Korean J. Math. **24** (2016), No. 3, pp. 319–330
http://dx.doi.org/10.11568/kjm.2016.24.3.319

A STUDY ON THE RECURRENCE RELATIONS OF 5-DIMENSIONAL ES-MANIFOLD

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Abstract. The manifold ${}^*g-ESX_n$ is a generalized n-dimensional Riemannian manifold on which the differential geometric structure is imposed by the unified field tensor ${}^*g^{\lambda\nu}$ through the ES-connection which is both Einstein and semi-symmetric. The purpose of the present paper is to study the algebraic geometric structures of 5-dimensional ${}^*g-ESX_5$. Particularly, in 5-dimensional ${}^*g-ESX_5$, we derive a new set of powerful recurrence relations in the first class.

1. Introduction

In Appendix II to his last book Einstein([3], 1950) proposed a new unified field theory that would include both gravitation and electromagnetism. Although the intent of this theory is physical, its exposition is mainly geometrical. It may be characterized as a set of geometrical postulates for the space time X_4. Characterizing Einstein's unified field theory as a set of geometrical postulates for X_4, Hlavatý([4], 1957) gave its mathematical foundation for the first time. Since then Hlavatý and number of mathematicians contributed for the development of this theory and obtained many geometrical cnosequences of these postulates.

The main purpose of the present paper is to study the algebraic geometric properties of 5-dimensional ${}^*g-ESX_5$ in the first class. In

Received March 23, 2016. Revised July 8, 2016. Accepted July 11, 2016.
2010 Mathematics Subject Classification: 83E50, 83C05, 58A05.
Key words and phrases: ES-manifold, recurrent relation.
This research was supported by Incheon National University Research Grant, 2014-2015.

particular, we derive a powerful recurrence relations in 5-dimensional $^*g - ESX_5$.

2. Preliminaries

This section is a brief collection of basic concepts, results, and notations needed in subsequent considerations. They are due to Chung ([1], 1963)), and Mishra([5], 1959) mostly due to Datta([2], 1964).

(a) n-dimensional $^*g$-unified field theory

Corresponding to the Einstein's $n - g$-UFT, our $n - {}^*g$-UFT is based on the following three principles.

PRINCIPLE A. Let X_n be an n-dimensional generalized Riemannian manifold referred to a real coordinate system x^ν, which obeys the coordinate transformations $x^\nu \to x^{\nu'}$ for which

$$det(\frac{\partial x'}{\partial x}) \neq 0 \tag{2.1}$$

In $n - g - UFT$ the manifold X_n is endowed with a real nonsymmetric tensor $g_{\lambda\mu}$, which may be decomposed into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{2.2}$$

where

$$\mathfrak{g} = det(g_{\lambda\mu}) \neq 0, \quad \mathfrak{h} = det(h_{\lambda\mu}) \neq 0, \quad \mathfrak{k} = det(k_{\lambda\mu}) \tag{2.3}$$

In $n - {}^*g - UFT$ the algebraic structure on X_n is imposed by the basic real tensor $^*g^{\lambda\nu}$ defined by

$$g_{\lambda\mu}{}^*g^{\lambda\nu} = g_{\mu\lambda}{}^*g^{\nu\lambda} = \delta_\mu^\nu \tag{2.4}$$

It may be also decomposed into its symmetric part $^*h^{\lambda\nu}$ and skew-symmetric part $^*k^{\lambda\nu}$:

$$^*g^{\lambda\nu} = {}^*h^{\lambda\nu} + {}^*k^{\lambda\nu} \tag{2.5}$$

Since $det({}^*h^{\lambda\nu}) \neq 0$, we may define a unique tensor $^*h_{\lambda\mu}$ by

$$^*h_{\lambda\mu}{}^*h^{\lambda\nu} = \delta_\mu^\nu \tag{2.6}$$

In $n - {}^*g$*-UFT we use both* ${}^*h^{\lambda\nu}$ *and* ${}^*h_{\lambda\mu}$ *as tensors for raising and/or lowering indices of all tensors in* X_n *in the usual manner.* We then have

$$(2.7)\qquad {}^*k_{\lambda\mu} = {}^*k^{\rho\sigma}\,{}^*h_{\lambda\rho}\,{}^*h_{\mu\sigma}, \qquad {}^*g_{\lambda\mu} = {}^*g^{\rho\sigma}\,{}^*h_{\lambda\rho}\,{}^*h_{\mu\sigma}$$

so that

$$(2.8)\qquad {}^*g_{\lambda\mu} = {}^*h_{\lambda\mu} + {}^*k_{\lambda\mu}$$

PRINCIPLE B The differential geometric structure on X_n is imposed by the tensor ${}^*g^{\lambda\nu}$ by means of a connection $\Gamma_\lambda{}^\nu{}_\mu$ defined by a system of equations

$$(2.9)\qquad D_\omega {}^*g^{\lambda\nu} = -2S_{\omega\alpha}{}^\nu\,{}^*g^{\lambda\alpha}$$

Here D_ω denotes the symbol of the covariant derivative with respect to $\Gamma_\lambda{}^\nu{}_\mu$ and $S_{\lambda\mu}{}^\nu$ is the torsion tensor of $\Gamma_\lambda{}^\nu{}_\mu$. Under certain conditions the system (2.9) admits a unique solutions $\Gamma_\lambda{}^\nu{}_\mu$.

PRINCIPLE C In order to obtain ${}^*g^{\lambda\nu}$ involved in the solution for $\Gamma_\lambda{}^\nu{}_\mu$ certain conditions are imposed. These conditions may be condensed to

$$(2.10)\qquad S_\lambda = S_{\lambda\alpha}{}^\alpha = 0, \quad R_{[\mu\lambda]} = \partial_{[\mu}Y_{\lambda]}, \quad R_{(\mu\lambda)} = 0$$

where Y_λ is an arbitrary vector, and $R_{\omega\mu\lambda}{}^\nu$ are the curvature tensors of X_n defined by

$$(2.11)\quad R_{\omega\mu\lambda}{}^\nu = 2(\partial_{[\mu}\Gamma_{|\lambda|}{}^\nu{}_{\omega]} + \Gamma_\alpha{}^\nu{}_{[\mu}\Gamma_{|\lambda|}{}^\alpha{}_{\omega]}), \qquad R_{\mu\lambda} = R_{\alpha\mu\lambda}{}^\alpha$$

(b) Some notations and results

The following quantities are frequently used in our further considerations:

$$(2.12)\qquad {}^*g = det({}^*g_{\lambda\mu}), \quad {}^*h = det({}^*h_{\lambda\mu}), \quad {}^*k = det({}^*k_{\lambda\mu})$$

$$(2.13)\qquad {}^*\mathfrak{g} = \frac{{}^*g}{{}^*h}, \quad {}^*\mathfrak{k} = \frac{{}^*k}{{}^*h}.$$

$$(2.14)\qquad K_p = {}^*k_{[\alpha_1}{}^{\alpha_1}\,{}^*k_{\alpha_2}{}^{\alpha_2}\cdots{}^*k_{\alpha_p]}{}^{\alpha^p}, \quad (p = 0, 1, 2, \cdots).$$

$$(2.15)\quad {}^{(0)*}k_\lambda{}^\nu = \delta_\lambda^\nu, \ {}^{(p)*}k_\lambda{}^\nu = {}^*k_\lambda{}^\alpha\ {}^{(p-1)*}k_\alpha{}^\nu \quad (p = 1, 2, \cdots).$$

In X_n it was proved in [1] that

(2.16) $K_0 = 1,\ K_n = {}^*k$ if n is even, and $\mathrm{K_n} = 0$ if n is odd.

$$^*g = 1 + K_2 + \cdots + K_{n-\sigma}. \tag{2.17}$$

$$\sum_{s=0}^{n-\sigma} K_s \ {}^{(n-s)*}k_\lambda{}^\nu = 0 \quad (p = 0, 1, 2, \cdots). \tag{2.18}$$

We also use the following useful abbreviations for an arbitrary tensor $T^{\cdots}_{\cdots}$ for $p = 1, 2, 3, \cdots$:

$$^{(p)}T^{\nu\cdots}_{\cdots} = ^{(p-1)}\ {}^*k^\nu{}_\alpha T^{\alpha\cdots}_{\cdots}. \tag{2.19}$$

(c) n-dimensional ES manifold $n - {}^*g$-UFT

In this subsection, we display an useful representation of the ES connection in $n - {}^*g$-UFT.

Definition 2.1. A connection $\Gamma_\lambda{}^\nu{}_\mu$ is said to be *semi-symmetric* if its torsion tensor $S_{\lambda\mu}{}^\nu$ is of the form

$$S_{\lambda\mu}{}^\nu = 2\delta^\nu_{[\lambda} X_{\mu]}. \tag{2.20}$$

for an arbitrary non-null vector X_μ.

A connection which is both semi-symmetric and Einstein is called an ES connection. An n-dimensional generalized Riemannian manifold X_n, on which the differential geometric structure is imposed by ${}^*g^{\lambda\nu}$ by means of an ES connection, is called an n-dimensional ${}^*g-ES$ manifold. We denote this manifold by ${}^*g - ESX_n$ in our further considerations.

Theorem 2.2. *Under the condition* (2.20), *the system of equations* (2.9) *is equivalent to*

$$\Gamma_\lambda{}^\nu{}_\mu = {}^*\left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + U^\nu{}_{\lambda\mu} + 2\delta^\nu_{[\lambda} X_{\mu]}. \tag{2.21}$$

where

$$U^\nu{}_{\lambda\mu} = -{}^*h_{\lambda\mu}{}^{(2)} X^\nu \tag{2.22}$$

Proof. Substituting (2.20) for $S_{\lambda\mu}{}^\nu$ into (2.9), we have the representation (2.21). □

3. Recurrence relations in $^*g - ESX_5$

In this section we derive several powerful recurrence relations, establishing a non-holonomic frame in $^*g - ESX_5$.

DEFINITION 3.1. The tensors $^*g_{\lambda\mu}$ is said to be
(1) of the first class, if $K_{n-\sigma} \neq 0$
(2) of the second class with jth category ($j >= 1$), if

$$K_2 j \neq 0, \quad K_{2j+2} = K_{2j+4} = \cdots = K_{n-\sigma} = 0 \tag{3.1}$$

(3) of the third class, if $K_2 = K_4 = \cdots = K_{n-\sigma} = 0$

In $5 - {}^*g$-UFT, we have three classes; namely the first class when $K_4 \neq 0$, the second class when $K_4 = 0, K_2 \neq 0$, and the third class when $K_4 = K_2 = 0$. In $5 - {}^*g$-UFT, the relation (2.17) gives

$$^*g = 1 + K_2 + K_4 \tag{3.2}$$

In this chapter we investigate only the first class of $5 - {}^*g$-UFT. Hence all considerations in this chapter are restricted to $n = 5$.

A. Basic vectors in the first class

REMARK 3.2. For the simplicity of our discussion, we assume in this and in what follows that

$$K_4 < 0 \tag{3.3}$$

The eigenvalues M and the corresponding eigenvectors A^ν in $^*g - ESX_n$, defined by

$$MA^\nu = {}^*k_\mu{}^\nu A^\mu, \quad (M : \text{a scalar}). \tag{3.4}$$

are called *basic scalars* and *basic vectors*, respectively.

THEOREM 3.3. *The basic scalars in $^*g - ESX_5$ may be given by*

$$\begin{aligned} &\underset{1}{M} = -\underset{2}{M} = \sqrt{-L-K} \neq 0 \\ &\underset{3}{M} = -\underset{4}{M} = \sqrt{L-K} \neq 0, \quad \underset{5}{M} = 0 \end{aligned} \tag{3.5}$$

where

$$K = \frac{K_2}{2}, \quad L = \sqrt{(\frac{K_2}{2})^2 - K_4} \tag{3.6}$$

Proof. In $5-{}^*g$-UFT, the characteristic equation is reduced to

$$M(M^4+K_2M^2+K_4)=0 \tag{3.7}$$

from which our assertion follows in virtue of (3.3) and (3.6). □

THEOREM 3.4. *There are five linearly independent basic vectors $\underset{1}{A}^\nu,\cdots \underset{5}{A}^\nu$ and they have the following properties:*

(a) *They are defined up to an arbitrary factor of proportionality.*

(b) *$\underset{1}{A}^\nu,\cdots \underset{4}{A}^\nu$ are null vectors, while $\underset{5}{A}^\nu$ is non-null.*

(c) *$\underset{1}{A}^\nu,\underset{2}{A}^\nu$ are perpendicular to $\underset{3}{A}^\nu,\underset{4}{A}^\nu$ and $\underset{5}{A}^\nu$ is also perpendicular to $\underset{1}{A}^\nu,\cdots \underset{4}{A}^\nu$.*

(d) *They satisfy the conditions*

$$^*h_{\lambda\mu}\underset{1}{A}^\nu\underset{2}{A}^\nu\neq 0,\quad {}^*h_{\lambda\mu}\underset{3}{A}^\nu\underset{4}{A}^\nu\neq 0 \tag{3.8}$$

Proof. Since the basic scalars $\underset{i}{M}$ are all distinct, (3.4) admits five linearly independent basic vectors $\underset{i}{A}^\nu$ which are defined up to an arbitrary factor of proportionality. The first half of statement (b) is a consequence of (3.4), (3.5), and

$$\underset{x}{M}{}^*h_{\lambda\mu}\underset{x}{A}^\lambda\underset{x}{A}^\mu={}^*k_{\lambda\mu}\underset{x}{A}^\lambda\underset{x}{A}^\mu=0,\qquad(x=1,\cdots,4) \tag{3.9}$$

Since $\underset{5}{M}+\underset{x}{M}\neq 0,\ (x=1,\cdots,4)$, statement (c) follows from (3.5) as in the following way:

$$\underset{x}{M}{}^*h_{\lambda\mu}\underset{x}{A}^\lambda\underset{y}{A}^\mu={}^*k_{\lambda\mu}\underset{x}{A}^\lambda\underset{y}{A}^\mu=-\underset{y}{M}{}^*h_{\mu\lambda}\underset{x}{A}^\lambda\underset{y}{A}^\mu,\quad(y=3,4) \tag{3.10}$$

In order to prove statement (d), consider a conic C with equation ${}^*h_{\lambda\mu}A^\lambda A^\mu=0$ on a projective plane P_2. In virtue of statement (b), $\underset{1}{A}^\nu$ and $\underset{2}{A}^\nu$ are two different points on C while ${}^*h_{\lambda\mu}\underset{1}{A}^\lambda=\underset{1}{A}_\mu$ is the tangent line to C at $\underset{1}{A}^\nu$. Since $det({}^*h_{\lambda\mu})\neq 0$, C is non-degenerate. Consequently ${}^*h_{\lambda\mu}\underset{1}{A}^\lambda=\underset{1}{A}_\mu$ and $\underset{2}{A}^\mu$ are not incident; that is, ${}^*h_{\lambda\mu}\underset{1}{A}^\lambda\underset{2}{A}^\mu\neq 0$. □

B. Nonholonomic frame of reference in the first class

In the first class, we have a set of 5 linearly independent basic vectors $\underset{i}{A}^\nu, (i = 1, \cdots, 5)$ and a unique reciprocal set $\overset{i}{A}_\lambda, (i = 1, \cdots, 5)$ satisfying

$$\overset{j}{A}_\lambda \underset{i}{A}^\lambda = \delta_i^j, \qquad \overset{i}{A}_\lambda \underset{i}{A}^\nu = \delta_\lambda^\nu \tag{3.11}$$

With these two set of vectors, we may construct a nonholonomic frame of reference as follows;

DEFINITION 3.5. If $T_{\lambda\cdots}^{\nu\cdots}$ are holonomic components of a tensor, then its nonholonomic components are defined by

$$T_{j\cdots}^{i\cdots} = T_{\lambda\cdots}^{\nu\cdots} \overset{i}{A}_\nu \cdots \underset{j}{A}^\lambda \cdots \tag{3.12}$$

An easy inspection shows that

$$T_{\lambda\cdots}^{\nu\cdots} = T_{j\cdots}^{i\cdots} \underset{i}{A}^\nu \cdots \overset{j}{A}_\lambda \cdots \tag{3.13}$$

THEOREM 3.6. *The nonholonomic components* $^*h_{ij}$ *and* $^*h^{ij}$ *are given by the matrix equation*

$$(^*h_{ij}) = (^*h^{ij}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.14}$$

Proof. (3.14) is a direct result of (3.5) and theorem (3.4) □

THEOREM 3.7. *We have*

$$\underset{i}{A}^\nu = \overset{j}{A}_\lambda {}^*h_{ij} {}^*h^{\lambda\nu}, \qquad \overset{j}{A}_\lambda = \underset{i}{A}^{\nu *} h^{ij*} h_{\lambda\nu} \tag{3.15}$$

so that

$$\underset{1}{A}^\nu = \overset{2}{A}{}^\nu, \quad \underset{2}{A}^\nu = \overset{1}{A}{}^\nu, \quad \underset{3}{A}^\nu = \overset{4}{A}{}^\nu \quad \underset{4}{A}^\nu = \overset{3}{A}{}^\nu, \quad \underset{5}{A}^\nu = \overset{5}{A}{}^\nu$$

$$(3.16)\quad \overset{1}{A}_\lambda = \underset{2}{A}{}_\lambda, \quad \overset{2}{A}_\lambda = \underset{1}{A}{}_\lambda, \quad \overset{3}{A}_\lambda = \underset{4}{A}{}_\lambda, \quad \overset{4}{A}_\lambda = \underset{3}{A}{}_\lambda, \quad \overset{5}{A}_\lambda = \underset{5}{A}{}_\lambda$$

Proof. In virtue of (2.6), (3.11) and (3.12), the first relation of (3.15) follows as in the following way;

$$(3.17)\quad \overset{j}{A}_\lambda {}^*h_{ij} {}^*h^{\lambda\nu} = \overset{j}{A}_\lambda ({}^*h_{\alpha\beta} \underset{i}{A}{}^\alpha \underset{j}{A}{}^\beta) {}^*h^{\lambda\nu} = \underset{i}{A}{}^\alpha ({}^*h_{\alpha\beta} {}^*h^{\lambda\nu}) \delta^\beta_\lambda = \underset{i}{A}{}^\nu.$$

(3.16) follows from (3.15) and (3.14). □

Theorem 3.8. *The nonholonomic components of* ${}^{(p)*}k_\lambda{}^\nu$, ${}^{(p)*}k_{\lambda\mu}$ *and* ${}^{(p)*}k^{\lambda\nu}$ *are given by*

$$(3.18)\quad {}^{(p)*}k_x{}^i = \underset{x}{M}{}^p \delta^i_x$$

$$(3.19)\quad {}^{(p)*}k_{xi} = \underset{x}{M}{}^p {}^*h_{xi}$$

$$(3.20)\quad {}^{(p)*}k^{xi} = \underset{x}{M}{}^p {}^*h^{xi}$$

Proof. Let $\underset{x}{A}{}^\nu$ be the basic vector corresponding to the basic scalar $\underset{x}{M}$. Then from (3.4), we have

$$(3.21)\quad {}^{(p)*}k_\lambda{}^\nu \underset{x}{A}{}^\lambda = \underset{x}{M}{}^p \underset{x}{A}{}^\nu \quad (p = 0, 1, 2, \cdots)$$

(3.18) follows immediately by multiplying $\overset{i}{A}_\nu$ to both sides of (3.21). The remaining relations may be obtained from (3.18) by lowering and/or raising indices. □

In the following theorem, we express the components of tensors ${}^*h_{\lambda\mu}$, ${}^{(p)*}k_\lambda{}^\nu$, ${}^{(p)*}k_{\lambda\mu}$, ${}^{(p)*}k^{\lambda\nu}$ in terms of basic vectors:

Theorem 3.9. *The representation of* ${}^*h_{\lambda\mu}$, ${}^{(p)*}k_\lambda{}^\nu$, ${}^{(p)*}k_{\lambda\mu}$, ${}^{(p)*}k^{\lambda\nu}$ *in terms of basic vectors are given by*

$$(3.22)\quad {}^*h_{\lambda\mu} = 2\overset{1}{A}_{(\lambda}\overset{2}{A}_{\mu)} + \overset{3}{A}_{(\lambda}\overset{4}{A}_{\mu)} + \overset{5}{A}_\lambda \overset{5}{A}_\mu$$

$$(3.23)\quad {}^{(p)*}k_\lambda{}^\nu = \underset{1}{M}{}^p(\overset{1}{A}_\lambda \underset{1}{A}{}^\nu + (-1)^p \overset{2}{A}_\lambda \underset{2}{A}{}^\nu) + \underset{3}{M}{}^p(\overset{3}{A}_\lambda \underset{3}{A}{}^\nu + (-1)^p \overset{4}{A}_\lambda \underset{4}{A}{}^\nu)$$

$$
{}^{(p)*}k_{\lambda\mu} = \begin{cases} 2\underset{1}{M}^p \overset{1}{A}_{(\lambda}\overset{2}{A}_{\mu)} + 2\underset{3}{M}^p \overset{3}{A}_{(\lambda}\overset{4}{A}_{\mu)}, & \text{if p is even} \\ 2\underset{1}{M}^p \overset{1}{A}_{[\lambda}\overset{2}{A}_{\mu]} + 2\underset{3}{M}^p \overset{3}{A}_{[\lambda}\overset{4}{A}_{\mu]}, & \text{if p is odd} \end{cases} \tag{3.24}
$$

$$
{}^{(p)*}k^{\lambda\nu} = \begin{cases} 2\underset{1}{M}^p \underset{1}{A}^{(\lambda}\underset{2}{A}^{\nu)} + 2\underset{3}{M}^p \underset{3}{A}^{(\lambda}\underset{4}{A}^{\nu)}, & \text{if p is even} \\ 2\underset{1}{M}^p \underset{1}{A}^{[\lambda}\underset{2}{A}^{\nu]} + 2\underset{3}{M}^p \underset{3}{A}^{[\lambda}\underset{4}{A}^{\nu]}, & \text{if p is odd} \end{cases} \tag{3.25}
$$

Proof. The representations $(3.22) - (3.25)$ follow from (3.13) in virtue of $(3.5), (3.14)$ and (3.16). □

C. Recurrence relations in the first class

In this subsection we derive several recurrence relations.

THEOREM 3.10. *In the first class, the tensor $T_{\omega\mu\nu}$, skew-symmetric in the first two indices, satisfies*

$$
\overset{(pq)r}{T}{}_{\omega\mu\nu} = \sum_{x,y,z} T_{xyz} \underset{x}{M}^{(p} \underset{y}{M}^{q)} \underset{z}{M}^r \overset{x}{A}_\omega \overset{y}{A}_\mu \overset{z}{A}_\nu \tag{3.26}
$$

$$
\overset{r(pq)}{T}{}_{\nu[\omega\mu]} = \sum_{x,y,z} T_{x[yz]} \underset{x}{M}^{(p} \underset{y}{M}^{q)} \underset{z}{M}^r \overset{x}{A}_\nu \overset{y}{A}_\omega \overset{z}{A}_\mu \tag{3.27}
$$

Proof. In virtue of (3.13) and (3.18), our assertion (3.26) may be derived as

$$
\begin{aligned}
\overset{(pq)r}{T}{}_{\omega\mu\nu} &= \sum_{x,y,z} \overset{(pq)r}{T}{}_{xyz} \overset{x}{A}_\omega \overset{y}{A}_\mu \overset{z}{A}_\nu \\
&= \sum_{x,y,z} \frac{1}{2} ({}^{(p)*}k_x{}^{i\,(q)*}k_y{}^j + {}^{(q)*}k_x{}^{i\,(p)*}k_y{}^j) {}^{(r)*}k_z{}^k \overset{x}{A}_\omega \overset{y}{A}_\mu \overset{z}{A}_\nu \\
&= \frac{1}{2} \sum_{x,y,z} T_{xyz} (\underset{x}{M}^p \underset{y}{M}^q + \underset{x}{M}^q \underset{y}{M}^p) \underset{z}{M}^r \overset{x}{A}_\omega \overset{y}{A}_\mu \overset{z}{A}_\nu
\end{aligned} \tag{3.28}
$$

The second relation may be proved similarly. □

Theorem 3.11. *The main recurrence relation in the first class is*

$$(3.29)\quad {}^{(p+5)*}k_\lambda{}^\nu = -K_2{}^{(p+3)*}k_\lambda{}^\nu - K_4{}^{(p+1)*}k_\lambda{}^\nu, \quad (p=0,1,2,\cdots)$$

Proof. Let $\underset{x}{M}$ be a basic scalar. In $5-{}^*g-ESX_5$, the characteristic equation is

$$\sum_{f=0}^{4} K_f \underset{x}{M}^{5-f} = 0 \tag{3.30}$$

Multiplying δ^i_x to both sides of (3.30) and making use of (3.18), we have

$$\sum_{f=0}^{4} K_f{}^{(5-f)*}k_x{}^i = 0 \tag{3.31}$$

whose holonomic form is

$$\sum_{f=0}^{4} K_f{}^{(5-f)*}k_\lambda{}^\alpha = 0 \tag{3.32}$$

The relation (3.29) immediately follows by multiplying ${}^{(p)*}k_\alpha{}^\nu$ to both sides of (3.32). □

The following theorem is simple consequences of (5.2) and (5.5).

Theorem 3.12. *The basic scalars $\underset{x}{M}$ satisfy*

$$\underset{1}{M} + \underset{2}{M} = \underset{3}{M} + \underset{4}{M} = 0 \tag{3.33}$$

$$\underset{1}{M}\underset{5}{M} = \underset{2}{M}\underset{5}{M} = \underset{3}{M}\underset{5}{M} = \underset{4}{M}\underset{5}{M} = 0 \tag{3.34}$$

$$\underset{1}{M}^2\underset{3}{M}^2 = \underset{1}{M}^2\underset{4}{M}^2 = \underset{2}{M}^2\underset{3}{M}^2 = \underset{2}{M}^2\underset{4}{M}^2 = K_4 \tag{3.35}$$

$$\underset{1}{M}^2 + \underset{3}{M}^2 = \underset{1}{M}^2 + \underset{4}{M}^2 = \underset{2}{M}^2 + \underset{3}{M}^2 = \underset{2}{M}^2 + \underset{4}{M}^2 = -K_2 \tag{3.36}$$

Proof. These relations follow easily from (3.5). □

In virtue of the above theorem, we have

THEOREM 3.13. *In the first class, the following identities hold for all values of x and y when $x \neq y$*

$$M_x^{(4} M_y^{1)} = -M_x^{(3} M_y^{2)} - K_2 M_x^{(2} M_y^{1)} \tag{3.37}$$

$$M_x^{(4} M_y^{3)} = K_4 M_x^{(2} M_y^{1)} \tag{3.38}$$

$$M_x^4 M_y^4 = K_4{}^2 M_x^2 M_y^2 + K_2 M_x^3 M_y^3 + 2K_4 M_x^{(3} M_y^{1)} \tag{3.39}$$

$$2M_x^{(4} M_y^{2)} = -M_x^3 M_y^3 - K_2 M_x^2 M_y^2 + K_4 M_x M_y \tag{3.40}$$

THEOREM 3.14. *(Recurrence relations in the first class)* *If $T_{\omega\mu\nu}$ is a tensor skew-symmetric in the first two indices, then the following recurrence relations hold in the first class of $5 - {}^*g - ESX_5$:*

$$\overset{(41)r}{T} = -\overset{(32)r}{T} - K_2 \overset{(21)r}{T} \tag{3.41}$$

$$\overset{(43)r}{T} = K_4 \overset{(21)r}{T} \tag{3.42}$$

$$\overset{44r}{T} = K_4 \overset{22r}{T} + K_2 \overset{33r}{T} + 2K_4 \overset{(31)r}{T} \tag{3.43}$$

$$2 \overset{(42)r}{T} = -\overset{33r}{T} - K_2 \overset{22r}{T} + K_4 \overset{11r}{T} \tag{3.44}$$

Proof. The above relations are consequences of (3.26), (3.27) and (3.37)–(3.40). For example, the relation (3.44) is proved as in the following way:

$$2 \overset{(42)r}{T} = 2 \overset{(42)r}{T}{}_{\omega\mu\nu} = 2 \sum_{x,y,z} T_{xyz} M_x^{(4} M_y^{2)} M_z^r \overset{x}{A}_\omega \overset{y}{A}_\mu \overset{z}{A}_\nu$$

$$= \sum_{x,y,z} T_{xyz} (-M_x^3 M_y^3 - K_2 M_x^2 M_y^2 + K_4 M_x M_y) M_z^r \overset{x}{A}_\omega \overset{y}{A}_\mu \overset{z}{A}_\nu$$

$$= -\overset{33r}{T}{}_{\omega\mu\nu} - K_2 \overset{22r}{T} + K_4 \overset{11r}{T} \tag{3.45}$$

□

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