Korean J. Math. **24** (2016), No. 3, pp. 375–395
http://dx.doi.org/10.11568/kjm.2016.24.3.375

GLOBAL ATTRACTORS AND REGULARITY FOR THE EXTENSIBLE SUSPENSION BRIDGE EQUATIONS WITH PAST HISTORY

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Abstract. In this paper, we study the long-time dynamical behavior for the extensible suspension bridge equations with past history. We prove the existence of the global attractors by using the contraction function method. Furthermore, the regularity of global attractor is achieved.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary Γ. We consider the asymptotic behavior of the solutions for the following extensible suspension bridge equation with linear memory:

$$\begin{cases} u_{tt} + u_t + \triangle^2 u + (\alpha - \beta \|\nabla u\|^2_{L^2(\Omega)})\triangle u \\ \qquad - \int_0^\infty \mu(s)\triangle^2 u(t-s)ds + ku^+ = g(x) \text{ in } \Omega \times \mathbb{R}^+, \\ u = \triangle u = 0 \text{ on } \Gamma \times \mathbb{R}^+ , \\ u(0) = u_0(x), u_t(0) = u_1(x), \ x \in \Omega, \end{cases} \tag{1.1}$$

Received May 9, 2016. Revised June 27, 2016. Accepted August 2, 2016.
2010 Mathematics Subject Classification: 35B40, 35B41, 35B45.
Key words and phrases: extensible suspension bridge equations; global attractors; a bounded absorbing set; past history.
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This work was partly supported by the NSFC (11561064,11361053), and partly supported by NWNU-LKQN-14-6.

where $u(x,t)$ is an unknown function, which represents the deflection of the road bed in the vertical plane, the real constant α represents the axial force acting at the ends of the road bed of the bridge in the reference configuration, namely, α is negative when the bridge is stretched, positive when it compressed, $k>0$ denotes the spring constant, $u^+ = max\{u,0\}$ is the positive part of u, μ is the memory kernel, β is a given positive constant, $g(x) \in L^2(\Omega)$ is a external force.

It is well known that Lazer and McKenna presented the following suspension bridge equation

$$u_{tt} + u_t + \triangle^2 u + ku^+ = g \tag{1.2}$$

as a new problem of nonlinear analysis [1]. Adding a nonlinear force $f(u)$ to the model (1.2), it becomes

$$u_{tt} + \delta u_t + \triangle^2 u + ku^+ + f(u) = g. \tag{1.3}$$

Zhong et al. [3] proved the existence of strong solutions and global attractors for (1.3). Similar models have been studied by several authors, see $[4-10]$ and references therein. For example, Ma et al. investigated the existence of global attractors in [5, 6] as well as uniform compact attractors in [7] for the coupled suspension bridge equation. Park and Kang [8, 9] respectively obtained the pullback attractors for the non-autonomous suspension bridge equations and the global attractors for the autonomous suspension bridge equations with nonlinear damping. Xu and Ma [10] proved the existence of random attractors for the floating beam equation with strong damping and white noise.

If taking into account the midplane stretching of the road bed due to its elongation, then the following equation was arrived at

$$u_{tt} + \delta u_t + \triangle^2 u + (\alpha - \|\nabla u\|^2_{L^2(\Omega)})\triangle u + ku^+ = g. \tag{1.4}$$

There are some classical results for (1.4), for details see $[11-13]$. Recently, Ma and Xu [14] studied the random attractors for the extensible suspension bridge equation with white noise. The model (1.1) is derived by considering the effect of the past history in (1.4). As far as the relative some problems to the past history, the asymptotic behavior of solutions have been discussed in many literatures, please refer the reader to $[15-18]$. For (1.1), in the case when $\alpha=\beta=0$ and without the damping term u_t, Kang [15] proved the existence of global attractors relying on the construction of a suitable Lyapunov functional in the space $H^2(\Omega)\cap H^1_0(\Omega)\times L^2(\Omega)\times L^2_\mu(\mathbb{R}^+; H^2(\Omega)\cap H^1_0(\Omega))$. But in some cases, the damping term appearing in the equation is important and significant from the view of the actual applications. Therefore, we in this paper focus on the asymptotic behavior of the solutions for the extensible suspension bridge equation with linear damping and memory, we investigate the existence of the global attractor for equation (1.1). Furthermore, the regularity of global attractor is shown.

We know that it is very vital to verify the compactness in proving the existence of the global attractor. For our problem, there are two essential difficulties in showing the compactness. One difficulty is caused by a geometric nonlinearity, it makes our energy estimates more complex, so we need to more accurate calculation. Another difficulty is the memory kernel itself, because there is no compact embedding in the history space, moreover, we can't use the finite rank method, that is, we can't use the term $(I - P_m)u$ as a test function to deal with our problem. For our purpose, we have to introduce a new variable and define a extend Hilbert space, as well as combine with the contraction function method.

This paper is organized as follows: In Section 2, we give some preliminaries for our consideration, including the notation we will use, the assumption on nonlinearity term and some general abstract results. In Section 3, we prove our main results about the existence of global attractors. In Section 4, we obtain the regularity of global attractors.

2. Preliminaries

In this section, we introduce some notations, functional spaces and preliminaries results that will be used.

In order to obtain our main results, we first transform the equation (1.1) into a determined autonomous dynamical system by introducing a new variable. For this purpose, as in [16], we define

$$\eta = \eta^t(x,s) = u(x,t) - u(x,t-s),\ (x,s) \in \Omega \times \mathbb{R}^+,\ t \geqslant 0. \tag{2.1}$$

By formal differentiation in (2.1) we obtain

$$\eta_t^t(x,s) = -\eta_s^t(x,s) + u_t(x,t),\ (x,s) \in \Omega \times \mathbb{R}^+,\ t \geqslant 0. \tag{2.2}$$

Then we have

$$\eta^0(x,s) = u_0(x,0) - u_0(x,-s),\ (x,s) \in \Omega \times \mathbb{R}^+.$$

By assuming that $\mu \in L^1(\mathbb{R}^+)$, the original problem (1.1) can be transformed into the equivalent autonomous system

$$\begin{cases} u_{tt} + u_t + (1 - \int_0^\infty \mu(s)ds)\triangle^2 u + (\alpha - \beta\|\nabla u\|^2_{L^2(\Omega)})\triangle u \\ \qquad + \int_0^\infty \mu(s)\triangle^2\eta^t(s)ds + ku^+ = g(x) \text{ in } \Omega \times \mathbb{R}^+, \\ \eta_t = -\eta_s + u_t,\ (x,t,s) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^+ , \end{cases} \tag{2.3}$$

with the boundary conditions

$$u = \Delta u = 0,\ (x,t) \in \Gamma \times \mathbb{R}^+,\ \eta = \triangle\eta = 0,\ (x,t,s) \in \Gamma \times \mathbb{R}^+ \times \mathbb{R}^+, \tag{2.4}$$

and the initial conditions

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \eta^t(x,0) = 0,\ \eta^0(x,s) = \eta_0(x,s), \tag{2.5}$$

where

$$\begin{cases} u_0(x) = u_0(x,0),\ x \in \Omega, \\ u_1(x) = \partial_t u_0(x,t)\mid_{t=0},\ x \in \Omega, \\ \eta_0(x,s) = u_0(x,0) - u_0(x,-s),\ (x,s) \in \Omega \times \mathbb{R}^+. \end{cases}$$

Throughout this paper we use the standard functional space and denote $(\cdot,\cdot)$ be a $L^2(\Omega)$-inner product and $\|\cdot\|_p$ be $L^p(\Omega)$ norm. Especially, we take

$$H = V_0 = L^2(\Omega),\ V = V_1 = H^2(\Omega) \cap H_0^1(\Omega),$$

equipped with respective inner product and norm,

$$(u,v)_V = (\triangle u, \triangle v),\ \|u\|_V = \|\triangle u\|_2.$$

Define

$$D(A) = \{u \in H^4(\Omega) :\ u|_{\partial\Omega} = \triangle u|_{\partial\Omega} = 0\},$$

where $Au = \triangle^2 u$, and equip this space with the inner product (Au, Av) and the norm $\|Au\|_2^2 = (Au, Au)$.

Obviously, we have the following continuous dense injections:

$$D(A) \subset V \subset H = H^* \subset V^*,$$

where $H^*,\ V^*$ is a dual space of $H,\ V$ respectively.

In order to consider the relative displacement η as a new variable, one introduces the weighted L^2-space

$$L^2_\mu(\mathbb{R}^+; V_i) = \{\eta : \mathbb{R}^+ \to V_i | \int_0^\infty \mu(s)\|\eta(s)\|^2_{V_i} ds < \infty\},$$

which is a Hilbert space endowed with inner product and norm

$$(u, v)_{\mu,V_i} = \int_0^\infty \mu(r)(u(r), v(r))_{V_i} dr,$$

$$\|u\|^2_{\mu,V_i} = (u, u)_{\mu,V_i} = \int_0^\infty \mu(r)\|u(r)\|^2_{V_i} dr,\ i = 0, 1, 2,$$

respectively, where $V_2 = D(A^{\frac{3}{4}}),\ V_3 = D(A)$. Finally, we introduce the following Hilbert space

$$\mathcal{H}_0 = V \times H \times L^2_\mu(\mathbb{R}^+; V),$$

$$\mathcal{H}_1 = D(A) \times V \times L^2_\mu(\mathbb{R}^+; D(A)).$$

Using the Poincar*é* inequality we obtain

$$\lambda_1\|v\|^2_2 \leqslant \|\triangle v\|^2_2,\ \forall v \in V,$$

where λ_1 denotes the first eigenvalue of $\triangle^2 v = \lambda v$ in Ω with $v = \triangle v = 0$ on Γ.

We present the following conditions about memory kernel

(H_1) $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+),\ \mu'(s) \leqslant 0 \leqslant \mu(s),\ \forall s \in \mathbb{R}^+$;

(H_2) $l = 1 - \int_0^\infty \mu(s)ds = 1 - \mu_0 > 0,\ \forall s \in \mathbb{R}^+$;

(H_3) $\mu'(s) + \delta\mu(s) \leqslant 0,\ \forall s \in \mathbb{R}^+,\ \delta > 0$.

In order to obtain the global attractors of the problem (2.3)-(2.5), we need the following theorem. The well-posedness of the problem (2.3)-(2.5) can be obtained by Faedo-Galerkin method (see [19]) and combining with a prior estimate of 3.1, we omit it and only give the following theorem:

THEOREM 2.1. *Assume that assumptions* $(H_1) - (H_3)$ *hold and* $g \in L^2(\Omega)$. *Problem* (2.3)$-$(2.5) *has a weak solution* $(u, u_t, \eta) \in C([0, T], \mathcal{H}_0)$ *with initial data* $(u_0, u_1, \eta_0) \in \mathcal{H}_0$, *satisfying*

$$u \in L^\infty(0, T; V),\ u_t \in L^\infty(0, T; H),\ \eta \in L^\infty(0, T; L^2_\mu(\mathbb{R}^+, V))$$

and the mapping $\{u_0, u_1, \eta_0\} \to \{u(t), u_t(t), \eta^t\}$ *is continuous in* $\mathcal{H}_0$. *In addition, if* $z^i(t) = (u^i(t), u^i_t(t), \eta^i)$ *be weak solution of problem* (2.3) $-$

(2.5) *corresponding to initial data* $z^i(0) = (u^i(0), u_1^i, \eta_0^i),\ i = 1, 2$. *Then one has*

$$\|z_1(t) - z_2(t)\|_{\mathcal{H}_0} \leqslant e^{ct}\|z_1(0) - z_2(0)\|_{\mathcal{H}_0},\ t \in [0, T],$$

for some constant $c > 0$.

The well-posedness of problem (2.3)-(2.5) implies that the family of operators $S(t) : \mathcal{H}_0 \to \mathcal{H}_0$ defined by

$$S(t)(u_0, u_1, \eta_0) = (u(t), u_t(t), \eta^t),\ t \geqslant 0,$$

where $(u(t), u_t(t), \eta^t)$ is the unique weak solution of the system (2.3)-(2.5), satisfies the semigroup properties and defines a nonlinear C_0-semigroup, which is locally Lipschitz continuous on $\mathcal{H}_0$.

Now, we recall some fundamentals of the theory of infinite dimensional systems in mathematical physics, these abstract results will be used in our consideration.

Definition 2.1. ([2]) A dynamical system $(\mathcal{H}, S(t))$ is dissipative if it possesses a bounded absorbing set, that is, a bounded set $\mathcal{B} \subset \mathcal{H}$ such that for any bounded set $B \subset \mathcal{H}$ there exists $t_B \geqslant 0$ satisfying

$$S(t)B \subset \mathcal{B},\ \forall t \geqslant t_B.$$

Definition 2.2. ([18]) *Let X be a Banach space and B be a bounded subset of X. We call a function* $\phi(\cdot, \cdot)$ *which is defined on* $X \times X$ *a contractive function on* $B \times B$ *if for any sequence* $\{x_n\}_{n=1}^\infty \subset B$, *there is a subsequence* $\{x_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$, *such that*

$$\lim_{k\to\infty}\lim_{l\to\infty}\phi(x_{n_k}, x_{n_l}) = 0. \tag{2.6}$$

Denote all such contractive functions on $B \times B$ by $\mathfrak{C}$.

Definition 2.3. ([18])Let $\{S(t)\}_{t\geqslant 0}$ be a semigroup on a Banach space $(X, \|\cdot\|)$ that has a bounded absorbing set B_0. Moreover, assume that for $\epsilon > 0$ there exist $T = T(B_0, \epsilon)$ and $\phi_T(\cdot, \cdot) \in \mathfrak{C}(B_0)$ such that

$$\|S(T)x - S(T)y\| \leqslant \epsilon + \phi_T(x, y),\ \forall (x, y) \in B_0,$$

where ϕ_T depends on T. Then $\{S(t)\}_{t\geqslant 0}$ is asymptotically compact in X, i.e., for any bounded sequence $\{y_n\}_{n=1}^\infty \subset X$ and $\{t_n\}$ with $t_n \to \infty$, $\{S(t_n)y_n\}_{n=1}^\infty$ is precompact in X.

Theorem 2.2. ([2]) *A dissipative dynamical system* $(\mathcal{H}, S(t))$ *has a compact global attractor if and only if it is asymptotically smooth.*

Our main result is the following:

THEOREM 2.3. *Assume that assumptions* $(H_1)-(H_3)$ *hold and* $g \in L^2(\Omega)$, $\alpha \in \mathbb{R}$, β, $k>0$, *then the dynamical system* $(\mathcal{H}_0, S(t))$ *corresponding to the system* (2.3) $-$ (2.5) *has a compact global attractor* $\mathcal{A} \subset \mathcal{H}_0$, *which attracts any bounded set in* $\mathcal{H}_0$ *with* $\|\cdot\|_{\mathcal{H}_0}$.

3. Global attractor in $\mathcal{H}_0$

In order to prove Theorem 2.3, we will apply the abstract results presented in Section 2. The first step is to show that the dynamical system $(\mathcal{H}_0, S(t))$ is dissipative. The second step is to verify the asymptotic compactness. Then the existence of a compact global attractor is guaranteed by Theorem 2.2.

3.1. A priori estimates in $\mathcal{H}_0$

First, taking the scalar product in H of the first equation of (2.3) with $v = u_t + \sigma u$, after a computation, we find

$$\frac{1}{2}\frac{d}{dt}(l\|\triangle u\|_2^2 + \|v\|_2^2 + k\|u^+\|_2^2) + \sigma l\|\triangle u\|_2^2 + (1-\sigma)(u_t, v)$$
$$+((\alpha - \beta\|\nabla u\|_2^2)\triangle u, v) + (\eta^t, u_t)_{\mu,V} + \sigma(\eta^t, u)_{\mu,V} + \sigma k\|u^+\|_2^2 = (g, v). \tag{3.1}$$

Exploiting $(H_2)-(H_3)$ and *Hölder* inequality, we have

$$(1-\sigma)(u_t, v) = (1-\sigma)\|v\|_2^2 - \sigma(1-\sigma)(u, v),$$

$$\begin{aligned}
(\eta^t, u_t)_{\mu,V} =&(\eta^t, \eta^t_t + \eta^t_s)_{\mu,V} = \frac{1}{2}\frac{d}{dt}\|\eta^t\|_{\mu,V}^2 + \int_0^\infty \mu(s)(\eta^t(s), \eta^t_s(s))_V ds \\
=&\frac{1}{2}\frac{d}{dt}\|\eta^t\|_{\mu,V}^2 + \frac{1}{2}\int_0^\infty \mu(s) d\|\eta^t(s)\|_V^2 \\
=&\frac{1}{2}\frac{d}{dt}\|\eta^t\|_{\mu,V}^2 - \frac{1}{2}\int_0^\infty \mu'(s)\|\eta^t(s)\|_V^2 ds \\
\geqslant&\frac{1}{2}\frac{d}{dt}\|\eta^t\|_{\mu,V}^2 + \frac{\delta}{2}\int_0^\infty \mu(s)\|\eta^t(s)\|_V^2 ds \\
=&\frac{1}{2}\frac{d}{dt}\|\eta^t\|_{\mu,V}^2 + \frac{\delta}{2}\|\eta^t\|_{\mu,V}^2,
\end{aligned}$$

$$\sigma(\eta^t, u)_{\mu,V} \geqslant -\frac{\delta}{4}\|\eta^t\|_{\mu,V}^2 - \frac{(1-l)\sigma^2}{\delta}\|\triangle u\|_2^2,$$

and

$$
\begin{aligned}
&((\alpha-\beta\|\nabla u\|_2^2)\triangle u, v)\\
=&((\alpha-\beta\|\nabla u\|_2^2)\triangle u, u_t+\sigma u)\\
=&-\frac{\alpha}{2}\frac{d}{dt}\|\nabla u\|_2^2-\sigma\alpha\|\nabla u\|_2^2+\frac{\beta}{4}\frac{d}{dt}\|\nabla u\|_2^4+\sigma\beta\|\nabla u\|_2^4\\
\geqslant&\frac{1}{2}\frac{d}{dt}\left(\sqrt{\frac{\beta}{2}}\|\nabla u\|_2^2-\frac{\sqrt{2}\alpha}{2\sqrt{\beta}}\right)^2+\sigma\left(\sqrt{\frac{\beta}{2}}\|\nabla u\|_2^2-\frac{\sqrt{2}\alpha}{2\sqrt{\beta}}\right)^2-\frac{\sigma\alpha^2}{2\beta}.
\end{aligned}
$$

Hence we conclude from (3.1) that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(l\|\triangle u\|_2^2+\|v\|_2^2+\left(\sqrt{\frac{\beta}{2}}\|\nabla u\|_2^2-\frac{\sqrt{2}\alpha}{2\sqrt{\beta}}\right)^2+\|\eta^t\|_{\mu,V}^2+k\|u^+\|_2^2\right)\\
&+\sigma l\left(1-\frac{(1-l)\sigma}{\delta l}\right)\|\triangle u\|_2^2+(1-\sigma)\|v\|_2^2-\sigma(1-\sigma)(u,v)\\
&+\sigma\left(\sqrt{\frac{\beta}{2}}\|\nabla u\|_2^2-\frac{\sqrt{2}\alpha}{2\sqrt{\beta}}\right)^2+\frac{\delta}{4}\|\eta^t\|_{\mu,V}^2+\sigma k\|u^+\|_2^2\leqslant(g,v)+\frac{\sigma\alpha^2}{2\beta}.
\end{aligned}
\tag{3.2}
$$

Choose σ small enough, such that

$$
1-\frac{(1-l)\sigma}{\delta}-\frac{\sigma}{2\lambda_1}\geqslant 1-\sigma,\ \frac{1}{2}-\sigma\geqslant\frac{1}{4},
$$

then combining with $H\ddot{o}lder$, $Young$ and Poincar$\acute{e}$ inequalities, we obtain

$$
\begin{aligned}
&\sigma l\left(1-\frac{(1-l)\sigma}{\delta l}\right)\|\triangle u\|_2^2+(1-\sigma)\|v\|_2^2-\sigma(1-\sigma)(u,v)\\
\geqslant&\sigma l\left(1-\frac{(1-l)\sigma}{\delta l}\right)\|\triangle u\|_2^2+(1-\sigma)\|v\|_2^2-\frac{\sigma}{\sqrt{\lambda_1}}\|\triangle u\|_2\|v\|_2\\
\geqslant&\sigma l\left(1-\frac{(1-l)\sigma}{\delta l}\right)\|\triangle u\|_2^2+(1-\sigma)\|v\|_2^2-\left(\frac{\sigma^2}{2\lambda_1}\|\triangle u\|_2^2+\frac{1}{2}\|v\|_2^2\right)\\
=&\sigma l\left(1-\frac{(1-l)\sigma}{\delta l}-\frac{\sigma}{2\lambda_1 l}\right)\|\triangle u\|_2^2+(\frac{1}{2}-\sigma)\|v\|_2^2\\
\geqslant&\sigma l(1-\sigma)\|\triangle u\|_2^2+\frac{1}{4}\|v\|_2^2.
\end{aligned}
\tag{3.3}
$$

In addition,

$$(g, v) = 2\|g\|_2^2 + \frac{1}{8}\|v\|_2^2. \tag{3.4}$$

Consequently, collecting with (3.2)-(3.4), there holds

$$\begin{aligned}
&\frac{d}{dt}\left(l\|\triangle u\|_2^2 + \|v\|_2^2 + \left(\sqrt{\frac{\beta}{2}}\|\nabla u\|_2^2 - \frac{\sqrt{2}\alpha}{2\sqrt{\beta}}\right)^2 + \|\eta^t\|_{\mu,V}^2 + k\|u^+\|_2^2\right)\\
&+2\sigma l(1-\sigma)\|\triangle u\|_2^2 + \frac{1}{2}\|v\|_2^2 + 2\sigma\left(\sqrt{\frac{\beta}{2}}\|\nabla u\|_2^2 - \frac{\sqrt{2}\alpha}{2\sqrt{\beta}}\right)^2\\
&+\frac{\delta}{2}\|\eta^t\|_{\mu,V}^2 + 2\sigma k\|u^+\|_2^2 \leqslant 4\|g\|_2^2 + \frac{\sigma\alpha^2}{\beta}.
\end{aligned} \tag{3.5}$$

Provided that $\sigma_0 = \min\left\{2\sigma(1-\sigma), \frac{1}{2}, \frac{\delta}{2}\right\}$, let

$$E(t) = l\|\triangle u\|_2^2 + \|v\|_2^2 + \left(\sqrt{\frac{\beta}{2}}\|\nabla u\|_2^2 - \frac{\sqrt{2}\alpha}{2\sqrt{\beta}}\right)^2 + \|\eta^t\|_{\mu,V}^2 + k\|u^+\|_2^2,$$

we have

$$\frac{d}{dt}E(t) + \sigma_0 E(t) \leqslant 4\|g\|_2^2 + \frac{\sigma\alpha^2}{\beta} = C_1.$$

By the Gronwall lemma, we get

$$E(t) \leqslant E(t_0)e^{-\sigma_0 t} + \frac{C_1}{\sigma_0},\ \forall t \geqslant 0.$$

Thus, we get the existence of bounded absorbing set in $\mathcal{H}_0$, this is the following results:

Lemma 3.1. *Assume that assumptions $(H_1)-(H_3)$ hold and $g \in L^2(\Omega)$, $\alpha \in \mathbb{R}$, β, $k > 0$, then the ball of $\mathcal{H}_0$, $B_0 = B_{\mathcal{H}_0}(0, \mu_0)$, centered at 0 of radius $\mu_0 = \sqrt{\frac{C_1}{\sigma_0}}$, is an absorbing set in $\mathcal{H}_0$ for the group $\{S(t)\}_{t\geqslant 0}$ generated by problem $(2.3)-(2.5)$, namely, for any bounded subset B in $\mathcal{H}_0$, $S(t)B \subset B_0$ for $t \geqslant t_0(B)$.*

On the other hand, from the above discussion, there exist a constant $\mu_1 > \mu_0$, such that

$$\|\triangle u\|_2^2 + \|v\|_2^2 + \|\eta^t\|_{\mu,V}^2 \leqslant \mu_1^2,\ \forall t \geqslant t_0. \tag{3.6}$$

3.3. Existence of global attractor

First we prove an important Lemma.

Lemma 3.2. *Under the hypotheses of Theorem 2.3, there exists a constant $\mu_2 > \mu_0$, such that*

$$\|\nabla\triangle u\|_2^2 + \|\nabla u_t\|_2^2 + \|\eta^t\|^2_{\mu,D(A^{\frac{3}{4}})} \leqslant \mu_2^2,\ \forall t \geqslant t_0. \tag{3.7}$$

Proof. Multiplying the first equation of (2.3) by $-\triangle\varsigma = -\triangle u_t - \sigma\triangle u$, and integrating over Ω, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(l\|\nabla\triangle u\|_2^2 + \|\nabla\varsigma\|_2^2 + k\|\nabla u^+\|_2^2\right) + \sigma l\|\nabla\triangle u\|_2^2 + (1-\sigma)(u_t, -\triangle\varsigma)\\
&\quad+(\eta^t, u_t)_{\mu,D(A^{\frac{3}{4}})} + \sigma(\eta^t, u)_{\mu,D(A^{\frac{3}{4}})} + \sigma k\|\nabla u^+\|_2^2 + (g, \triangle\varsigma)\\
&= -((\alpha - \beta\|\nabla u\|_2^2)\triangle u, -\triangle\varsigma).
\end{aligned} \tag{3.8}$$

Similar to the previous estimates, we see that

$$(1-\sigma)(u_t, -\triangle\varsigma) = (1-\sigma)\|\nabla\varsigma\|_2^2 - \sigma(1-\sigma)(u, \nabla\varsigma),$$

$$(\eta^t, u_t)_{\mu,D(A^{\frac{3}{4}})} \geqslant \frac{1}{2}\frac{d}{dt}\|\eta^t\|^2_{\mu,D(A^{\frac{3}{4}})} + \frac{\delta}{2}\|\eta^t\|^2_{\mu,D(A^{\frac{3}{4}})},$$

and

$$\sigma(\eta^t, u)_{\mu,D(A^{\frac{3}{4}})} \geqslant -\frac{\delta}{4}\|\eta^t\|^2_{\mu,D(A^{\frac{3}{4}})} - \frac{(1-l)\sigma^2}{\delta}\|\nabla\triangle u\|_2^2.$$

Like the estimate of (3.2), there holds

$$\begin{aligned}
&\sigma l\left(1 - \frac{(1-l)\sigma}{\delta l}\right)\|\nabla\triangle u\|_2^2 + (1-\sigma)\|\nabla\varsigma\|_2^2 - \sigma(1-\sigma)(\nabla u, \nabla\varsigma)\\
&\geqslant \sigma l(1-\sigma)\|\nabla\triangle u\|_2^2 + \frac{1}{4}\|\nabla\varsigma\|_2^2.
\end{aligned}$$

Then we get from (3.8)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(l\|\nabla\triangle u\|_2^2 + \|\nabla\varsigma\|_2^2 + \|\eta^t\|^2_{\mu,D(A^{\frac{3}{4}})} + k\|\nabla u^+\|_2^2\right) + \sigma l(1-\sigma)\|\nabla\triangle u\|_2^2\\
&\quad+\frac{1}{4}\|\nabla\varsigma\|_2^2 + \frac{\delta}{4}\|\eta^t\|^2_{\mu,D(A^{\frac{3}{4}})} + k\|\nabla u^+\|_2^2 \leqslant -((\alpha - \beta\|\nabla u\|_2^2)\triangle u, \varsigma).
\end{aligned} \tag{3.9}$$

From (3.6), $Young$, *Hölder* and Poincaré inequalities, it follows that

$$
\begin{aligned}
&|-((\alpha-\beta\|\nabla u\|_2^2)\triangle u,-\triangle\varsigma)|\\
&\quad\leqslant((\alpha-\beta\|\nabla u\|_2^2)\triangle u,-\triangle u_t-\sigma\triangle u)\\
&\quad=-\frac{1}{2}\frac{d}{dt}(\alpha-\beta\|\triangle u\|_2^2)\|\triangle u\|_2^2+\beta\|\nabla u\|_2\|\nabla u_t\|_2\|\triangle u\|_2^2\\
&\quad-\sigma(\alpha-\beta\|\nabla u\|_2^2)\|\triangle u\|_2^2\\
&\quad\leqslant-\frac{1}{2}\frac{d}{dt}(\alpha-\beta\|\nabla u\|_2^2)\|\triangle u\|_2^2-\sigma(\alpha-\beta\|\nabla u\|_2^2)\|\triangle u\|_2^2 \qquad (3.10)\\
&\quad+\frac{\sigma}{2}\|\nabla u_t\|_2^2+\frac{\beta^2\mu_1^6}{2\sigma}\\
&\quad\leqslant-\frac{1}{2}\frac{d}{dt}(\alpha-\beta\|\nabla u\|_2^2)\|\triangle u\|_2^2-\sigma(\alpha-\beta\|\nabla u\|_2^2)\|\triangle u\|_2^2\\
&\quad+\frac{\sigma}{2}\|\nabla\varsigma\|_2^2+\frac{\sigma^3}{2\lambda_1}\|\nabla\triangle u\|_2^2+\frac{\beta^2\mu_1^6}{2\sigma},
\end{aligned}
$$

in above inequality, we use the fact that $\|\nabla u_t\|_2^2=\|\nabla\varsigma-\sigma\nabla u\|_2^2\leqslant\|\nabla\varsigma\|_2^2+\sigma^2\|\nabla u\|_2^2$, where δ is a proper positive constant.
Combining with (3.9) – (3.10), we can obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(l\|\nabla\triangle u\|_2^2+\|\nabla\varsigma\|_2^2+\|\eta^t\|^2_{\mu,D(A^{\frac{3}{4}})}+(\alpha-\beta\|\nabla u\|_2^2)\|\triangle u\|_2^2)+k\|\nabla u^+\|_2^2\right)\\
&+l\left(2\sigma(1-\sigma)-\frac{\sigma^3}{l\lambda_1}\right)\|\nabla\triangle u\|_2^2+\left(\frac{1}{2}-\sigma\right)\|\nabla\varsigma\|_2^2+\frac{\delta}{2}\|\eta^t\|^2_{\mu,D(A^{\frac{3}{4}})}\\
&+2\sigma(\alpha-\beta\|\nabla u\|_2^2)\|\triangle u\|_2^2+2\sigma k\|\nabla u^+\|_2^2\leqslant\frac{\beta^2\mu_1^6}{\sigma}.
\end{aligned}
\tag{3.11}
$$

Taking σ small enough, such that

$$2\sigma(1-\sigma)-\frac{\sigma^3}{l\lambda_1}>0,\ \frac{1}{2}-\sigma>0.$$

Thus, denote

$$Y(t)=l\|\nabla\triangle u\|_2^2+\|\nabla\varsigma\|_2^2+\|\eta^t\|^2_{\mu,D(A^{\frac{3}{4}})}+(\alpha-\beta\|\nabla u\|_2^2)\|\triangle u\|_2^2+k\|\nabla u^+\|_2^2,$$

we have

$$\frac{d}{dt}Y(t)+\alpha_0Y(t)\leqslant C_2.$$

where $\alpha_0=\min\left\{2\sigma(1-\sigma)-\frac{\sigma^3}{l\lambda_1},\frac{1}{2}-\sigma,\frac{\delta}{2}\right\},\ C_2=\frac{\beta^2\mu_1^6}{\sigma}$.

By the Gronwall lemma, we get

$$Y(t) \leqslant e^{-\alpha_0 t} Y(0) + \frac{C_2}{\alpha_0}.$$

Because of $Y(t) \geqslant \|\nabla \triangle u\|_2^2 + \|\nabla u_t\|_2^2 + \|\eta^t\|^2_{\mu, D(A^{\frac{3}{4}})}$, we have (3.7). $\square$

Next we show an essential inequality to prove Theorem 2.3.

Lemma 3.3. *Under the hypotheses of Theorem 2.3, given a bounded set $B \subset \mathcal{H}_0$, let $z_1 = (u, u_t, \eta)$ and $z_2 = (v, v_t, \xi)$ be two weak solutions of problem (2.3) – (2.5) such that $z_1(0) = (u_0, u_1, \eta_0)$ and $z_2(0) = (v_0, v_1, \xi_0)$ are in B. Then*

$$\begin{aligned} &\|z_1(t) - z_2(t)\|^2_{\mathcal{H}_0} \\ \leqslant & e^{-\alpha_1 t}\|z_1(0) - z_2(0)\|^2_{\mathcal{H}_0} + C_3 \int_0^t e^{-\alpha_1 (t-s)} \|u(s) - v(s)\|^2_{2(p+1)} ds \\ &+ C_4 \int_0^t e^{-\alpha_1 (t-s)} \|\nabla u(s) - \nabla v(s)\|_2^2 ds, \ \forall t \geqslant 0, \end{aligned} \tag{3.12}$$

where $\alpha_1 > 0$ is a small constant and $p,\ C_3,\ C_4$ are positive constants.

Proof. Let us fix a bounded set $B \subset \mathcal{H}_0$. We set $w = u - v$ and $\zeta = \eta - \xi$. Then (w, ζ) satisfy

$$\begin{cases} w_{tt} + w_t + l\triangle^2 w + (\alpha - \beta\|\nabla u\|_2^2)\triangle w + \int_0^\infty \mu(s)\Delta^2 \zeta^t(s) ds \\ +k(u^+ - v^+) = 0, \\ \zeta_t = -\zeta_s + w_t, \end{cases} \tag{3.13}$$

with initial condition

$$w(0) = u_0 - v_0,\ w_t(0) = u_1 - v_1,\ \zeta^0 = \eta_0 - \xi_0.$$

Taking the scalar product in H of the first equation of (3.13) with $\varsigma = w_t + \sigma w$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(l\|\triangle w\|_2^2 + \|\varsigma\|_2^2) + \sigma l \|\triangle w\|_2^2 + (1-\sigma)(w_t, \varsigma) + (\zeta^t, w_t)_{\mu,V} + \sigma(\zeta^t, w)_{\mu,V} \\ &= -(k(u^+ - v^+), \varsigma) - ((\alpha - \beta\|\nabla u\|_2^2)\triangle w, \varsigma). \end{aligned} \tag{3.14}$$

Combining with previous discussion, we can obtain

$$(1-\sigma)(w_t, \varsigma) = (1-\sigma)\|\varsigma\|_2^2 - \sigma(1-\sigma)(w, \varsigma),$$

$$(\zeta^t, w_t)_{\mu,V} \geqslant \frac{1}{2}\frac{d}{dt}\|\zeta^t\|_{\mu,V}^2 + \frac{\delta}{2}\|\zeta^t\|_{\mu,V}^2,$$

and

$$\sigma(\zeta^t, w)_{\mu,V} \geqslant -\frac{\delta}{4}\|\zeta^t\|_{\mu,V}^2 - \frac{(1-l)\sigma^2}{\delta}\|\triangle w\|_2^2.$$

Like the estimate of (3.2), there holds

$$\begin{aligned}
&\sigma l\left(1-\frac{(1-l)\sigma}{\delta l}\right)\|\triangle w\|_2^2 + (1-\sigma)\|\varsigma\|_2^2 - \sigma(1-\sigma)(w,\varsigma)\\
&\geqslant \sigma l(1-\sigma)\|\triangle w\|_2^2 + \frac{1}{4}\|\varsigma\|_2^2.
\end{aligned}$$

Then we get from (3.14)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(l\|\triangle w\|_2^2 + \|\varsigma\|_2^2 + \|\zeta^t\|_{\mu,V}^2) + \sigma l(1-\sigma)\|\triangle w\|_2^2 + \frac{1}{4}\|\varsigma\|_2^2 + \frac{\delta}{4}\|\zeta^t\|_{\mu,V}^2\\
&\quad\leqslant -((\alpha-\beta\|\nabla u\|_2^2)\triangle w,\varsigma) - (k(u^+-v^+),\varsigma).
\end{aligned}\tag{3.15}$$

From (3.6), (3.7), *Young* and *Hölder* inequalities, we see that

$$\begin{aligned}
&|-((\alpha-\beta\|\nabla u\|_2^2)\triangle w,\varsigma)|\\
\leqslant&((\alpha-\beta\|\nabla u\|_2^2)\triangle w, w_t+\sigma w)\\
=&-\frac{1}{2}\frac{d}{dt}(\alpha-\beta\|\nabla u\|_2^2)\|\nabla w\|_2^2 + \beta\|\nabla u\|_2\|\nabla u_t\|_2\|\nabla w\|_2^2\\
&-\sigma(\alpha-\beta\|\nabla u\|_2^2)\|\nabla w\|_2^2\\
\leqslant&-\frac{1}{2}\frac{d}{dt}(\alpha-\beta\|\nabla u\|_2^2)\|\nabla w\|_2^2 + \beta\mu_1\mu_2\|\nabla w\|_2^2\\
&-\sigma(\alpha-\beta\|\nabla u\|_2^2)\|\nabla w\|_2^2.
\end{aligned}\tag{3.16}$$

Thanks to the *Young* and Poincaré inequalities, we obtain

$$\begin{aligned}
&|-(k(u^+-v^+),\varsigma)|\\
=&\left|-\int_\Omega k(u^+-v^+)(w_t+\sigma w)dx\right|\\
\leqslant&\int_\Omega k(u^+-v^+)w_t dx + \int_\Omega k(u^+-v^+)\sigma w dx\\
\leqslant&\frac{k^2}{\sigma}\|u^+-v^+\|_2^2 + \frac{\sigma}{4}\|w_t\|_2^2 + \sigma kL\|w\|_2^2\\
\leqslant&\left(\frac{k^2Lc_0}{\sigma} + \sigma kLc_0 + \frac{\sigma^3c_0}{4}\right)\|w\|_{2(p+1)}^2 + \frac{\sigma}{4}\|\varsigma\|_2^2,
\end{aligned}\tag{3.17}$$

in above inequality, we use the fact that $|u^+(t)-v^+(t)| \leqslant L|u(t)-v(t)| \leqslant L|w(t)|$, $\|w_t\|_2^2 = \|\varsigma-\sigma w\|_2^2 \leqslant \|\varsigma\|_2^2+\sigma^2\|w\|_2^2$, where L is a proper positive constant and $c_0 > 0$ is an embedding constant for $L^{2(p+1)}(\Omega) \hookrightarrow L^2(\Omega)$. Integrating with (3.16)-(3.17), we get from (3.15)

$$\begin{aligned}
&\frac{d}{dt}(l\|\triangle w\|_2^2 + \|\varsigma\|_2^2 + \|\zeta^t\|_{\mu,V}^2 + (\alpha-\beta\|\nabla u\|_2^2)\|\nabla w\|_2^2)) \\
&\quad + l(2\sigma(1-\sigma))\|\triangle w\|_2^2 + \left(\frac{1}{2}-\frac{\sigma}{2}\right)\|\varsigma\|_2^2 + \frac{\delta}{2}\|\zeta^t\|_{\mu,V}^2 \\
&\quad + 2\sigma(\alpha-\beta\|\nabla u\|_2^2)\|\triangle w\|_2^2 \\
&\leqslant \left(\frac{2k^2Lc_0}{\sigma} + 2\sigma kLc_0 + \frac{\sigma^3 c_0}{2}\right)\|w\|_{2(p+1)}^2 + 2\beta\mu_1\mu_2\|\nabla w\|_2^2.
\end{aligned} \tag{3.18}$$

Choosing σ small enough, such that

$$2\sigma(1-\sigma) > 0,\ \frac{1}{2}-\frac{\sigma}{2} > 0.$$

Thus, denote

$$W(t) = l\|\triangle w\|_2^2 + \|\varsigma\|_2^2 + \|\zeta^t\|_{\mu,V}^2 + (\alpha-\beta\|\nabla u\|_2^2)\|\nabla w\|_2^2,$$

we have

$$\frac{d}{dt}E(t) + \alpha_1 E(t) \leqslant C_3\|w\|_{2(p+1)}^2 + C_4\|\nabla w\|_2^2,$$

where $\alpha_1 = \min\left\{2\sigma(1-\sigma), \frac{1-\sigma}{2}, \frac{\delta}{2}\right\}$, $C_3 = \frac{2k^2Lc_0}{\delta_0} + 2\sigma kLc_0 + \frac{\sigma^3 c_0}{2}$, $C_4 = 2\beta\mu_1\mu_2$, which implies that

$$\begin{aligned}
E(t) \leqslant e^{-\alpha_1 t}E(0) + C_3\int_0^t e^{-\alpha_1(t-s)}\|w\|_{2(p+1)}^2 ds \\
+C_4\int_0^t e^{-\alpha_1(t-s)}\|\nabla u(s)-\nabla v(s)\|_2^2 ds.
\end{aligned}$$

Because of $E(t) \geqslant \|z_1(t)-z_2(t)\|_{\mathcal{H}_0}^2$, we have (3.12). □

Lemma 3.4. *Under assumptions of Theorem* 2.3, *the dynamical system* $(\mathcal{H}_0, S(t))$ *corresponding to problem* (2.3) – (2.5) *is asymptotically smooth.*

Proof. Let B be a bounded subset of $\mathcal{H}_0$ positively invariant with respect to S(t). Denote by C_B several positive constants that are dependent on B but not on t. For $z_0^1,\ z_0^2 \in B$, $S(t)z_0^1 = (u(t), u_t(t), \eta^t)$ and

$S(t)z_0^2 = (v(t), v_t(t), \xi^t)$ are the solutions of $(2.3)-(2.5)$. Then given $\epsilon > 0$, from inequality (3.7), we can choose $T > 0$ such that

$$\begin{aligned} &\|S(t)z_0^1 - S(t)z_0^2\|_{\mathcal{H}_0} \\ &\leqslant \epsilon + C_B \left(\int_0^T \|u(s) - v(s)\|_{2(p+1)}^2 ds + \int_0^T \|\nabla u(s) - \nabla v(s)\|_2^2 ds \right)^{\frac{1}{2}}, \end{aligned} \tag{3.19}$$

where $C_B > 0$ is a constant which depends only on the size of B. $\square$

The condition $p > 0$ implies that $2 < 2(p+1) < \infty$. Taking $\theta = \frac{1}{2}(1-\frac{1}{p+1})$ and applying *Gagliardo – Nirenberg* interpolation inequality, we have

$$\|u(t) - v(t)\|_{2(p+1)} \leqslant C\|\triangle(u(t) - v(t))\|_2^{\theta}\|u(t) - v(t)\|_2^{1-\theta}.$$

Since $\|\triangle u(t)\|_2$ and $\|\triangle v(t)\|_2$ are uniformly bounded, there exists a constant $C_B > 0$ such that

$$\|u(t) - v(t)\|_{2(p+1)}^2 \leqslant C_B\|u(t) - v(t)\|_2^{2(1-\theta)}. \tag{3.20}$$

Then, from (3.19) and (3.20) we obtain

$$\|S(t)z_0^1 - S(t)z_0^2\|_{\mathcal{H}_0} \leqslant \epsilon + \phi_T(z_0^1, z_0^2),$$

with

$$\phi_T(z_0^1, z_0^2) = C_B \left(\int_0^T \|u(s) - v(s)\|_2^{2(1-\theta)} ds + \int_0^T \|\nabla u(s) - \nabla v(s)\|_2^2 \right)^{\frac{1}{2}}.$$

The following proof $\phi_T \in \mathfrak{C}$, namely ϕ_T satisfies (2.6) .

Given a sequence $(z_0^n) = (u_0^n, u_1^n, \eta_0^n) \in B$, let us write $S(t)(z_0^n) = (u^n(t), u_t^n(t), \eta^{n,t})$. Since B is positively invariant by $S(t),\ t \geqslant 0$, it follows that sequence $(u^n(t), u_t^n(t), \eta^{n,t})$ is uniformly bounded in $\mathcal{H}_0$. On the other hand, (u^n, u_t^n) is bounded in $C([0, T], V \times H), T > 0$.

By the compact embedding $V \subset H$, there exists a subsequence (u^{n_k}) that converges strongly in $C([0, T], H)$. Thus,

$$\lim_{k\to\infty}\lim_{l\to\infty} \int_0^T \|u^{n_k}(s) - u^{n_l}(s)\|_2^{2(1-\theta)} ds = 0.$$

Furthermore, since B is a bounded positively invariant set in $\mathcal{H}$, without loss of generality, we assume that

$$u_n \to u \textit{ weakly star in } L^\infty(0, T; H_0^2(\Omega)), \tag{3.21}$$

By the compact embedding theorem, from (3.21), we have

$$u_n \to u \text{ strongly in } L^2(0,T;H_0^1(\Omega)), \tag{3.22}$$

So we obtain

$$\lim_{k\to\infty}\lim_{l\to\infty}\int_0^T \|\nabla u^{n_k}(s)-\nabla u^{n_l}(s)\|_2^2 ds = 0.$$

and consequently (2.6) holds.

Proof of Theorem 2.3. Lemma 3.1 and Lemma 3.3 imply that $(\mathcal{H}_0, S(t))$ is a dissipative dynamical system which is asymptotically smooth. Then it has compact global attractor from Theorem 2.2.

4. Asymptotic regular estimates

THEOREM 4.1. *Under assumptions of Theorem* 2.3. *Then the global attractor* $\mathcal{A}$ *is a bounded subset of* $\mathcal{H}_1$.

In order to prove Theorem 4.1, we fix a bounded set $B \subset \mathcal{H}_0$ and for $z = (u_0, u_1, \eta_0) \in B$, we split the solution $S(t)z = (u(t), u_t(t), \eta^t)$ of problem (2.3) – (2.5) into the sum

$$S(t)z = D(t)z + K(t)z,$$

where $D(t)z = z_1(t)$ and $K(t)z = z_2(t)$, namely $z = (u, u_t, \eta^t) = z_1 + z_2$, furthermore,

$$u = v + w,\ \eta^t = \zeta^t + \xi^t,$$

$$z_1 = (v, v_t, \zeta^t),\ z_2 = (w, w_t, \xi^t),$$

where $z_1(t)$ satisfy

$$\begin{cases} v_{tt} + v_t + l\triangle^2 v + (\alpha - \beta\|\nabla u\|_2^2)\triangle v + \int_0^\infty \mu(s)\Delta^2\zeta^t(s)ds + \gamma v = 0, \\ \zeta_t^t = -\zeta_s^t + v_t, \\ v(x,t)|_{\partial\Omega} = 0,\ v(x,\tau) = u_\tau(x), \\ \zeta^t(x,s)|_{\partial\Omega} = 0,\ \zeta^\tau(x,s) = \eta_\tau(x,s), \end{cases} \tag{4.1}$$

and $z_2(t)$ satisfy

$$\begin{cases} w_{tt}+w_t+l\triangle^2 w+(\alpha-\beta\|\nabla u\|_2^2)\triangle w+\int_0^\infty \mu(s)\Delta^2\xi^t(s)ds-\gamma v+ku^+ \\ =g, \\ \xi_t^t=-\xi_s^t+w_t, \\ w(x,t)|_{\partial\Omega}=0,\ w(x,\tau)=0, \\ \xi^t(x,s)|_{\partial\Omega}=0,\ \xi^\tau(x,s)=\xi_\tau(x,s)=0. \end{cases} \tag{4.2}$$

The well-posedness of the problem (4.1) and (4.2) can be obtained by Faedo-Galerkin method.

Furthermore, combining with a prior estimate of 3.1, about the solution $z_1(t)$ of equation (4.1) has the following result.

Lemma 4.2. *Under assumptions of Theorem* 2.3, *there exists a constant* $k_0>0$, *such that the solution of* (4.1) *satisfy*

$$\|D(t)z\|_{\mathcal{H}_0}^2\leqslant Ce^{-k_0t},$$

where C is a constant.

About the solution of equation (4.2), we have the following results.

Lemma 4.3. *Under assumptions of Theorem* 2.3, *there exists a constant* $N>0$, *such that the solution of* (4.2) *satisfy*

$$\|K(t)z\|_{\mathcal{H}_1}^2\leqslant N.$$

Proof. Taking the scalar product in H of the first equation of (4.2) with $A\varsigma=Aw_t+\sigma Aw$, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(l\|Aw\|_2^2+\|\triangle\varsigma\|_2^2)+\sigma l\|Aw\|_2^2+(1-\sigma)(w_t,A\varsigma)+((\alpha-\beta\|\nabla u\|_2^2)\triangle w,A\varsigma)\\ &\quad+(\xi^t,w_t)_{\mu,D(A)}+\sigma(\xi^t,w)_{\mu,D(A)}+(ku^+,A\varsigma)=(g,A\varsigma)+(\gamma v,A\varsigma).\end{aligned} \tag{4.3}$$

Similar to the previous discussion, there yields

$$(1-\sigma)(w_t,A\varsigma)=(1-\sigma)\|\triangle\varsigma\|_2^2-\sigma(1-\sigma)(Aw,\varsigma),$$

$$(\xi^t,w_t)_{\mu,D(A)}\geqslant\frac{1}{2}\frac{d}{dt}\|\xi^t\|_{\mu,D(A)}^2+\frac{\delta}{2}\|\xi^t\|_{\mu,D(A)}^2,$$

$$\sigma(\xi^t,w)_{\mu,D(A)}\geqslant-\frac{\delta}{4}\|\xi^t\|_{\mu,D(A)}^2-\frac{(1-l)\sigma^2}{\delta}\|Aw\|_2^2.$$

Then we get from (4.3)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(l\|Aw\|_2^2+\|\triangle\varsigma\|_2^2+\|\xi^t\|_{\mu,D(A)}^2)+\sigma l\left(1-\frac{(1-l)\sigma}{\delta l}\right)\|Aw\|_2^2\\&\quad+(1-\sigma)\|\triangle\varsigma\|_2^2+\frac{\delta}{4}\|\xi^t\|_{\mu,D(A)}^2-\sigma(1-\sigma)(Aw,\varsigma)+(ku^+,A\varsigma)\\&\leqslant(g,A\varsigma)+(\gamma v,A\varsigma)-((\alpha-\beta\|\nabla u\|_2^2)\triangle w,A\varsigma).\end{aligned}\tag{4.4}$$

Furthermore, similar to the estimate of (3.2), we obtain

$$\begin{aligned}&\sigma l\left(1-\frac{(1-l)\sigma}{\delta l}\right)\|Aw\|_2^2+(1-\sigma)\|\triangle\varsigma\|_2^2-\sigma(1-\sigma)(Aw,\varsigma)\\&\geqslant\sigma l(1-\sigma)\|Aw\|_2^2+\frac{1}{4}\|\triangle\varsigma\|_2^2.\end{aligned}\tag{4.5}$$

In line with the *Hölder*, *Young*, *Cauchy* inequalities and (3.6), (3.7), it follows that

$$\begin{aligned}(ku^+,A\varsigma)=&(ku^+,Aw_t+\sigma Aw)\\=&\frac{d}{dt}(ku^+,Aw)-(ku_t^+,Aw)+\sigma(ku^+,Aw)\\\geqslant&\frac{d}{dt}(ku^+,Aw)+\sigma(ku^+,Aw)-k\|u_t\|_2\|Aw\|_2\\\geqslant&\frac{d}{dt}(ku^+,Aw)+\sigma(ku^+,Aw)-k\mu_1\|Aw\|_2\\\geqslant&\frac{d}{dt}(ku^+,Aw)+\sigma(ku^+,Aw)-\frac{\sigma}{2}\|Aw\|_2^2-\frac{k^2\mu_1^2}{2\sigma},\ t\geqslant t_0,\end{aligned}\tag{4.6}$$

$$\begin{aligned}&|-((\alpha-\beta\|\nabla u\|_2^2)\triangle w,A\varsigma)|\\\leqslant&((\alpha-\beta\|\nabla u\|_2^2)\triangle w,Aw_t+\sigma Aw)\\=&-\frac{1}{2}\frac{d}{dt}(\alpha-\beta\|\nabla u\|_2^2)\|\nabla\triangle w\|_2^2+\beta\|\nabla u\|_2\|\nabla u_t\|_2\|\nabla\triangle w\|_2^2\\&-\sigma(\alpha-\beta\|\nabla u\|_2^2)\|\nabla\triangle w\|_2^2\\\leqslant&-\frac{1}{2}\frac{d}{dt}(\alpha-\beta\|\nabla u\|_2^2)\|\nabla\triangle w\|_2^2+2\beta\mu_1\mu_2^3\\&-\sigma(\alpha-\beta\|\nabla u\|_2^2)\|\nabla\triangle w\|_2^2,\end{aligned}\tag{4.7}$$

$$
\begin{aligned}
(\gamma v, A\varsigma) = (\gamma \triangle v, \triangle \varsigma) &\leqslant \gamma \|\triangle v\|_2 \|\triangle \varsigma\|_2 \leqslant \gamma \|\triangle v\|_2^2 + \frac{\gamma}{4}\|\triangle \varsigma\|_2^2 \\
&\leqslant \gamma \mu_1^2 + \frac{\gamma}{4}\|\triangle \varsigma\|_2^2,
\end{aligned} \tag{4.8}
$$

and

$$
(g, A\varsigma) = (g, Aw_t + \sigma Aw) = \frac{d}{dt}(g, Aw) + \sigma(g, Aw). \tag{4.9}
$$

Thus, collecting (4.5) – (4.9), from (4.4) yields

$$
\begin{aligned}
&\frac{d}{dt}(l\|Aw\|_2^2 + \|\triangle\varsigma\|_2^2 + (\alpha - \beta\nabla u\|_2^2)\|\nabla\triangle w\|_2^2 + \|\xi^t\|_{\mu,D(A)}^2 \\
&+2(ku^+, Aw) - 2(g, Aw)) + 2l\left(\sigma(1-\sigma) - \frac{\sigma}{2l}\right)\|Aw\|_2^2 \\
&+\frac{1-\gamma}{2}\|\triangle\varsigma\|_2^2 + 2\sigma(\alpha - \beta\nabla u\|_2^2)\|\nabla\triangle w\|_2^2 + \frac{\delta}{2}\|\xi^t\|_{\mu,D(A)}^2 \\
&+2\sigma(ku^+, Aw) - 2\sigma(g, Aw) \leqslant \frac{k^2\mu_1^2}{\sigma} + 2\gamma\mu_1^2 + 4\beta\mu_1\mu_2^3.
\end{aligned} \tag{4.10}
$$

Taking $\sigma_0 = min\left\{2\sigma(1-\sigma) - \frac{\sigma}{l}, \frac{1-\gamma}{2}, \sigma, \frac{\delta}{2}\right\}$, we can obtain from (4.10)

$$
\begin{aligned}
&\frac{d}{dt}(l\|Aw\|_2^2 + \|\triangle\varsigma\|_2^2 + (\alpha - \beta\nabla u\|_2^2)\|\nabla\triangle w\|_2^2 + \|\xi^t\|_{\mu,D(A)}^2 + 2(ku^+, Aw) \\
&-2(g, Aw)) + \sigma_0(l\|Aw\|_2^2 + \|\triangle\varsigma\|_2^2 + (\alpha - \beta\nabla u\|_2^2)\|\nabla\triangle w\|_2^2 + \|\xi^t\|_{\mu,D(A)}^2 \\
&+2(ku^+, Aw) - 2(g, Aw)) \leqslant \frac{k^2\mu_1^2}{\sigma} + 2\gamma\mu_1^2 + 4\beta\mu_1\mu_2^3,
\end{aligned} \tag{4.11}
$$

On the other hand, by the *Hölder* inequality, the Sobolev embedding theorem and (3.6), it follows that

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{l}{2}\|Aw\|_2^2 + 2(ku^+, Aw)\right) \\
&=\frac{d}{dt}\|\sqrt{\frac{l}{2}}Aw + \sqrt{\frac{2}{l}}u^+\|_2^2 - \frac{4k^2}{l}\int_\Omega |u^+||u_t^+|dx \\
&\geqslant\frac{d}{dt}\|\sqrt{\frac{l}{2}}Aw + \sqrt{\frac{2}{l}}ku^+\|_2^2 - \frac{4k^2}{l}\|u\|_2\|u_t\|_2 \\
&\geqslant\frac{d}{dt}\|\sqrt{\frac{l}{2}}Aw + \sqrt{\frac{2}{l}}u^+\|_2^2 - \frac{4k^2\mu_1^2}{l},
\end{aligned} \tag{4.12}
$$

and

$$\frac{d}{dt}\left(\frac{l}{2}\|Aw\|_2^2-2(g,Aw)\right)=\frac{d}{dt}\|\sqrt{\frac{l}{2}}Aw-\sqrt{\frac{2}{l}}g\|_2^2,\ t\geqslant t_0. \tag{4.13}$$

Therefore, integrating with (4.12) − (4.13), we get from (4.11)

$$\begin{aligned}&\frac{d}{dt}(\|\sqrt{\frac{l}{2}}Aw+\sqrt{\frac{2}{l}}u^+\|_2^2+\|\sqrt{\frac{l}{2}}Aw-\sqrt{\frac{l}{2}}g\|_2^2+\|\triangle\varsigma\|_2^2\\&+(\alpha-\beta\nabla u\|_2^2)\|\nabla\triangle w\|_2^2+\|\xi^t\|_{\mu,D(A)}^2)+\sigma_0(\|\sqrt{\frac{l}{2}}Aw+\sqrt{\frac{2}{l}}u^+\|_2^2\\&+\|\sqrt{\frac{l}{2}}Aw-\sqrt{\frac{2}{l}}g\|_2^2+\|\triangle\varsigma\|_2^2+(\alpha-\beta\nabla u\|_2^2)\|\nabla\triangle w\|_2^2+\|\xi^t\|_{\mu,D(A)}^2)\\&\leqslant C_5,\end{aligned} \tag{4.14}$$

where $C_5=k^2\mu_1^2(\frac{1}{\sigma}+\frac{4}{l})+\frac{2\sigma_0}{l}(k^2\mu_1^2+\|g\|_2^2)+2\gamma\mu_1^2+4\beta\mu_1\mu_2^3$.

Applying the Gronwall lemma, we can obtain there exist a constant N such that

$$\|Aw\|_2^2+\|\triangle w_t\|_2^2+\|\xi^t\|_{\mu,D(A)}^2\leqslant N.$$

Proof of Theorem 4.1. By Lemma 4.2 and 4.3, $(u,u_t,\eta^t)\in\mathcal{H}_1$ and

$$\|Au\|_2^2+\|\triangle u_t\|_2^2+\|\eta^t\|_{\mu,D(A)}^2\leqslant N.$$

Now since $u(t,x)$ satisfies (2.3) − (2.5) with initial data (u_0,u_1,η_0), we can obtain

$$\|(u_0,u_1,\eta_0)\|_{\mathcal{H}_1}\leqslant\widehat{N}.$$

Thus $\mathcal{A}$ is a bounded subset of $\mathcal{H}_1$. □

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