathbb{R}^d, \quad (4.30)$$

where, for each $x \in \mathbb{R}^d$, both the inner integral and the outer integral are absolutely convergent, but possible not the double integral.

Proof. We put

$$i(a, x) = \int_{\mathbb{R}^d} \Phi_g(f)(a, b) g_{a,b}(x) \mathcal{A}_k(b) db,$$

and

$$I(x) = \frac{1}{C_g} \int_0^{+\infty} i(a, x) \frac{da}{a}.$$

First, we shall prove that, for each $x \in \mathbb{R}^d$, the integrals $i(a, x)$ and $I(x)$ are absolutely convergent, and we have

$$I(x) = \int_{\mathbb{R}^d} \mathcal{H}^W(f)(\lambda) F_\lambda(x) \mathcal{C}_k^W(\lambda) d\lambda. \quad (4.31)$$

As f is in $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and g_a in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, then from Proposition 4.2 and Definition 4.3, for $b \in \mathbb{R}^d$, we have

$$\Phi_g(f)(a, b) g_{a,b}(x) = \check{f} *_{\mathcal{H}^W} \overline{g_a}(b) \mathcal{T}_x^W(g_a)(b).$$

Proposition 3.9 and the relation (3.37) imply that the functions $b \rightarrow \check{f} *_{\mathcal{H}^W} \overline{g_a}(b)$ and $b \rightarrow \mathcal{T}_x^W(g_a)(b)$ belong to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$. Then, Hölder’s inequality shows that the integral $i(a, x)$ is absolutely convergent.

On the other hand, from (3.40), the Plancherel formula (3.33), Proposition 3.9 and the relations (3.36),(3.4),(3.3), we obtain

$$\begin{aligned}
 i(a, x) &= \int_{\mathbb{R}^d} \check{f} *_{\mathcal{H}^W} \overline{g_a}(b) \mathcal{T}_x^W(g_a)(b) \mathcal{A}_k(b) db \\
 &= \int_{\mathbb{R}^d} \mathcal{H}^W(\check{f} *_{\mathcal{H}^W} \overline{g_a})(\lambda) \overline{\mathcal{H}^W(\mathcal{T}_x^W(\overline{g_a}))(\lambda)} \mathcal{C}_k^W(\lambda) d\lambda \\
 &= \int_{\mathbb{R}^d} \mathcal{H}^W(\check{f} *_{\mathcal{H}^W} \overline{g_a})(\lambda) \overline{F_\lambda(x)} \overline{\mathcal{H}^W(\overline{g_a})(\lambda)} \mathcal{C}_k^W(\lambda) d\lambda \\
 &= \int_{\mathbb{R}^d} \mathcal{H}^W(\check{f})(\lambda) \mathcal{H}^W(\overline{g_a})(\lambda) F_{-\lambda}(x) \overline{\mathcal{H}^W(\overline{g_a})(\lambda)} \mathcal{C}_k^W(\lambda) d\lambda \\
 &= \int_{\mathbb{R}^d} \mathcal{H}^W(\check{f})(\lambda) \overline{\mathcal{H}^W(\check{g}_a)(\lambda)} \mathcal{H}^W(\check{g}_a)(\lambda) F_{-\lambda}(x) \mathcal{C}_k^W(\lambda) d\lambda.
 \end{aligned}$$

Then,

$$i(a, x) = \int_{\mathbb{R}^d} \mathcal{H}^W(f)(\lambda) |\mathcal{H}^W(g_a)(\lambda)|^2 F_\lambda(x) \mathcal{C}_k^W(\lambda) d\lambda. \tag{4.32}$$

Thus, from Fubini-Tonelli's theorem and the relation (2.5), we get

$$\begin{aligned}
 &\frac{1}{C_g} \int_0^{+\infty} |i(a, x)| \frac{da}{a} \\
 &\leq |W|^{\frac{1}{2}} \int_{\mathbb{R}^d} |\mathcal{H}^W(f)(\lambda)| \left(\frac{1}{C_g} \int_0^{+\infty} |\mathcal{H}^W(g_a)(\lambda)|^2 \frac{da}{a} \right) \mathcal{C}_k^W(\lambda) d\lambda.
 \end{aligned}$$

But, from Definition 4.1, for almost all $\lambda \in \mathbb{R}^d$, we have

$$\frac{1}{C_g} \int_0^{+\infty} |\mathcal{H}^W(g_a)(\lambda)|^2 \frac{da}{a} = 1. \tag{4.33}$$

Then,

$$\frac{1}{C_g} \int_0^{+\infty} |i(a, x)| \frac{da}{a} \leq |W|^{\frac{1}{2}} \|\mathcal{H}^W(f)\|_{\mathcal{C}_k^W, 1} < +\infty. \tag{4.34}$$

This inequality implies that the integral $I(x)$ is absolutely convergent. We prove now the relation (4.31). From the relation (4.32), we have

$$I(x) = \frac{1}{C_g} \int_0^{+\infty} \int_{\mathbb{R}^d} \mathcal{H}^W(f)(\lambda) |\mathcal{H}^W(g_a)(\lambda)|^2 F_\lambda(x) \mathcal{C}_k^W(\lambda) d\lambda \frac{da}{a}.$$

First, we apply Fubini-Tonelli's theorem to the second member and we use the relation (4.34). Next, we apply Fubini's theorem and we obtain

$$I(x) = \int_{\mathbb{R}^d} \mathcal{H}^W(f)(\lambda) \left(\frac{1}{C_g} \int_0^{+\infty} |\mathcal{H}^W(g_a)(\lambda)|^2 \frac{da}{a} \right) F_\lambda(x) \mathcal{C}_k^W(\lambda) d\lambda.$$

We deduce the relation (4.31) from (4.33).

As the function f belongs to $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W \cap L^\infty_{\mathcal{A}_k}(\mathbb{R}^d)^W$ then, the inversion formula (3.35) and Remark 3.6 imply the relation (4.30). \square

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