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QUANTITATIVE ESTIMATES FOR GENERALIZED TWO DIMENSIONAL BASKAKOV OPERATORS

Neha Bhardwaj and Naokant Deo

Abstract. In this paper, we obtain quantitative estimates for generalized double Baskakov operators. We calculate global results for these operators using Lipschitz-type spaces and estimate the error using modulus of continuity.

1. Introduction

The Baskakov operator $V_n(f;x)$ was introduced by V.A. Baskakov [2] given by:

$$V_n(f;x) = \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

where $p_{n,k}(x) = \begin{pmatrix} n+k-1 \\ k \end{pmatrix} x^k (1+x)^{-(n+k)}, f \in C_B[0,\infty), C_B[0,\infty)$ is the set of functions which are bounded on the set.
By now a number of results about the operator have been obtained [1, 3–7]. In this paper, we address the investigation for the multivariate Baskakov operator defined as follows.

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Let $P_n(f;x,y)$ are the following two variate Baskakov operators:

$$P_n(f;x,y)=\sum_{k,l=0}^{\infty}p_{n,k}(x)\,p_{n,l}(y)f\left(\frac{k}{n},\frac{l}{n}\right), \tag{1.2}$$

where $0\leq k\leq n, 0\leq l\leq n, f(x,y)\in C([0,\infty)\times[0,\infty))$.

Ozarslan and Duman [9] have introduced a different approach in order to get a faster approximation without preserving the test functions. In [8], Özarslan and Aktuğlu have calculated quantitative global estimates for double Szasz-Mirakjan operators. In this paper, same method is used for generalized double Baskakov operators.

Consider the classical Baskakov operators defined by (1.1). Since for $f_i=t^i, i=0,1,2$

$$V_n(f_0;x)=1, V_n(f_1;x)=x, V_n(f_2;x)=\left(1+\frac{1}{n}\right)x^2+\frac{x}{n}.$$

Following the similar arguments as used in [9], the best error estimation among all the general double Baskakov operators can be obtained from the case by taking

$$a_n=1, b_n=e_n=0, c_n=1+\frac{1}{n}, d_n=\frac{1}{n}$$

for all $n\in N$ where $(a_n),(b_n),(c_n),(d_n)$ and (e_n) are sequences of non-negative real numbers satisfying the conditions given in [9].

Now observe that

$$u_n(x)=\frac{2a_nx-d_n}{2c_n}=\frac{2nx-1}{2(n+1)}\in[0,\infty),$$

where u_n is a functional sequence, $u_n: I\to A$ where A denote $\mathbb{R}^+$ and assume that I be subinterval of A.

So, $u_n(x)\in\mathbb{R}^+$ if and only if $x\geq\frac{1}{2n}$ and $n\geq 1$. Hence, choosing

$$I=\left[\frac{1}{2},\infty\right)\subset\mathbb{R}^+.$$

The best error estimation among all the general double Baskakov operators can be obtained from the case

$$u_n(x)=\frac{2nx-1}{2(n+1)}, v_n(y)=\frac{2ny-1}{2(n+1)}; n\in\mathbb{N}$$

for all $f \in C_B([0,\infty) \times C_B[0,\infty))$ and $x, y \in \left[\frac{1}{2}, \infty\right)$. Hence, (1.2) becomes

$$(1.3) \qquad Q_n(f;x,y) : P_n(f;u_n(x), v_n(y)) = \sum_{k,l=0}^{h} \binom{n+k-1}{k} \binom{n+l-1}{l} f\left(\frac{k}{n}, \frac{l}{n}\right) (u_n(x))^k (1+u_n(x))^{-(n+k)} (v_n(y))^l (1+v_n(y))^{-(n+l)},$$
$$f \in C_B([0,\infty) \times [0,\infty)).$$

For the operators $Q_n(f;x,y)$, we have following Lemma:

LEMMA 1.1. *Let $\mathbf{x} = (x,y), \boldsymbol{t} = (t,s); e_{i,j}(x) = x^i y^j, i,j = 0,1,2$ and $\psi_x^2(t) = \|t-x\|^2$. Then, for each $x, y \geq 0$ and $n > 1$, we have*

(*i*) $Q_n(e_{0,0};x,y) = 1$

(*ii*) $Q_n(e_{1,0};x,y) = u_n(x)$

(*iii*) $Q_n(e_{0,1};x,y) = v_n(y)$

(*iv*) $Q_n(e_{2,0}+e_{0,2};x,y) = \left(1+\frac{1}{n}\right)(u_n^2(x)+v_n^2(y)) + \frac{u_n(x)+v_n(y)}{n}$

(*v*) $Q_n(\psi_x^2(t);x,y) = (u_n(x)-x)^2 + (v_n(y)-y)^2 + \frac{1}{n}(u_n^2(x)+v_n^2(y)+u_n(x)+v_n(y))$.

2. Global Results

We have used following definitions in this paper for global results of the operators $Q_n(f;x,y)$.
Otto Szasz [10] earlier considered this space of bivariate extension of Lipschitz-type space, given as:

$$Lip_M^*(\alpha) := \left\{ f \in C([0,\infty) \times [0,\infty)) : |f(\mathbf{t}) - f(\mathbf{x})| \leq M \frac{\|\mathbf{t}-\mathbf{x}\|^{\alpha}}{(\|\mathbf{t}\| + x + y)^{\alpha/2}} ; t, s; x, y \in (0,\infty) \right\},$$

where $\mathbf{t} = (t,s), \mathbf{x} = (x,y)$ and M is any positive constant and $0 < \alpha \leq 1$.

For all $f \in C([0,\infty) \times [0,\infty))$, the modulus of f denoted by $\omega(f;\delta)$ is defined as

$$\omega(f;\delta) := \sup\left\{ |f(t,s) - f(x,y)| : \sqrt{(t-x)^2 + (s-y)^2} < \delta, (t,s), \right. \\ \left. (x,y) \in [0,\infty) \times [0,\infty) \right\}.$$

Now, for the space $Lip_M^*(\alpha)$ with $0 < \alpha \leq 1$, we have the following approximation result.

THEOREM 2.1. *For any* $f \in Lip_M^*(\alpha), \alpha \in (0,1]$, *and for each* $x, y \in (0,\infty), n \in N$, *we have*

$$|Q_n(f;x,y) - f(x,y)| \leq \frac{M}{(x+y)^{\alpha/2}} \left[(u_n(x) - x)^2 + (v_n(y) - y)^2 \right. \\ \left. + \frac{1}{n}\left(u_n^2(x) + v_n^2(y) + u_n(x) + v_n(y)\right) \right]^{\alpha/2}. \tag{2.1}$$

Proof. Let $\alpha = 1$. For each $x, y \in (0,\infty)$ and for $f \in Lip_M^*(1)$, we have

$$\begin{aligned} |Q_n(f;x,y) - f(x,y)| &\leq Q_n(|f(t,s) - f(x,y)|;x,y) \\ &\leq MQ_n\left(\frac{\|\mathbf{t} - \mathbf{x}\|}{(\|\mathbf{t}\| + x + y)^{1/2}}; x, y \right) \\ &\leq \frac{M}{(x+y)^{1/2}} Q_n(\|\mathbf{t} - \mathbf{x}\|; x, y). \end{aligned}$$

Applying Cauchy-Schwarz inequality, we get

$$|Q_n(f;x,y) - f(x,y)| \leq \frac{M}{(x+y)^{1/2}} \sqrt{Q_n(\psi_x^2(t);x,y)}$$

$$= \frac{M}{(x+y)^{1/2}} \sqrt{(u_n(x) - x)^2 + (v_n(y) - y)^2 + \frac{1}{n}(u_n^2(x) + v_n^2(y) + u_n(x) + v_n(y))}.$$

Now, let $0<\alpha<1$. Then for each $x,y\in(0,\infty)$ and for $f\in Lip_M^*(\alpha)$, we obtain

$$\begin{aligned}
\left|Q_n(f;x,y)-f(x,y)\right| &\le Q_n\left(\left|f(t,s)-f(x,y)\right|;x,y\right)\\
&\le MQ_n\left(\frac{\|\mathbf{t}-\mathbf{x}\|^\alpha}{(\|\mathbf{t}\|+x+y)^{\alpha/2}};x,y\right)\\
&\le \frac{M}{(x+y)^{\alpha/2}}Q_n\left(\|\mathbf{t}-\mathbf{x}\|^\alpha;x,y\right).
\end{aligned}$$

For Holder inequality with $p=\frac{2}{\alpha}$ and $q=\frac{2}{2-\alpha}$, for any $f\in Lip_M^*(\alpha)$, we have

$$\begin{aligned}
\left|Q_n(f;x,y)-f(x,y)\right| &\le \frac{M}{(x+y)^{\alpha/2}}\left[Q_n\left(\psi_x^2(t);x,y\right)\right]^{\alpha/2}\\
&= \frac{M}{(x+y)^{\alpha/2}}\Bigg[\left(u_n(x)-x\right)^2+\left(v_n(y)-y\right)^2\\
&\quad+\frac{1}{n}\left(u_n^2(x)+v_n^2(y)+u_n(x)+v_n(y)\right)\Bigg]^{\alpha/2}
\end{aligned}$$

which is the required result. □

Lemma 2.2. *For each $x,y>0$,*

$$\begin{aligned}
Q_n&\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right)\\
&\le \frac{1}{\sqrt{x}}\sqrt{\left(u_n(x)-x\right)^2+\frac{u_n^2(x)+u_n(x)}{n}}\\
&\quad+\frac{1}{\sqrt{y}}\sqrt{\left(v_n(y)-y\right)^2+\frac{v_n^2(y)+v_n(y)}{n}}.
\end{aligned}\tag{2.2}$$

Proof. We have $\sqrt{c+d} \leq \sqrt{c} + \sqrt{d}\,(c, d \geq 0)$, therefore

$$Q_n\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2}; x, y\right)$$

$$= \sum_{k,l=0}^{\infty} \binom{n+k-1}{k} \binom{n+l-1}{l} \sqrt{\left(\sqrt{\frac{k}{n}}-\sqrt{x}\right)^2+\left(\sqrt{\frac{l}{n}}-\sqrt{y}\right)^2}$$

$$\times \left(u_n\left(x\right)\right)^k (1+u_n\left(x\right))^{-(n+k)} (v_n\left(y\right))^l (1+v_n\left(y\right))^{-(n+l)}$$

$$\leq \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left|\sqrt{\frac{k}{n}}-\sqrt{x}\right| \left(u_n\left(x\right)\right)^k (1+u_n\left(x\right))^{-(n+k)}$$

$$+ \sum_{l=0}^{\infty} \binom{n+l-1}{l} \left|\sqrt{\frac{l}{n}}-\sqrt{y}\right| (v_n\left(y\right))^l (1+v_n\left(y\right))^{-(n+l)}$$

$$= \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{\left|\frac{k}{n}-x\right|}{\sqrt{\frac{k}{n}}+\sqrt{x}} \left(u_n\left(x\right)\right)^k (1+u_n\left(x\right))^{-(n+k)}$$

$$+ \sum_{l=0}^{\infty} \binom{n+l-1}{l} \frac{\left|\frac{l}{n}-y\right|}{\sqrt{\frac{l}{n}}+\sqrt{y}} (v_n\left(y\right))^l (1+v_n\left(y\right))^{-(n+l)}$$

$$= \frac{1}{\sqrt{x}} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left|\frac{k}{n}-x\right| \left(u_n\left(x\right)\right)^k (1+u_n\left(x\right))^{-(n+k)}$$

$$+ \frac{1}{\sqrt{y}} \sum_{l=0}^{\infty} \binom{n+l-1}{l} \left|\frac{l}{n}-y\right| (v_n\left(y\right))^l (1+v_n\left(y\right))^{-(n+l)}.$$

Using the Cauchy-Schwarz inequality,

$$Q_n\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2}; x, y\right)$$

$$\leq \frac{1}{\sqrt{x}} \sqrt{\sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{k}{n}-x\right)^2 \left(u_n\left(x\right)\right)^k (1+u_n\left(x\right))^{-(n+k)}}$$

$$+ \frac{1}{\sqrt{y}} \sqrt{\sum_{l=0}^{\infty} \binom{n+l-1}{l} \left(\frac{l}{n}-y\right)^2 (v_n\left(y\right))^l (1+v_n\left(y\right))^{-(n+l)}}.$$

Using Lemma 1.1,

$$Q_n\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right)$$

$$\leq \frac{1}{\sqrt{x}}\sqrt{\left(u_n\left(x\right)-x\right)^2+\frac{u_n^2\left(x\right)+u_n\left(x\right)}{n}}+\frac{1}{\sqrt{y}}\sqrt{\left(v_n\left(y\right)-y\right)^2+\frac{v_n^2\left(y\right)+v_n\left(y\right)}{n}}$$

which is the desired result. □

THEOREM 2.3. *Let $g\left(x,y\right)=f\left(x^2,y^2\right)$. Then we have for each $x,y>0$,*

$$\left|Q_n\left(f;x,y\right)-f\left(x,y\right)\right|\leq 2\omega\left(g;\delta_n\left(x,y\right)\right),$$

where $\delta_n\left(x,y\right)=\frac{1}{\sqrt{x}}\sqrt{\left(u_n\left(x\right)-x\right)^2+\frac{u_n^2(x)+u_n(x)}{n}}+\frac{1}{\sqrt{y}}\sqrt{\left(v_n\left(y\right)-y\right)^2+\frac{v_n^2(y)+v_n(y)}{n}}$.

Proof. We have

$$\left|Q_n\left(f;x,y\right)-f\left(x,y\right)\right|\leq Q_n\left(\left|f\left(t,s\right)-f\left(x,y\right)\right|;x,y\right)$$

$$=Q_n\left(\left|g\left(\sqrt{t},\sqrt{s}\right)-g\left(\sqrt{x},\sqrt{y}\right)\right|;x,y\right)$$

$$\leq Q_n\left(\omega\left(g;\sqrt{\left(t-x\right)^2+\left(s-y\right)^2}\right);x,y\right)$$

$$=\sum_{k,l=0}^{\infty}\binom{n+k-1}{k}\binom{n+l-1}{l}\omega\left(g;\sqrt{\left(\sqrt{\frac{k}{n}}-\sqrt{x}\right)^2+\left(\sqrt{\frac{l}{n}}-\sqrt{y}\right)^2};x,y\right)$$

$$\times\left(u_n\left(x\right)\right)^k\left(1+u_n\left(x\right)\right)^{-(n+k)}\left(v_n\left(y\right)\right)^l\left(1+v_n\left(y\right)\right)^{-(n+l)}$$

$$=\sum_{k,l=0}^{\infty}\binom{n+k-1}{k}\binom{n+l-1}{l}$$

$$\omega\left(g;\frac{\sqrt{\left(\sqrt{\frac{k}{n}}-\sqrt{x}\right)^2+\left(\sqrt{\frac{l}{n}}-\sqrt{y}\right)^2}}{Q_n\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right)}\right.$$

$$\left.\times Q_n\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right);x,y\right).$$

Now, we have

$$\omega\left(f;\lambda\delta\right)\leq\left(1+\lambda\right)\omega\left(f;\delta\right).$$

Therefore,

$$\begin{aligned}
&|Q_n(f;x,y) - f(x,y)| \\
&\leq \omega\left(g; Q_n\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2 + \left(\sqrt{s}-\sqrt{y}\right)^2}; x,y\right)\right) \\
&\quad \times \sum_{k,l=0}^{\infty} \binom{n+k-1}{k}\binom{n+l-1}{l} \\
&\quad \left[1 + \frac{\sqrt{\left(\sqrt{\frac{k}{n}}-\sqrt{x}\right)^2 + \left(\sqrt{\frac{l}{n}}-\sqrt{y}\right)^2}}{Q_n\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2 + \left(\sqrt{s}-\sqrt{y}\right)^2}; x,y\right)}\right] \\
&\quad (u_n(x))^k (1+u_n(x))^{-(n+k)} (v_n(y))^l (1+v_n(y))^{-(n+l)} \\
&\leq 2\omega\left(g; Q_n\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2 + \left(\sqrt{s}-\sqrt{y}\right)^2}; x,y\right)\right).
\end{aligned}$$

Now, using Lemma 2.2, completes the proof. □

Theorem 2.4. *Let* $g(x,y) = f(x^2, y^2)$. *Let*

$$g \in Lip_M(\alpha) := \left\{ g \in C_{\mathbf{B}}([0,\infty)\times[0,\infty)) : |g(\boldsymbol{t}) - g(\mathbf{x})| \leq M\|\boldsymbol{t}-\mathbf{x}\|^{\alpha} \; ; t,s;x,y \in (0,\infty) \right\},$$

where $\boldsymbol{t} = (t,s)$, $\mathbf{x} = (x,y)$ *and* M *is any positive constant and* $0 < \alpha \leq 1$.
Then,

$$|Q_n(f;x,y) - f(x,y)| \leq M\delta_n^{\alpha}(x,y), \tag{2.3}$$

where $\delta_n(x,y) = \frac{1}{\sqrt{x}}\sqrt{(u_n(x)-x)^2 + \frac{u_n^2(x)+u_n(x)}{n}} + \frac{1}{\sqrt{y}}\sqrt{(v_n(y)-y)^2 + \frac{v_n^2(y)+v_n(y)}{n}}$.

Proof. We have

$$|Q_n(f;x,y)-f(x,y)| \leq Q_n(|f(t,s)-f(x,y)|;x,y)$$
$$= Q_n\left(\left|g\left(\sqrt{t},\sqrt{s}\right)-g\left(\sqrt{x},\sqrt{y}\right)\right|;x,y\right)$$
$$\leq MQ_n\left(\left(\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2\right)^{\alpha/2};x,y\right)$$
$$= M\sum_{k,l=0}^{\infty}\binom{n+k-1}{k}\binom{n+l-1}{l}\left(\left(\sqrt{\frac{k}{n}}-\sqrt{x}\right)^2+\left(\sqrt{\frac{l}{n}}-\sqrt{y}\right)^2\right)^{\alpha/2}$$
$$(u_n(x))^k(1+u_n(x))^{-(n+k)}(v_n(y))^l(1+v_n(y))^{-(n+l)}.$$

For Holder inequality with $p=\frac{2}{\alpha}$ and $q=\frac{2}{2-\alpha}$, we have

$$|Q_n(f;x,y)-f(x,y)| \leq M\left[Q_n\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right)\right]^{\alpha}.$$

By using Lemma 2.2, completes the proof. □

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Neha Bhardwaj
Department of Mathematics
Amity Institute of Applied Sciences
Amity University
Noida 201301, India
E-mail: neha_bhr@yahoo.co.in

Naokant Deo
Department of Applied Mathematics
Delhi Technological University
(Formerly Delhi College of Engineering)
Bawana Road, Delhi 110042, India
E-mail: dr_naokant_deo@yahoo.com