Korean J. Math. **24** (2016), No. 2, pp. 297–305
http://dx.doi.org/10.11568/kjm.2016.24.2.297

A FIXED POINT APPROACH TO THE STABILITY OF THE FUNCTIONAL EQUATION RELATED TO DISTANCE MEASURES

Hwan-Yong Shin† and Gwang Hui Kim*

Abstract. In this paper, by using fixed point theorem, we obtain the stability of the following functional equations

$$f(pr, qs) + g(ps, qr) = \theta(p, q, r, s)f(p, q)h(r, s)$$
$$f(pr, qs) + g(ps, qr) = \theta(p, q, r, s)g(p, q)h(r, s),$$

where G is a commutative semigroup, $\theta : G^4 \to \mathbb{R}_k$ a function and f, g, h are functionals on G^2.

1. Introduction

Let $I = (0, 1)$ denote the open unit interval and $\mathbb{R}$ and $\mathbb{C}$ be the set of real and complex numbers, respectively, $\mathbb{R}_+ = \{x \in \mathbb{R} \,|\, x > 0\}$ and $\mathbb{R}_k = \{x \in \mathbb{R} \,|\, x > k > 1\}$ be a set of positive real numbers.

In [2], Chung, Kannappan, Ng and Sahoo characterized symmetrically compositive sum-form distance measures with a measurable generating function. The following functional equation

$$f(pr, qs) + f(ps, qr) = f(p, q)\, f(r, s) \qquad (FE)$$

Received February 11, 2016. Revised May 13, 2016. Accepted May 17, 2016.
2010 Mathematics Subject Classification: 39B82, 39B52.
Key words and phrases: distance measure, superstability, stability of functional equation, fixed point theorem.
†The first author of this work was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant number: 2015R1D1A1A01058083).
∗Corresponding Author.

holding for all $p, q, r, s \in I$ was instrumental in the characterization of symmetrically compositive sum-form distance measures.

They obtained that the general solution of equation (FE) is represented by $f(p,q) = M_1(p)\, M_2(q) + M_1(q)\, M_2(p)$ where $M_1, M_2 : \mathbb{R} \to \mathbb{C}$ are multiplicative functions. Further, either M_1 and M_2 are both real or M_2 is the complex conjugate of M_1. The converse is also true.

The stability of the functional equation (FE), as well as the four generalizations of (FE), namely,

$$f(pr, qs) + f(ps, qr) = f(p, q)g(r, s), \qquad (FE_{fg})$$
$$f(pr, qs) + f(ps, qr) = g(p, q)f(r, s), \qquad (FE_{gf})$$
$$f(pr, qs) + f(ps, qr) = g(p, q)g(r, s), \qquad (FE_{gg})$$
$$f(pr, qs) + f(ps, qr) = g(p, q)h(r, s) \qquad (FE_{gh})$$
$$f(pr, qs) + g(ps, qr) = h(p, q)k(r, s) \qquad (FE_{fghk})$$

for all $p, q, r, s \in G$, were studied by Kim and Sahoo in ([16], [17]). For other functional equations similar to (FE), the interested reader should refer to [5], [6], [20], [21], [22]

It should be noted that many well known functional equations like dAlembert functional equation,Wilson functional equation, Jensen functional equation can be obtained from the functional equation (FE_{fghk}). For instance, letting $r = s = 1$ in (FE_{fghk}), one obtains the equation

$$f(p, q) + g(p, q) = k(1, 1)h(p, q) \qquad (1.1)$$

When $f(p, q) = (p + q), g(p, q) = (pq)$, and $k(1, 1)h(p, q) = 2(p)(q)$, then the equation (1.1) yields the well known dAlembert functional equation. Similarly, when $f(p, q) = (p+q), g(p, q) = (pq)$, and $k(1, 1)h(p, q) = (p)(q)$, then (1.1) yields the Wilson functional equation. Letting $f(p, q) = (p+q), g(p, q) = (pq)$, and $k(1, 1)h(p, q) = 2(p)$ it is easy to see that (1.1) reduces to Jensen functional equation. For stability of related functional equations, see papers ([7], [8], [9], [10], [12], [11], and [13]).

The superstability for some functional equations of the compositive function form on two variables is found in [1]

In papers [19], Lee and Kim investigates the superstability of the generalized functional equation of (FE) as following:

$$\sum_{i=0}^{n-1} f(P \cdot \sigma_i(Q)) = f(P)f(Q), \tag{FFE}$$

$$\sum_{i=0}^{n-1} f(P \cdot \sigma_i(Q)) = g(P)h(Q).$$

where f is an measure, P and Q in set of n-ary discrete complete probability, and σ_i is a permutation for each $i = 0, 1, \cdots, n-1$.

J. Tabor [24] investigated the cocycle property, that is, θ is a cocycle which satisfies $\theta(a, bc) + \theta(b, c) = \theta(ab, c) + \theta(a, b)$,

In papers([14], [18]), Kim and Lee investigates the superstability of the generalized characterization of symmetrically compositive sum-form related to distance measures with a cocycle property:

$$f(pr, qs) + f(ps, qr) = \theta(pq, rs)\, f(p, q)\, f(r, s) \tag{CDM}$$

$$f(pr, qs) + g(ps, qr) = \theta(pq, rs)h(p, q)k(r, s),$$

for all $p, q, r, s \in G$ and f is functionals on G^2, θ is a cocycle, which can be represented as exponential functional equation in reduction.

For examples, if $f(x, y) = \frac{1}{x} + \frac{1}{y}$ and $\theta(x, y) = 2$, then $f(pr, qs) + f(ps, qr) = f(p, q)\, f(r, s)$, and also if $f(x, y) = a^{\ln xy}$, and $\theta(x, y) = 2$ then f, θ satisfy the equation $f(pr, qs) + f(ps, qr) = \theta(pq, rs)\, f(p, q)\, f(r, s)$.

This paper aims to investigate the stability of the following equations as the general mapping without the cocycle condition of θ by using fixed point theorem:

$$f(pr, qs) + g(ps, qr) = \theta(p, q, r, s)f(p, q)h(r, s) \tag{GFE_1}$$

$$f(pr, qs) + g(ps, qr) = \theta(p, q, r, s)g(p, q)h(r, s) \tag{GFE_2}$$

In fact, if $f, g, h : (0, \infty) \to \mathbb{R}$, $\theta : \mathbb{R}_+^4 \to \mathbb{R}_k$ be functions such that $f(p, q) = g(p, q) = h(p, q) = \big(\dfrac{1}{p} + \dfrac{1}{q}\big)^2$ and $\theta(p, q, r, s) = \dfrac{(pr + qs)^2 + (ps + qr)^2}{(pr + qs + ps + qr)^2}$, then f, g, h satisfy above equations.

We now introduce one of the fundamental results of fixed point theory by J. B. Diaz and B. Margolis [3], which is using as main tools for proofs of the stability of the functional equation.

Fixed Point Theorem 1. *Suppose we are given a complete generalized metric space (X, d) and a strictly contractive mapping $J : X \to X$, with the Lipschitz constant L. Then, for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*

(a) *$d(J^n x, J^{n+1}x) < \infty$, for all $n \geq n_0$;*
(b) *The sequence $(J^n x)$ is convergent to a fixed point y^* of J;*
(c) *y^* is the unique fixed point of J in the set $Y = \{y \in X \quad | \quad d(J^{n_0}y, y) < \infty\}$;*
(d) *$d(y, y^*) \leq \dfrac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

2. Stability of the equations (GFE)

Let $(G, \cdot)$ be a commutative semigroup. We will construct a strictly contractive mapping with the Lipschitz constant satisfying Fixed Point Theorem in introduction.

Theorem 1. *Let $h, \phi : G^2 \to \mathbb{R}$ be functions and $r, s \in G$ be arbitrary fixed elements such that $|h(r, s)| \geq M > \dfrac{L}{k} > 0$ and $\phi(pr, qs) \leq L\phi(p, q)$ for $p, q \in G$. If $f, g, h : G^2 \to \mathbb{R}$ be functions such that*

$$|f(pr, qs) + g(ps, qr) - \theta(p, q, r, s)g(p, q)h(r, s)| \leq \phi(p, q) \; \forall \; p, q \in G, \tag{2.1}$$

then there exists a unique function g_0 satisfying $f(pr, qs) + g_0(ps, qr) = \theta(p, q, r, s)g_0(p, q)h(r, s)$ and

$$|g(p, q) - g_0(p, q)| \leq \frac{\phi(p, q)}{kM - L} \tag{2.2}$$

for all $p, q \in G$.

Proof. First, we define a set

$$X = \{y : G^2 \to \mathbb{R}\}$$

and introduce a generalized metric on X as follows:

$$d(y_1, y_2) = inf\{C \in [0, \infty)||y_1(p, q) - y_2(p, q)| \leq C\phi(p, q), \forall p, g \in G\} \tag{2.3}$$

(Here, we give a proof for the triangle inequality. Assume that $d(y_1, y_3) > d(y_1, y_2) + d(y_2, y_3)$ would hold for some $y_1, y_2, y_3 \in X$. Then, there should exist an $(p_0, q_0) \in G^2$ with

$$\begin{aligned} |y_1(p_0, q_0) - y_3(p_0, q_0)| &> \{d(y_1, y_2) + d(y_2, y_3)\}\phi(p_0, q_0) \\ &= d(y_1, y_2)\phi(p_0, q_0) + d(y_2, y_3)\phi(p_0, q_0) \end{aligned}$$

In view of (2.3), this inequality would yield

$$|y_1(p_0, q_0) - y_3(p_0, q_0)| > |y_1(p_0, q_0) - y_2(p_0, q_0)| + |y_2(p_0, q_0) - y_3(p_0, q_0)|$$

a contradiction.)

Our task is to show that (X, d) is complete. Let $\{y_n\}$ be a Cauchy sequence in (X, d). Then, for any $\varepsilon > 0$ there exists an integer $N_\varepsilon > 0$ such that $d(y_m, y_n) \leq \varepsilon$ for all $m, n \geq N_\varepsilon$. In view of (2.3), we have

$$\begin{aligned} &\forall \varepsilon > 0 \quad \exists N_\varepsilon \in \mathbb{N} \quad \forall m, n \geq N_\varepsilon \\ &\qquad \forall (p, q) \in G^2 : |y_m(p, q) - y_n(p, q)| \leq \varepsilon\phi(p, q). \end{aligned} \tag{2.4}$$

If (p, q) is fixed, (2.4) implies that $\{y_n(p, q)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\{y_n(p, q)\}$ converges for each $(p, q) \in G^2$. Thus, we can define a function $y : G^2 \to \mathbb{R}$ by

$$y(p, q) := \lim_{n\to\infty} y_n(p, q).$$

Since this definition is well defined, we have $y \in X$.

If we let m increase to infinity, in follows from (2.4) that

$$\begin{aligned} &\forall \varepsilon > 0 \quad \exists N_\varepsilon \in \mathbb{N} \quad \forall m, n \geq N_\varepsilon \\ &\qquad \forall (p, q) \in G^2 : |y(p, q) - y_n(p, q)| \leq \varepsilon\phi(p, q). \end{aligned} \tag{2.5}$$

By considering (2.3), we get

$$\forall \varepsilon > 0 \quad \exists N_\varepsilon \in \mathbb{N} \quad \forall n \geq N_\varepsilon : d(y, y_n) \leq \varepsilon.$$

This means that the Cauchy sequence $\{y_n\}$ converges to y in (X, d). Hence, (X, d) is complete.

Let $(r, s) \in G^2$ be an arbitrary fixed element. We now define an operator $\Lambda : X \to X$

$$(\Lambda y)(p, q) := \frac{f(pr, qs) + y(ps, qr)}{\theta(p, q, r, s)h(r, s)} \tag{2.6}$$

for all $y \in X$ and $(p, q) \in G^2$. We get $\Lambda y \in X$.

We assert that Λ is strictly contractive on X. Given any $y_1, y_2 \in X$, let $C_{y_1y_2} \in [0,\infty]$ be an arbitrary constant with $d(y_1,y_2) \leq C_{y_1y_2}$, that is,

$$|y_1(p,q) - y_2(p,q)| \leq C_{y_1y_2}\phi(p,q)$$

for any $(p,q) \in G^2$. Then we have the following inequality

$$\begin{aligned} |(\Lambda y_1)(p,q) - (\Lambda y_2)(p,q)| &= \frac{|y_1(ps,qr) - y_2(ps,qr)|}{\theta(p,q,r,s)|h(r,s)|} \\ &\leq \frac{C_{y_1y_2}\phi(ps,qr)}{kM} \leq \frac{L}{kM}C_{y_1,y_2}\phi(p,q) \end{aligned}$$

for all $(p,q) \in G^2$, that is, $d(\Lambda y_1, \Lambda y_2) \leq \dfrac{L}{kM}C_{y_1y_2}$. Hence, we may conclude that $d(\Lambda y_1, \Lambda y_2) \leq \dfrac{L}{kM}d(y_1,y_2)$ for any $y_1, y_2 \in X$ and we note that $0 < \dfrac{L}{kM} < 1$.

By (2.1), we get the following inequality

$$\begin{aligned} |(\Lambda g)(p,q) - g(p,q)| &= \left|\frac{f(px_0,qy_0) + g(py_0,qx_0)}{\theta(p,q,r,s)h(r,s)} - g(p,q)\right| \\ &\leq \frac{\phi(p,q)}{kM} \end{aligned}$$

for all $p,q \in X$. This implies that

$$d(\Lambda g, g) \leq \frac{1}{kM} < \infty.$$

Therefore, it follows from 1 (b) that there exists a unique function $g_0 : G^2 \to \mathbb{R}$ such that $\Lambda^n g \to g_0$ in (X,d) and $\Lambda g_0 = g_0$.

Finally, Theorem 1(d) implies that

$$d(g,g_0) \leq \frac{1}{1 - \frac{L}{kM}}d(\Lambda g, g) \leq \frac{1}{kM - L}.$$

Therefore

$$|g(p,q) - g_0(p,q)| \leq \frac{1}{kM - L}\phi(p,q)$$

for all $(p,q) \in G^2$.

□

Corollary 1. *Let $h, \phi : G^2 \to \mathbb{R}$ be functions and $r, s \in G$ be arbitrary fixed elements such that $|h(r,s)| \geq M > \dfrac{L}{k} > 0$ and $\phi(pr, qs) \leq L\phi(p,q)$ for $p, q \in G$. If $f, g, h : G^2 \to \mathbb{R}$ be functions such that*

$$|f(pr, qs) + g(ps, qr) - g(p,q)h(r,s)| \leq \phi(p,q) \; \forall \; p, q \in G, \tag{2.7}$$

then there exists a unique function g_0 satisfying $f(pr, qs) + g_0(ps, qr) = g_0(p,q)h(r,s)$ and

$$|g(p,q) - g_0(p,q)| \leq \frac{\phi(p,q)}{kM - L} \tag{2.8}$$

for all $p, q \in G$.

Proof. Letting $\theta = 1$ and applying Theorem 1, we get the desired result, as claimed. □

Theorem 2. *Let $h, \phi : G^2 \to \mathbb{R}$ be functions and $r, s \in G$ be arbitrary fixed elements such that $|h(r,s)| \geq M > \dfrac{L}{k} > 0$ and $\phi(pr, qs) \leq L\phi(p,q)$ for all $p, q \in G$. If $f, g, h : G^2 \to \mathbb{R}$ be functions such that*

$$|f(pr, qs) + g(ps, qr) - \theta(p,q,r,s)f(p,q)h(r,s)| \leq \phi(p,q) \; \forall \; p, q \in G,$$

then there exists a unique function f_0 satisfying $f_0(pr, qs) + g(ps, qr) = \theta(p,q,r,s)f_0(p,q)h(r,s)$ for each fixed $p, q \in G$ such that

$$|f(p,q) - f_0(p,q)| \leq \frac{\phi(p,q)}{kM - L} \tag{2.9}$$

for all $p, q \in G$.

Proof. In the proof of Theorem 1, we define a contractive mapping $\Lambda : X \to X$

$$(\Lambda y)(p,q) = \frac{y(pr, qs) + g(ps, qr)}{\theta(p,q,r,s)h(r,s)}, \quad \forall p, q \in G. \tag{2.10}$$

for some fixed elements $r, s \in G$. By the similar proof of Theorem 2, one can obtain the desired result. □

Corollary 2. *Let $h, \phi : G^2 \to \mathbb{R}$ be functions such that $|h(r,s)| \geq M > L > 0$ and $\phi(pr, qs) \leq L\phi(p,q)$ for all $r, s \in G$. If $f, g, h : G^2 \to \mathbb{R}$ be functions such that*

$$|f(pr, qs) + g(ps, qr) - f(p,q)h(r,s)| \leq \phi(p,q) \; \forall \; p, q, r, s \in G,$$

then there exists a unique function f_0 satisfying $f_0(pr, qs) + g(ps, qr) = f_0(p, q)h(r, s)$ for all $p, q \in G$ and for each fixed $r, s \in G$ such that

$$|f(p,q) - f_0(p,q)| \leq \frac{\phi(p,q)}{M-L} \tag{2.11}$$

for all $p, q \in G$.

Proof. Letting $\theta = 1$ and applying Theorem 2, we get the desired result, as claimed. □

REMARK 1. For all results,

(1) Putting $\phi(p, q) = \phi(r, s) = c$: *constant*, then we obtains same types results.

(2) Applying $\theta(p, q, r, s) = \theta(pq, rs)$: cocycle, and also $\theta(p, q, r, s) = c$: *constant*, we will obatin similar types results.

References

[1] J. Brzdek, A. Najdecki and B. Xu, *Two general theorems on superstability of functional equations*, Aequationes Math., Doi: 10.1007/s00010-014-0266-6.[1].

[2] J. K. Chung, P. Kannappan, C. T. Ng and P. K. Sahoo, *Measures of distance between probability distributions*, J. Math. Anal. Appl. **138** (1989), 280–292.

[3] J. B. Diaz and B. Margolis, *A fixed point theorem of the alternative, for contractions on a generalized complete metric space*, Bulletin of the American Mathematical Society **74** (1968), 305–309.

[4] M. Hosszú, *On the functional equation $f(x+y,z)+f(x,y) = f(x,y+z)+f(y,z)$*, Periodica Math. Hungarica **1** (3) (1971), 213–216.

[5] Pl. Kannappan and P. K. Sahoo, *Sum form distance measures between probability distributions and functional equations*, Int. J. of Math. & Stat. Sci. **6** (1997), 91–105.

[6] Pl. Kannappan, P. K. Sahoo and J. K. Chung, *On a functional equation associated with the symmetric divergence measures*, Utilitas Math. **44** (1993), 75–83.

[7] G. H. Kim, *The Stability of the d'Alembert and Jensen type functional equations*, Jour. Math. Anal & Appl. **325** (2007), 237–248.

[8] Pl. Kannappan and G. H. Kim, *On the stability of the generalized cosine functional equations*, Ann. Acad. Pedagog. Crac. Stud. Math. **1** (2001), 49–58.

[9] G. H. Kim, *On the Stability of the pexiderized trigonometric functional equation*, Appl. Math. Compu. **203** (2008), 99–105.

[10] G. H. Kim and Y.H. Lee, *The superstability of the Pexider type trigonometric functional equation*, Math. Ineq. & Appl., submitted.

[11] G. H. Kim and Y.H. Lee, *Boundedness of approximate trigonometric functional equations*, Appl. Math. Lett. **331** (2009), 439–443.

[12] G. H. Kim, *On the Stability of trigonometric functional equations*, Ad. Diff. Eq. Vol 2007, Article ID 90405, (2007).

[13] G. H. Kim and Sever S. Dragomir, *On the Stability of generalized d'Alembert and Jensen functional equation*, Intern. Jour. Math. & Math. Sci., Article ID 43185, DOI 10.1155 (2006), 1–12.
[14] G. H. Kim and Y. W. Lee *Superstability of Pexiderized functional equations arising from distance measures*, J. Nonlinear Sci. Appl. **9** (2016), 413–423.
[15] G. H. Kim and P. K. Sahoo, *Stability of a Pexider type functional equation related to distance measures*, Jour. Math. Ineq. **9** (4) (2015), 11691179.
[16] G. H. Kim and P. K. Sahoo, *Stability of a functional equation related to distance measure - I*, Appl. Math. Lett. **24** (2011), 843–849.
[17] G. H. Kim and P. K. Sahoo, *Stability of a functional equation related to distance measure - II*, Ann. Funct. Anal. **1** (2010) 26–35.
[18] Y. W. Lee and G. H. Kim *Superstability of the functional equation with a cocycle related to distance measures*, Math. Ineq. & Appl., (2014), 2014:393 doi:10.1186/1029-242X-2014-393
[19] Y. W. Lee and G. H. Kim *Superstability of the functional equation related to distance measures*, Jour. Ineq. & Appl. 20152015:352, DOI: 10.1186/s13660-015-0880-4
[20] T. Riedel and P. K. Sahoo, *On a generalization of a functional equation associated with the distance between the probability distributions*, Publ. Math. Debrecen **46** (1995), 125–135.
[21] T. Riedel and P. K. Sahoo, *On two functional equations connected with the characterizations of the distance measures*, Aequationes Math. **54** (1998), 242–263.
[22] P. K. Sahoo, *On a functional equation associated with stochastic distance measures*, Bull. Korean Math. Soc. **36** (1999), 287–303.
[23] T. Riedel and P. K. Sahoo, *On a generalization of a functional equation associated with the distance between the probability distributions*, Publ. Math. Debrecen **46** (1995), 125–135.
[24] J. Tabor, *Hyers theorem and the cocycle property*, Fumctional equations-Results and Advaces, Kluwer Academic Publ.(Z. Daróczy and Z. Páles), (2002), 275-290.

Hwan-Yong Shin
Department of Mathematics
Chungnam National University
Daejeon 34134, Republic of Korea
E-mail: hyshin31@cnu.ac.kr

Gwang Hui Kim
Department of Mathematics
Kangnam University
Yongin 16979, Republic of Korea
E-mail: ghkim@kangnam.ac.kr