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COMPUTATION OF λ-INVARIANT

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Abstract. We give an explicit formula for the computation of Iwasawa λ-invariants and an example of the computation using our method.

1. Introduction

Let K be an imaginary quadratic field and p be an odd prime. It is well-known(see [1] and [2]) that there exist non-negative integers $\lambda_p(K)$ and $\nu_p(K)$ such that the exact power of p dividing the class number $h(K_n)$ is equal to $\lambda_p(K)n + \nu_p(K)$ for all suficiently large n. Here K_n is the n-th layer of the cyclotomic $\mathbb{Z}_p$-extension of K. Fukuda [3] computed $\lambda_p(K)$ using theorems of Gold and Iwasawa's construction of p-adic L function attached to K. In a paper [6], we gave another method to compute $\lambda_p(K)$ using Sinnott's construction of p-adic L function and Kida's formula. Examples of computation of $\lambda_p(K)$ were given for $p = 3$ in the paper. In this paper, we compute $\lambda_p(K)$ for primes greater than 5 using our method in the paper [6].

2. Computation of λ-invariant

We briefly explain our method in the paper [6] for computing $\lambda_p(K)$. Let Λ be the ring of $\mathbb{Z}_p$-valued measures on $\mathbb{Z}_p$. Then Λ is isomorphic to

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the ring $\mathbb{Z}_p[[T-1]]$;explicitly, if $\alpha \in \Lambda$, then the power series associated to α is defined by

$$F(T) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \left(\begin{array}{c} x \\ n \end{array} \right) d\alpha (T-1)^n,$$

where $\left(\begin{array}{c} x \\ n \end{array} \right) = \frac{x(x-1)\cdots(x-n+1)}{n!}$.

Let $c > 1$ be an integer prime to p and the conductor of a nontrivial first kind character χ of K, and let $\varepsilon : \mathbb{Z} \to \mathbb{Z}_p$ be the function defined by $\varepsilon(a) = \chi(a)$, if a is not divisible by c, and $\varepsilon(a) = \chi(a)(1-c)$ if a is divisible by c. Define

$$F_\varepsilon(T) = \frac{\sum_{a=1}^{f} \varepsilon(a) T^a}{1 - T^f},$$

where f is any multiple of the minimal period of ε. It is known that $F_\varepsilon(T)$ lies in $\mathbb{Z}_p[[T-1]]$. Hence it corresponds to a measure in Λ. Let $G(T)$ be the power series in $\mathbb{Z}_p[[T-1]]$ corresponding to the measure

$$(\sum_{\eta \in V} \alpha \circ \eta|_U) \circ \phi,$$

where V is the group of $p-1$-th roots of unity in $\mathbb{Z}_p$, $U = 1 + p\mathbb{Z}_p$ and ϕ is the isomorphism $\phi : \mathbb{Z}_p \simeq U$ given by $\phi(y) = (1+p)^y$.

If $F(T)$ is an element of $\mathbb{Z}_p[[T-1]]$, write $F(T) = p^\mu F_0(T), F_0(T) = \sum_{n \geq 0} a_n (T-1)^n$, where $a_n \not\equiv 0 \mod p$ for some n. Then the λ-invariant of $F(T)$ is defined by

$$\lambda(F(T)) = min\{n : a_n \not\equiv 0 \mod p\}$$

Sinnott [7] proved that

$$\lambda_p(K) = \lambda(G(T))$$

when $p \geq 5$. Moreover we have Kida's formula [5]:

$$p\lambda(G(T)) = \lambda(\sum_{\eta \in V} \alpha \circ \eta|_U).$$

In the paper [6], we computed the power series $Q(T)$ corresponding to the measure $\sum_{\eta \in V} \alpha \circ \eta|_U$.

Theorem 1.

$$Q(T) = \sum_{\eta \in V} \frac{\sum_{a \equiv \eta^{-1}}^{f} \varepsilon(a) T^{a\eta}}{1 - T^{f\eta}},$$

where f is a multiple of the minimal period of ε and p.

Proof. See the proof of Theorem 2 in [6]. □

To compute $\lambda(Q(T))$ explicitly, we need to replace η by an integer i_η.

LEMMA 1. *Let $f(T)$ be in $\mathbb{Z}_p[[T-1]]$. Then*

$$\lambda(f(T)) = \lambda(f(T^\beta))$$

for $\beta \in 1 + p\mathbb{Z}_p$.

Proof. Note that if $f(T)$ is the power series associated to a measure α, then $f(T^\beta)$ is the power series associated to a measure $\alpha \circ \beta^{-1}$. So $f(T^\beta)$ is in $\mathbb{Z}_p[[T-1]]$. We may write $f(T) = \sum_{n=0}^{\infty} a_n(T-1)^n$. By the definition of λ we see that $a_n \equiv 0 \mod p$ for $n < \lambda(f(T))$ and $a_{\lambda(f(T))} \not\equiv 0 \mod p$. Since

$$\begin{aligned} T^\beta &= \sum_{n=0}^{\infty} \binom{\beta}{n} (T-1)^n \equiv 1 + \beta(T-1) + \text{higher terms} \\ &\equiv T + \text{higher terms} (\text{mod } p), \end{aligned}$$

it is easy to check that $\lambda(f(T)) = \lambda(f(T^\beta))$. □

For $\eta \in V$, let $1 \le i_\eta, j_\eta \le (p-1)$ be integers such that $\eta \equiv i_\eta \mod p$ and $i_\eta j_\eta \equiv 1 \bmod p$. Now we give a formula to compute λ-invariants for imaginary quadratic fields.

THEOREM 2. *For primes $p \ge 5$, we have*

$$\lambda_p(K) = \frac{1}{p}\lambda(\sum_{\eta \in V} \frac{\sum_{a \equiv j_\eta}^{f} \varepsilon(a) T^{a i_\eta}}{1 - T^{f i_\eta}}).$$

Proof.

$$\begin{aligned} \lambda_p(K) &= \lambda(G(T)) = \frac{1}{p}\lambda(\sum_{\eta \in V} \alpha \circ \eta|_U) \\ &= \frac{1}{p}\lambda(Q(T)) = \frac{1}{p}\lambda(\sum_{\eta \in V} \frac{\sum_{a \equiv j_\eta}^{f} \varepsilon(a) T^{a i_\eta}}{1 - T^{f i_\eta}}) \end{aligned}$$

The last equality comes from Lemma 1 with $\beta = \eta^{-1} i_\eta$. □

We give an example.

Example 1. For $K = \mathbb{Q}(\sqrt{-127})$ and $p = 5$, we can choose $c = 2, f = 1270$. Moreover, $\varepsilon(a) = (\frac{a}{127})(-1)^{a+1}$, where $(\frac{*}{*})$ is the Jacobi symbol. Hence we have

$$\begin{aligned}
\lambda_5(\mathbb{Q}(\sqrt{-127})) &= \frac{1}{5}\lambda(\frac{\sum_{a\equiv 1(5)}^{1270}\varepsilon(a)T^a}{1-T^{1270}} + \frac{\sum_{a\equiv 3(5)}^{1270}\varepsilon(a)T^{2a}}{1-T^{2*1270}} \\
&+ \frac{\sum_{a\equiv 2(5)}^{1270}\varepsilon(a)T^{3a}}{1-T^{3*1270}} + \frac{\sum_{a\equiv 4(5)}^{1270}\varepsilon(a)T^{4a}}{1-T^{4*1270}}). \\
&= \frac{1}{5}\lambda((T-1)^{10} + (T-1)^{11} + \text{higher terms (mod p)}) = 2,
\end{aligned}$$

which agrees with the Table 1 of [4]. We used Maple for the second equality.

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