Korean J. Math. **24** (2016), No. 3, pp. 409–445
http://dx.doi.org/10.11568/kjm.2016.24.3.409

APPLICATION OF CONVOLUTION THEORY ON NON-LINEAR INTEGRAL OPERATORS

Satwanti Devi and A. Swaminathan

Abstract. The class $\mathcal{W}_{\beta}^{\delta}(\alpha,\gamma)$ defined in the domain $|z|<1$ satisfying

$$\mathrm{Re}\, e^{i\phi}\Big(\,(1-\alpha+2\gamma)(f/z)^{\delta}+\Big(\alpha-3\gamma+\gamma\Big[\,(1-1/\delta)\,(zf'/f)\\ +1/\delta\,\Big(1+zf''/f'\Big)\Big]\Big)\,\;(f/z)^{\delta}(zf'/f)-\beta\Big)>0,$$

with the conditions $\alpha\geq 0$, $\beta<1$, $\gamma\geq 0$, $\delta>0$ and $\phi\in\mathbb{R}$ generalizes a particular case of the largest subclass of univalent functions, namely the class of Bazilevič functions. Moreover, for $0<\delta\leq\frac{1}{(1-\zeta)}$, $0\leq\zeta<1$, the class $\mathcal{C}_{\delta}(\zeta)$ be the subclass of normalized analytic functions such that

$$\mathrm{Re}\;(1/\delta\,(1+zf''/f')+(1-1/\delta)\,(zf'/f))>\zeta,\quad |z|<1.$$

In the present work, the sufficient conditions on $\lambda(t)$ are investigated, so that the non-linear integral transform

$$V_{\lambda}^{\delta}(f)(z)=\left(\int_0^1\lambda(t)\,(f(tz)/t)^{\delta}\,dt\right)^{1/\delta},\quad |z|<1,$$

carries the functions from $\mathcal{W}_{\beta}^{\delta}(\alpha,\gamma)$ into $\mathcal{C}_{\delta}(\zeta)$. Several interesting applications are provided for special choices of $\lambda(t)$. These results are useful in the attempt to generalize the two most important extremal problems in this direction using duality techniques and provide scope for further research.

Received July 7, 2016. Revised August 22, 2016. Accepted August 25, 2016.
2010 Mathematics Subject Classification: 33C05, 30C45, 30C55, 30C80.
Key words and phrases: Duality techniques, Integral transforms, Convex functions, Starlike functions, Hypergeometric functions, Hadamard Product.

1. Introduction

Let $\mathcal{A}$ be the class of all normalized analytic functions f defined in the region $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the condition $f(0) = f'(0) - 1 = 0$ and $\mathcal{S} \subset \mathcal{A}$ be the class of all univalent functions in $\mathbb{D}$. We are interested in the following problem.

PROBLEM 1. Given $\lambda(t) : [0, 1] \to \mathbb{R}$ be a non-negative integrable function with the condition $\int_0^1 \lambda(t)dt = 1$, then for f in a particular class of analytic functions, the generalized integral transform defined by

$$(1.1) \quad V_\lambda^\delta(f)(z) := \left(\int_0^1 \lambda(t) \left(\frac{f(tz)}{t} \right)^\delta dt \right)^{1/\delta}, \quad \delta > 0 \quad \text{and} \quad z \in \mathbb{D}$$

is in one of the subclasses of $\mathcal{S}$.

This problem, for the case $\delta = 1$ was first stated by R. Fournier and S. Ruscheweyh [10] by examining the characterization of two extremal problems. They considered the functions f in the class $\mathcal{P}_\beta$, where

$$\mathcal{P}_\beta = \left\{ f \in \mathcal{A} : \text{Re} \left(e^{i\alpha}(f'(z) - \beta) \right) > 0, \quad \alpha \in \mathbb{R}, \quad z \in \mathbb{D} \right\}.$$

such that the integral operator $V_\lambda(f)(z) : V_\lambda^1(f)(z)$ is in the class $\mathcal{S}^*$ of functions that map $\mathbb{D}$ onto domain that are starlike with respect to origin using duality techniques. Same problem was solved by R.M. Ali and V.Singh [3] for functions f in the class $\mathcal{P}_\beta$ so that the integral operator $V_\lambda(f)(z)$ is in the class $\mathcal{C}$ of functions that map $\mathbb{D}$ onto domain that are convex.

The integral operator $V_\lambda(f)(z)$ contains some of the well-known operator such as Bernardi, Komatu and Hohlov as its special cases for particular choices of $\lambda(t)$, which has been extensively studied by various authors (for details see [3, 5, 6] and references therein). Generalization of the class $\mathcal{P}_\beta$ for studying the above problem with reference to the operator $V_\lambda(f)(z)$ were considered by several researchers in the recent past and interesting applications were obtained. For most general result in this direction, see [6] and references therein.

R. Fournier and S.Ruscheweyh [10] were interested in the two extremal problems to characterize the weight functions for which $L_\Lambda(\mathcal{M}) = 0$, where $\mathcal{M}$ is any subclass of $\mathcal{S}$ as it is not possible to solve the problem for the case $\mathcal{M} = \mathcal{S}$. After their consideration of the class $\mathcal{K}$ of close-to-convex functions in [10], further study has been carried out by many researchers in considering the class $\mathcal{K}$ only. Since the class $\mathcal{B}$ of Bazilevič functions (see [16] for the definition of this class) is the largest subclass of $\mathcal{S}$ and contains the class $\mathcal{K}$ it would be interesting to study the extremal problem for the class $\mathcal{B}$. Hence in [7], the authors proposed the following problem.

Problem 2. To characterize the weight functions and study the two extremal problems given by R. Fournier and S.Ruscheweyh [10] for the class $\mathcal{B}$ of Bazilevič functions.

Problem 1 for the generalized integral operator $V_{\lambda}^{\delta}(f)(z)$ relating starlikeness was investigated by A. Ebadian et al. in [9] by considering the class

$$P_{\alpha}(\delta,\beta) := \left\{ f \in \mathcal{A},\, \exists\, \phi \in \mathbb{R} :\, \mathrm{Re}\, e^{i\phi}\left((1-\alpha)\left(\frac{f}{z}\right)^{\delta} + \alpha\left(\frac{f}{z}\right)^{\delta}\left(\frac{zf'}{f}\right) - \beta \right) > 0,\, z \in \mathbb{D} \right\}$$

with $\alpha \geq 0$, $\beta < 1$ and $\delta > 0$. The authors of the present work have generalized the starlikeness criteria [7] by considering the following subclass of $\mathcal{S}^*$

$$f \in \mathcal{S}_s^*(\zeta) \iff z^{1-\delta} f^{\delta} \in \mathcal{S}^*(\xi), \tag{1.2}$$

for $\xi = 1 - \delta + \delta\zeta$ and $0 \leq \xi < 1$ where $\mathcal{S}^*(\xi)$ is the class having the analytic characterization

$$\mathrm{Re}\left(\frac{zf'}{f}\right) > \xi, \quad 0 \leq \xi < 1, \quad z \in \mathbb{D}.$$

Note that $\mathcal{S}^* := \mathcal{S}^*(0)$. In [7], this problem was investigated by considering the integral operator acting on the most generalized class of $\mathcal{P}_{\beta}$, related to the present context, which is defined as follows.

$$\mathcal{W}_{\beta}^{\delta}(\alpha,\gamma) := \left\{ f \in \mathcal{A} : \mathrm{Re}\, e^{i\phi}\left((1-\alpha+2\gamma)\left(\frac{f}{z}\right)^{\delta} + \left(\alpha - 3\gamma + \gamma\left[\left(1-\frac{1}{\delta}\right)\left(\frac{zf'}{f}\right) + \frac{1}{\delta}\left(1+\frac{zf''}{f'}\right)\right]\right)\left(\frac{f}{z}\right)^{\delta}\left(\frac{zf'}{f}\right) - \beta \right) > 0,\ z \in \mathbb{D},\ \phi \in \mathbb{R} \right\}.$$

Here, $\alpha \geq 0$, $\beta < 1$, $\gamma \geq 0$ and $\phi \in \mathbb{R}$. Note that $\mathcal{W}_{\beta}^{\delta}(\alpha,0) \equiv P_{\alpha}(\delta,\beta)$ is the class considered by A. Ebadian et al in [9], $R_{\alpha}(\delta,\beta) :\equiv \mathcal{W}_{\beta}^{\delta}(\alpha+\delta+$

$\delta\alpha, \delta\alpha)$ is a closely related class and $\mathcal{W}_\beta^1(\alpha,\gamma) \equiv \mathcal{W}_\beta(\alpha,\gamma)$ introduced by R.M. Ali et al in [1].

As the investigation of this generalization provided fruitful results, we are interested in considering further geometric properties of the generalized integral operator given by (1.1) for $f \in \mathcal{W}_\beta^\delta(\alpha,\gamma)$. Motivated, by the well-known Alexander theorem [8],

$$f \in \mathcal{C}(\xi) \Longleftrightarrow zf' \in \mathcal{S}^*(\xi),$$

where $\mathcal{C}(0) = \mathcal{C}$, we consider the subclass

$$f \in \mathcal{C}_\delta(\zeta) \Longleftrightarrow (z^{2-\delta} f^{\delta-1} f') \in \mathcal{S}^*(\xi), \tag{1.3}$$

where $\xi := 1-\delta+\delta\zeta$ with the conditions $1-\frac{1}{\delta} \leq \zeta < 1$, $0 \leq \xi < 1$ and $\delta \geq 1$. In the sequel, the term ξ is used to denote $(1-\delta+\delta\zeta)$. From the above expression, it is easy to observe that the class $\mathcal{C}_\delta(\zeta)$ and $\mathcal{C}(\xi)$ are equal, when $\delta = 1$.

The class $\mathcal{C}_\delta(\zeta)$ given in (1.3) is related to the class of α- convex of order ζ $(0 \leq \zeta < 1)$ that were introduced in the work of P. T. Mocanu [13] and defined analytically as

$$\mathrm{Re}\left((1-\alpha)\left(\frac{zf'(z)}{f(z)}\right) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > \zeta, \quad (1-\zeta) \leq \alpha < \infty.$$

Clearly the class $\mathcal{C}_\delta(\zeta)$ is nothing but the subclass of $\mathcal{S}$ consisting of $1/\delta$-convex functions of order ζ.

Having provided all the required information from the literature, in what follows, we obtain sharp estimates for the parameter β so that the generalized integral operator (1.1) maps the function from $\mathcal{W}_\beta^\delta(\alpha,\gamma)$ into $\mathcal{C}_\delta(\zeta)$, where $0 < \delta \leq \frac{1}{(1-\zeta)}$ and $0 \leq \zeta < 1$. Duality techniques, given in [15] provide the platform for the entire study of this manuscript. One of the particular tool in this regard is the convolution or Hadamard product of two functions $f_1(z) = \sum_{n=0}^\infty a_n z^n$ and $f_2(z) = \sum_{n=0}^\infty b_n z^n$, $z \in \mathbb{D}$, given by $(f_1 * f_2)(z) = \displaystyle\sum_{n=0}^\infty a_n b_n z^n$.

Furthermore, consider the complex parameters c_i $(i = 0,1,\ldots,p)$ and d_j $(j = 0,1,\ldots,q)$ with $d_j \neq 0,-1,\ldots$ and $p \leq q+1$. Then, in the region $\mathbb{D}$, the generalized hypergeometric function is given by

$${}_pF_q\left(\begin{matrix} c_1,\ldots,c_p \\ d_1,\ldots,d_q \end{matrix}; z\right) = \sum_{n=0}^\infty \frac{(c_1)_n \ldots,(c_p)_n}{(d_1)_n \ldots,(d_q)_n n!} z^n, \quad z \in \mathbb{D},$$

that can also be represented as ${}_pF_q(c_1,\ldots,c_p;d_1,\ldots,d_q;z)$ or ${}_pF_q$. In particular, ${}_2F_1$ is the well-known Gaussian hypergeometric function. For any natural number n, the Pochhammer symbol or shifted factorial $(\varepsilon)_n$ is defined as $(\varepsilon)_0 = 1$ and $(\varepsilon)_n = \varepsilon(\varepsilon+1)_{n-1}$.

The paper is organized as follows: Necessary and sufficient conditions are obtained in Section 3 that ensures $V_\lambda^\delta(\mathcal{W}_\beta^\delta(\alpha,\gamma)) \subset \mathcal{C}_\delta(\zeta)$. The simpler sufficient criterion are derived in Section 4, which are further implemented to find many interesting applications involving various integral operators for special choices of $\lambda(t)$.

2. Preliminaries

The parameters μ, $\nu \geq 0$ introduced in [1] are used for further analysis that are defined by the relations

$$\mu\nu = \gamma \quad \text{and} \quad \mu + \nu = \alpha - \gamma. \tag{2.1}$$

Clearly (2.1) leads to two cases.

(i) $\gamma = 0 \implies \mu = 0,\ \nu = \alpha \geq 0$.
(ii) $\gamma > 0 \implies \mu > 0,\ \nu > 0$.

Define the auxiliary function

$$\psi_{\mu,\nu}^\delta(z) := \sum_{n=0}^\infty \frac{\delta^2}{(\delta+n\mu)(\delta+n\nu)} z^n = \int_0^1\int_0^1 \frac{1}{(1-u^{\nu/\delta}/v^{\mu/\delta}z)} du dv, \tag{2.2}$$

which by a simple computation gives

$$\Phi_{\mu,\nu}^\delta(z) := \left(z\psi_{\mu,\nu}^\delta(z)\right)' = \sum_{n=0}^\infty \frac{(n+1)\delta^2}{(\delta+n\nu)(\delta+n\mu)} z^n \tag{2.3}$$

$$\text{and} \qquad \Upsilon_{\mu,\nu}^\delta(z) := \left(z\left(z\psi_{\mu,\nu}^\delta(z)\right)'\right)' = \sum_{n=0}^\infty \frac{(n+1)^2\delta^2}{(\delta+n\nu)(\delta+n\mu)} z^n. \tag{2.4}$$

Taking $\gamma = 0$ ($\mu = 0$, $\nu = \alpha \geq 0$), let $q_{0,\alpha}^\delta(t)$ be the solution of the differential equation

(2.5)

$$\frac{d}{dt}\left(t^{\delta/\alpha} q^{\delta}_{0,\alpha}(t)\right) = \frac{\delta t^{\delta/\alpha-1}}{\alpha}\left(\left(1-\frac{1}{\delta}\right)\frac{(1-\xi\,(1+t))}{(1-\xi)\,(1+t)^2}+\left(\frac{1}{\delta}\right)\frac{(1-t-\xi\,(1+t))}{(1-\xi)\,(1+t)^3}\right),$$

with the initial condition $q^{\delta}_{\alpha}(0)=1$. Then the solution of (2.5) is given by

$$q^{\delta}_{0,\alpha}(t) = \frac{\delta t^{-\delta/\alpha}}{\alpha}\int_0^t\left(\left(1-\frac{1}{\delta}\right)\frac{(1-\xi\,(1+s))}{(1-\xi)\,(1+s)^2}+\left(\frac{1}{\delta}\right)\frac{(1-s-\xi\,(1+s))}{(1-\xi)\,(1+s)^3}\right)s^{\delta/\alpha-1}ds.$$

Also, for the case $\gamma>0$ ($\mu>0$, $\nu>0$), let $q^{\delta}_{\mu,\nu}(t)$ be the solution of the differential equation

(2.6)

$$\frac{d}{dt}\left(t^{\delta/\nu} q^{\delta}_{\mu,\nu}(t)\right) = \frac{\delta^2 t^{\delta/\nu-1}}{\mu\nu}\int_0^1\left(\left(1-\frac{1}{\delta}\right)\frac{(1-\xi(1+st))}{(1-\xi)(1+st)^2}+\left(\frac{1}{\delta}\right)\frac{(1-st-\xi(1+st))}{(1-\xi)(1+st)^3}\right)s^{\delta/\mu-1}ds,$$

with the initial condition $q^{\delta}_{\mu,\nu}(0)=1$. Then the solution of (2.6) is given as

$$q^{\delta}_{\mu,\nu}(t)=\frac{\delta^2}{\mu\nu}\int_0^1\int_0^1\left(\left(1-\frac{1}{\delta}\right)\frac{(1-\xi\,(1+trs))}{(1-\xi)\,(1+trs)^2}+\left(\frac{1}{\delta}\right)\frac{(1-trs-\xi\,(1+trs))}{(1-\xi)\,(1+trs)^3}\right)r^{\delta/\nu-1}s^{\delta/\mu-1}drds, \tag{2.7}$$

Furthermore, for given $\lambda(t)$ and $\delta>0$, we introduce the functions

$$\Lambda^{\delta}_{\nu}(t) := \int_t^1 \frac{\lambda(s)}{s^{\delta/\nu}}ds, \quad \nu>0, \tag{2.8}$$

and

$$\Pi^{\delta}_{\mu,\nu}(t) := \begin{cases} \displaystyle\int_t^1 \frac{\Lambda^{\delta}_{\nu}(s)}{s^{\delta/\mu-\delta/\nu+1}}ds & \gamma>0\,(\mu>0,\nu>0), \\ \Lambda^{\delta}_{\alpha}(t) & \gamma=0\,(\mu=0,\nu=\alpha\geq 0) \end{cases} \tag{2.9}$$

which are positive on $t \in (0,1)$ and integrable on $t \in [0,1]$.

For $\delta = 1$, these information coincide with the one given in [2]. (2.8) and (2.9) are also considered in [7]. In [7], the investigations are related to $V^{\delta}_{\lambda}(f)(z) \in \mathcal{S}^{*}_{s}(\zeta)$, whenever $f \in \mathcal{W}^{\delta}_{\beta}(\alpha,\gamma)$, whereas various other inclusion properties, in particular, $V^{\delta}_{\lambda}(f)(z) \in \mathcal{W}^{\delta_1}_{\beta_1}(\alpha_1,\gamma_1)$, whenever $f \in \mathcal{W}^{\delta_2}_{\beta_2}(\alpha_2,\gamma_2)$ are investigated in [7].

3. Main results

The following result establishes both the necessary and sufficient conditions that ensure $F_{\delta}(z) := V^{\delta}_{\lambda}(f)(z) \in \mathcal{C}_{\delta}(\zeta)$, whenever $f \in \mathcal{W}^{\delta}_{\beta}(\alpha,\gamma)$.

THEOREM 3.1. *Let $\mu\geq 0$, $\nu\geq 0$ are given by the relation in* (2.1) *and $\left(1-\frac{1}{\delta}\right) \leq \zeta \leq \left(1-\frac{1}{2\delta}\right)$ where $\delta \geq 1$. Let $\beta<1$ satisfy the condition*

$$\frac{\beta-\frac{1}{2}}{1-\beta} = -\int_0^1 \lambda(t) q^{\delta}_{\mu,\nu}(t)dt, \tag{3.1}$$

where $q^{\delta}_{\mu,\nu}(t)$ is defined by the differential equation (2.5) *for $\gamma = 0$ and* (2.6) *for $\gamma > 0$. Further assume that the functions given in* (2.8) *and* (2.9) *attains*

$$\lim_{t\to 0^+} t^{\delta/\nu}\Lambda^{\delta}_{\nu}(t) \to 0 \quad \text{and} \quad \lim_{t\to 0^+} t^{\delta/\mu}\Pi^{\delta}_{\mu,\nu}(t) \to 0.$$

Then for $f(z) \in \mathcal{W}^{\delta}_{\beta}(\alpha,\gamma)$, the function $F_{\delta} := V^{\delta}_{\lambda}(f(z)) \in \mathcal{C}_{\delta}(\zeta)$ iff, $\mathcal{M}_{\Pi^{\delta}_{\mu,\nu}}(h_{\xi})(z) \geq 0$, where

$$\mathcal{M}_{\Pi^{\delta}_{\mu,\nu}}(h_{\xi})(z) := \begin{cases} \displaystyle\int_0^1 t^{\delta/\mu-1}\Pi^{\delta}_{\mu,\nu}(t)\, h_{\xi,\delta,z}(t)dt, & \gamma>0\,(\mu>0,\,\nu>0), \\ \displaystyle\int_0^1 t^{\delta/\alpha-1}\Lambda^{\delta}_{\alpha}(t)\, h_{\xi,\delta,z}(t)dt, & \gamma=0\,(\mu=0,\,\nu=\alpha\geq 0), \end{cases}$$

and

$$h_{\xi,\delta,z}(t) := \left(1-\frac{1}{\delta}\right)\left(\operatorname{Re}\frac{h_\xi(tz)}{tz}-\frac{1-\xi(1+t)}{(1-\xi)(1+t)^2}\right) + \left(\frac{1}{\delta}\right)\left(\operatorname{Re} h'_\xi(tz)-\frac{1-t-\xi(1+t)}{(1-\xi)(1+t)^3}\right)$$

for the function

$$h_\xi(z) := z\left(\frac{1+\frac{\epsilon+2\xi-1}{2(1-\xi)}z}{(1-z)^2}\right), \quad |\epsilon|=1 \tag{3.2}$$

and $\xi := 1-\delta(1-\zeta)$, $0\le\xi\le 1/2$. *The value of* β *is sharp.*

Proof. From (1.3), it is clear that

$$F_\delta \in \mathcal{C}_\delta(\zeta) \Longleftrightarrow \left(z^{2-\delta}(F_\delta)^{\delta-1}F'_\delta\right) \in \mathcal{S}^*(\xi) \tag{3.3}$$

where $\xi := 1-\delta(1-\zeta)$. Thus, to prove $\mathcal{M}_{\Pi^\delta_{\mu,\nu}}(h_\xi)\ge 0$ using the given hypothesis, it is required to show that the function $z^{2-\delta}(F_\delta)^{\delta-1}F'_\delta$ is univalent and satisfy the order of starlikeness condition, and conversely. Let

$$H(z) :=(1-\alpha+2\gamma)\left(\frac{f}{z}\right)^\delta + \left(\alpha-3\gamma+\gamma\left[\left(1-\frac{1}{\delta}\right)\left(\frac{zf'}{f}\right)+\frac{1}{\delta}\left(1+\frac{zf''}{f'}\right)\right]\right)\left(\frac{f}{z}\right)^\delta\left(\frac{zf'}{f}\right). \tag{3.4}$$

Using the relation (2.1) in (3.4) gives

$$H(z) = \frac{\mu\nu}{\delta^2}z^{1-\delta/\mu}\left(z^{\delta/\mu-\delta/\nu+1}\left(z^{\delta/\nu}\left(\frac{f}{z}\right)^\delta\right)'\right)'.$$

Further, set $G(z) = (H(z)-\beta)/(1-\beta)$, then there exist some $\phi\in\mathbb{R}$, such that $\operatorname{Re}\left(e^{i\phi}G(z)\right)>0$. Hence by the duality principle [15, p. 22], we may confine to the function $f(z)$ for which $G(z)=(1+xz)/(1+yz)$, where $|x|=|y|=1$, which directly implies

$$\frac{\mu\nu}{\delta^2}\, z^{1-\delta/\mu}\left(z^{\delta/\mu-\delta/\nu+1}\left(z^{\delta/\nu}\left(\frac{f}{z}\right)^{\delta}\right)'\right)' = (1-\beta)\frac{1+xz}{1+yz}+\beta,$$

or equivalently,

(3.5)

$$\begin{aligned}\left(\frac{f(z)}{z}\right)^{\delta} &= \frac{\delta^2}{\mu\nu z^{\delta/\nu}}\left(\int_0^z \frac{1}{\eta^{\delta/\mu-\delta/\nu+1}}\left(\int_0^{\eta}\frac{1}{\omega^{1-\delta/\mu}}\left((1-\beta)\frac{1+x\omega}{1+y\omega}+\beta\right)d\omega\right)d\eta\right)\\ &= \beta+(1-\beta)\left(\left(\frac{1+xz}{1+yz}\right)*\sum_{n=0}^{\infty}\frac{\delta^2 z^n}{(\delta+n\nu)(\delta+n\mu)}\right).\end{aligned}$$

If $A(z)$ is taken as $\left(\frac{f(z)}{z}\right)^{\delta}$, then using (2.2), (2.3) and (2.4), in $A(z)$, $zA'(z)$ and $z(zA'(z))'$ respectively, gives

$$\left(z\left(z\left(\frac{f(z)}{z}\right)^{\delta}\right)'\right)' = \left(\beta+(1-\beta)\left(\frac{1+xz}{1+yz}\right)\right)*\Upsilon^{\delta}_{\mu,\nu}(z). \tag{3.6}$$

Since

$$\left(z\left(\frac{f(z)}{z}\right)^{\delta}\right)' = (1-\delta)\left(\frac{f(z)}{z}\right)^{\delta}+\delta\left(\frac{f(z)}{z}\right)^{\delta}\left(\frac{zf'(z)}{f(z)}\right), \tag{3.7}$$

this gives

(3.8)

$$\left(z\left(\frac{f(z)}{z}\right)^{\delta}\left(\frac{zf'(z)}{f(z)}\right)\right)' = \left(1-\frac{1}{\delta}\right)\left(z\left(\frac{f(z)}{z}\right)^{\delta}\right)'+\frac{1}{\delta}\left(z\left(z\left(\frac{f(z)}{z}\right)^{\delta}\right)'\right)',$$

taking the logarithmic derivative on both sides of the integral operator (1.1) and differentiating further with a simple computation involving (3.6) gives

$$\begin{aligned}&\left(z\left(\frac{z(F_\delta)'}{F_\delta}\right)\left(\frac{F_\delta}{z}\right)^{\delta}\right)'\\ &=(1-\beta)\left(\int_0^1\lambda(t)\left(\left(1-\frac{1}{\delta}\right)\Phi^{\delta}_{\mu,\nu}(tz)+\left(\frac{1}{\delta}\right)\Upsilon^{\delta}_{\mu,\nu}(tz)\right)dt+\frac{\beta}{(1-\beta)}\right)*\left(\frac{1+xz}{1+yz}\right).\end{aligned}$$

The Noshiro-Warschawski's Theorem (for details see [8, Theorem 2.16]) states that the function $z^{2-\delta}(F_\delta)^{\delta-1}(F_\delta)'$ defined in the region $\mathbb{D}$ is univalent if $\left(z^{2-\delta}(F_\delta)^{\delta-1}(F_\delta)'\right)'$ is contained in the half plane not containing the origin. Hence, from the result based on duality principle [15, Pg. 23], it follows that

$$0 \neq \left(z\left(\frac{z(F_\delta)'}{F_\delta}\right)\left(\frac{F_\delta}{z}\right)^{\delta}\right)'$$

is true if, and only if

$$\mathrm{Re}\,(1-\beta)\left(\int_0^1 \lambda(t)\left(\left(1-\frac{1}{\delta}\right)\Phi^{\delta}_{\mu,\nu}(tz)+\left(\frac{1}{\delta}\right)\Upsilon^{\delta}_{\mu,\nu}(tz)\right)dt+\frac{\beta}{(1-\beta)}\right)>\frac{1}{2}$$

or equivalently,

$$\mathrm{Re}\,(1-\beta)\left(\int_0^1 \lambda(t)\left(\left(1-\frac{1}{\delta}\right)\Phi^{\delta}_{\mu,\nu}(tz)+\left(\frac{1}{\delta}\right)\Upsilon^{\delta}_{\mu,\nu}(tz)\right)dt+\frac{\beta-\frac{1}{2}}{(1-\beta)}\right)>0.$$

Now, substituting (3.1) in the above inequality implies

(3.9)
$$\mathrm{Re}\int_0^1 \lambda(t)\left(\left(1-\frac{1}{\delta}\right)\Phi^{\delta}_{\mu,\nu}(tz)+\left(\frac{1}{\delta}\right)\Upsilon^{\delta}_{\mu,\nu}(tz)-q^{\delta}_{\mu,\nu}(t)\right)dt>0.$$

From equation (2.3) and (2.4), it is easy to see that

$$\left(1-\frac{1}{\delta}\right)\Phi^{\delta}_{\mu,\nu}(tz)+\left(\frac{1}{\delta}\right)\Upsilon^{\delta}_{\mu,\nu}(tz)=\sum_{n=0}^{\infty}\frac{\delta(n+1)(n+\delta)(tz)^n}{(\delta+n\nu)(\delta+n\mu)},$$

whose integral representation is given as

$$\begin{aligned}(3.10)\qquad &\left(1-\frac{1}{\delta}\right)\Phi^{\delta}_{\mu,\nu}(tz)+\left(\frac{1}{\delta}\right)\Upsilon^{\delta}_{\mu,\nu}(tz)\\ &=\frac{\delta^2}{\mu\nu}\int_0^1\int_0^1\left(\frac{\left(1-\frac{1}{\delta}\right)}{(1-trsz)^2}+\frac{\frac{1}{\delta}(1+trsz)}{(1-trsz)^3}\right)r^{\delta/\nu-1}s^{\delta/\mu-1}drds.\end{aligned}$$

Thus, using (2.7) and (3.10) in (3.9) and on further using the fact that $\mathrm{Re}\left(\frac{1}{1-rstz}\right)^2\geq\frac{1}{(1+rst)^2}$ for $z\in\mathbb{D}$, directly implies

$$\begin{aligned}
\text{Re}\int_0^1 \lambda(t)\Bigg(\int_0^1\!\!\int_0^1\Bigg(\left(1-\frac{1}{\delta}\right)\frac{1}{(1-trsz)^2}+\left(\frac{1}{\delta}\right)\frac{1+trsz}{(1-trsz)^3}\Bigg)r^{\delta/\nu-1}s^{\delta/\mu-1}drds\\
-\int_0^1\!\!\int_0^1\Bigg(\left(1-\frac{1}{\delta}\right)\frac{1-\xi\,(1+trs)}{(1-\xi)\,(1+trs)^2}+\left(\frac{1}{\delta}\right)\frac{1-trs-\xi\,(1+trs)}{(1-\xi)\,(1+trs)^3}\Bigg)\\
\times r^{\delta/\nu-1}s^{\delta/\mu-1}drds\Bigg)dt
\end{aligned}$$

$$\begin{aligned}
\geq \int_0^1 \lambda(t)\Bigg(\int_0^1\!\!\int_0^1\Bigg(\left(1-\frac{1}{\delta}\right)\frac{1}{(1+trs)^2}+\left(\frac{1}{\delta}\right)\frac{1-trs}{(1+trs)^3}\Bigg)r^{\delta/\nu-1}s^{\delta/\mu-1}drds\\
-\int_0^1\!\!\int_0^1\Bigg(\left(1-\frac{1}{\delta}\right)\frac{1-\xi\,(1+trs)}{(1-\xi)\,(1+trs)^2}+\left(\frac{1}{\delta}\right)\frac{1-trs-\xi\,(1+trs)}{(1-\xi)\,(1+trs)^3}\Bigg)\\
\times r^{\delta/\nu-1}s^{\delta/\mu-1}drds\Bigg)dt
\end{aligned}$$

$$\begin{aligned}
=\int_0^1 \lambda(t)\Bigg(\int_0^1\!\!\int_0^1\Bigg(\left(1-\frac{1}{\delta}\right)\frac{\xi trs}{(1-\xi)\,(1+trs)^2}+\left(\frac{1}{\delta}\right)\frac{2\xi trs}{(1-\xi)\,(1+trs)^3}\Bigg)\\
\times r^{\delta/\nu-1}s^{\delta/\mu-1}drds\Bigg)dt
\end{aligned}$$

$$=\int_0^1 \lambda(t)\Bigg(\int_0^1\!\!\int_0^1\left(1+trs+\frac{1}{\delta}(1-trs)\right)\frac{\xi trs}{(1-\xi)\,(1+trs)^3}r^{\delta/\nu-1}s^{\delta/\mu-1}drds\Bigg)dt>0.$$

Thus, $\text{Re}\left(z^{2-\delta}(F_\delta)^{\delta-1}(F_\delta)'\right)'>0$, means that the function $z^{2-\delta}(F_\delta)^{\delta-1}(F_\delta)'$ is univalent in $\mathbb{D}$.

In the next part of the theorem the following two cases are discussed to show the order of starlikeness condition for the function $z^{2-\delta}(F_\delta)^{\delta-1}(F_\delta)'$.

Case (i). Let $\gamma=0\,(\mu=0,\nu=\alpha\geq 0)$. The function $H(z)$ defined in (3.4) decreases to

$$H(z)=\frac{\alpha}{\delta}z^{1-\delta/\alpha}\left(z^{\delta/\alpha}\left(\frac{f}{z}\right)^{\delta}\right)'.$$

Thus using duality principle, it is easy to see that

$$\left(\frac{f}{z}\right)^{\delta}=\beta+\frac{\delta(1-\beta)}{\alpha z^{\delta/\alpha}}\int_0^z \omega^{\delta/\alpha-1}\left(\frac{1+x\omega}{1+y\omega}\right)d\omega, \tag{3.11}$$

where $|x|=|y|=1$ and $z\in\mathbb{D}$. A famous result from the theory of convolution [15, P. 94] states that, if

$$g\in\mathcal{S}^*(\xi)\Longleftrightarrow\frac{1}{z}(g*h_\xi)(z)\neq 0, \tag{3.12}$$

where $h_\xi(z)$ is defined in (3.2).

For the function $f(z)\in\mathcal{W}_\beta^\delta(\alpha,0)$, the generalized integral operator F_δ defined in (1.1), belongs to the class $\mathcal{C}_\delta(\zeta)$ with the conditions $\left(1-\frac{1}{\delta}\right)\leq\zeta\leq\left(1-\frac{1}{2\delta}\right)$ and $\delta\geq 1$, is equivalent of getting $z\left(\frac{F_\delta}{z}\right)^\delta\left(\frac{z(F_\delta)'}{F_\delta}\right)\in\mathcal{S}^*(\xi)$, where ξ is defined by the hypothesis, $\xi:=1-\delta(1-\zeta)$ and $0\leq\xi\leq 1/2$. Therefore, (3.3) and (3.12) leads to

$$z\left(\frac{F_\delta}{z}\right)^\delta\left(\frac{z(F_\delta)'}{F_\delta}\right)\in\mathcal{S}^*(\xi)\Longleftrightarrow 0\neq\frac{1}{z}\left(z\left(\frac{F_\delta}{z}\right)^\delta\left(\frac{z(F_\delta)'}{F_\delta}\right)*h_\xi(z)\right).$$

Further, using logarithmic derivative of (1.1) in the above expression gives

$$\begin{aligned}0&\neq\int_0^1\lambda(t)\left(\frac{f(tz)}{tz}\right)^\delta\left(\frac{tzf'(tz)}{f(tz)}\right)dt*\frac{h_\xi(z)}{z}\\ &=\int_0^1\frac{\lambda(t)}{1-tz}dt*\left(\frac{f(z)}{z}\right)^\delta\left(\frac{zf'(z)}{f(z)}\right)*\frac{h_\xi(z)}{z}.\end{aligned} \tag{3.13}$$

Now, using a simple computation involving $z(f/z)'$, it is easy to see that (3.13) is equivalent to

$$\begin{aligned}0&\neq\int_0^1\frac{\lambda(t)}{1-tz}dt*\left(\left(1-\frac{1}{\delta}\right)\left(\frac{f(z)}{z}\right)^\delta+\frac{1}{\delta}\left(z\left(\frac{f(z)}{z}\right)^\delta\right)'\right)*\frac{h_\xi(z)}{z}\\ &=\int_0^1\lambda(t)\left(\left(1-\frac{1}{\delta}\right)\frac{h_\xi(tz)}{tz}+\frac{1}{\delta}h_\xi'(tz)\right)dt*\left(\frac{f(z)}{z}\right)^\delta.\end{aligned}$$

Substituting the value of $(f/z)^\delta$ from (3.11) will give

$$\begin{aligned}0\neq&\left(\int_0^1\lambda(t)\left(\left(1-\frac{1}{\delta}\right)\frac{h_\xi(tz)}{tz}+\frac{1}{\delta}h_\xi'(tz)\right)dt\right)\\ &\qquad*\left(\beta+\frac{\delta(1-\beta)}{\alpha z^{\delta/\alpha}}\int_0^z\omega^{\delta/\alpha-1}\left(\frac{1+x\omega}{1+y\omega}\right)d\omega\right)\end{aligned}$$

$$= (1-\beta)\left(\int_0^1 \lambda(t)\left(\frac{\delta}{\alpha z^{\delta/\alpha}}\int_0^z \omega^{\delta/\alpha-1}\left(\left(1-\frac{1}{\delta}\right)\frac{h_\xi(t\omega)}{t\omega}+\frac{1}{\delta}h'_\xi(t\omega)\right)d\omega\right)dt + \frac{\beta}{1-\beta}\right) * \frac{1+xz}{1+yz}.$$

Again from [15, Pg. 23] the above expression is true if, and only if,

$$\mathrm{Re}\,(1-\beta)\left(\int_0^1 \lambda(t)\left(\frac{\delta}{\alpha z^{\delta/\alpha}}\int_0^z \omega^{\delta/\alpha-1}\left(\left(1-\frac{1}{\delta}\right)\frac{h_\xi(t\omega)}{t\omega}+\frac{1}{\delta}h'_\xi(t\omega)\right)d\omega\right)dt + \frac{\beta}{1-\beta}\right) > \frac{1}{2}$$

or equivalently,

$$\mathrm{Re}\,(1-\beta)\left(\int_0^1 \lambda(t)\left(\frac{\delta}{\alpha z^{\delta/\alpha}}\int_0^z \omega^{\delta/\alpha-1}\left(\left(1-\frac{1}{\delta}\right)\frac{h_\xi(t\omega)}{t\omega}+\frac{1}{\delta}h'_\xi(t\omega)\right)d\omega\right)dt + \frac{\beta-\frac{1}{2}}{1-\beta}\right) > 0.$$

Using the condition on β given in (3.1), the above inequality reduces to

$$\mathrm{Re}\int_0^1 \lambda(t)\left(\frac{\delta}{\alpha z^{\delta/\alpha}}\int_0^z \omega^{\delta/\alpha-1}\left(\left(1-\frac{1}{\delta}\right)\frac{h_\xi(t\omega)}{t\omega}+\frac{1}{\delta}h'_\xi(t\omega)\right)d\omega - q^\delta_{0,\alpha}(t)\right)dt \geq 0.$$

Changing the variable $t\omega = u$, integrating by parts with respect to t and on further using (2.5) and (2.8), the above inequality gives

$$\mathrm{Re}\int_0^1 \Lambda^\delta_\alpha(t)\frac{d}{dt}\left(\frac{\delta}{\alpha z^{\delta/\alpha}}\int_0^{tz} u^{\delta/\alpha-1}\left(\left(1-\frac{1}{\delta}\right)\frac{h_\xi(u)}{u}+\frac{1}{\delta}h'_\xi(u)\right)du - t^{\delta/\alpha}q^\delta_{0,\alpha}(t)\right)dt \geq 0$$

or equivalently,

$$\mathrm{Re}\int_0^1 t^{\delta/\alpha-1}\Lambda_\alpha^\delta(t)\left[\left(1-\frac{1}{\delta}\right)\left(\frac{h_\xi(tz)}{tz}-\frac{1-\xi(1+t)}{(1-\xi)(1+t)^2}\right)\right.$$
$$\left.+\left(\frac{1}{\delta}\right)\left(h'_\xi(tz)-\frac{1-t-\xi(1+t)}{(1-\xi)(1+t)^3}\right)\right]dt\geq 0.$$

Case (ii). Let $\gamma>0$ ($\mu>0$, $\nu>0$). Using the conditions (3.3) and (3.12), the integral transform $V_\lambda^\delta(\mathcal{W}_\beta^\delta(\alpha,\gamma))\subset\mathcal{C}_\delta(\zeta)$, for $1-\frac{1}{\delta}\leq\zeta\leq\left(1-\frac{1}{2\delta}\right)$, $\delta\geq 1$ is equivalent of getting

$$0\neq\frac{1}{z}\left(z\left(\frac{F_\delta}{z}\right)^\delta\left(\frac{z(F_\delta)'}{F_\delta}\right)*h_\xi(z)\right),$$

where $\xi=1-\delta(1-\zeta)$ and $0\leq\xi\leq 1/2$. Hence using (3.7) and (3.13), a simple computation similar to case (i) reduces the above expression to

$$0\neq\int_0^1\lambda(t)\left(\left(1-\frac{1}{\delta}\right)\frac{h_\xi(tz)}{tz}+\frac{1}{\delta}h'_\xi(tz)\right)dt*\left(\frac{f(z)}{z}\right)^\delta.$$

Using (3.7) in the above inequality provides

$$0\neq\int_0^1\lambda(t)\left(\left(1-\frac{1}{\delta}\right)\frac{h_\xi(tz)}{tz}+\frac{1}{\delta}h'_\xi(tz)\right)dt*\left[\beta+(1-\beta)\left(\frac{1+xz}{1+yz}\right)\right]$$
$$*\,\psi_{\mu,\nu}^\delta(z)$$
$$=(1-\beta)\left(\int_0^1\lambda(t)\left(\left(1-\frac{1}{\delta}\right)\frac{h_\xi(tz)}{tz}+\frac{1}{\delta}h'_\xi(tz)\right)dt+\frac{\beta}{1-\beta}\right)$$
$$*\,\psi_{\mu,\nu}^\delta(z)*\left(\frac{1+xz}{1+yz}\right).$$

which is true if, and only if,

$$\mathrm{Re}\,(1-\beta)\left(\int_0^1\lambda(t)\left(\left(1-\frac{1}{\delta}\right)\frac{h_\xi(tz)}{tz}+\frac{1}{\delta}h'_\xi(tz)\right)dt+\frac{\beta}{1-\beta}\right)$$
$$*\psi_{\mu,\nu}^\delta(z)>\frac{1}{2}$$

or equivalently,

$$\mathrm{Re}\,(1-\beta)\left(\int_0^1\lambda(t)\left(\left(1-\frac{1}{\delta}\right)\frac{h_\xi(tz)}{tz}+\frac{1}{\delta}h'_\xi(tz)\right)dt+\frac{\beta-\frac{1}{2}}{1-\beta}\right)$$
$$*\psi_{\mu,\nu}^\delta(z)>0.$$

Using the condition on β given in (3.1), the above inequality becomes

$$\mathrm{Re}\int_0^1 \lambda(t)\left(\left(1-\frac{1}{\delta}\right)\frac{h_\xi(tz)}{tz}+\frac{1}{\delta}h'_\xi(tz)-q^\delta_{\mu,\nu}(t)\right)dt * \psi^\delta_{\mu,\nu}(z)\geq 0$$

which on further using (2.2) leads to

$$\mathrm{Re}\int_0^1 \lambda(t)\left(\left(1-\frac{1}{\delta}\right)\frac{h_\xi(tz)}{tz}+\frac{1}{\delta}h'_\xi(tz)-q^\delta_{\mu,\nu}(t)\right)dt \\ *\int_0^1\int_0^1 \frac{1}{(1-u^{\nu/\delta}v^{\mu/\delta}z)}dudv\geq 0$$

or equivalently,

$$\mathrm{Re}\int_0^1 \lambda(t)\left(\frac{\delta^2}{\mu\nu}\int_0^1\int_0^1\left(\left(1-\frac{1}{\delta}\right)\frac{h_\xi(tzrs)}{tzrs}+\frac{1}{\delta}h'_\xi(tzrs)\right)r^{\delta/\nu-1}s^{\delta/\mu-1}drds \\ -q^\delta_{\mu,\nu}(t)\right)dt\geq 0.$$

Changing the variable $tr=\omega$, integrating with respect to t and using (2.8) leads to

$$\mathrm{Re}\int_0^1\Lambda^\delta_\nu(t)\frac{d}{dt}\left(\frac{\delta^2}{\mu\nu}\int_0^t\int_0^1\left(\left(1-\frac{1}{\delta}\right)\frac{h_\xi(\omega zs)}{\omega zs}+\frac{1}{\delta}h'_\xi(\omega zs)\right)\omega^{\delta/\nu-1}s^{\delta/\mu-1}dsd\omega \\ -t^{\delta/\nu}q^\delta_{\mu,\nu}(t)\right)dt>0.$$

Further, using (2.6) reduces the above inequality to

$$\mathrm{Re}\int_0^1\Lambda^\delta_\nu(t)t^{\delta/\nu-1}\left(\int_0^1\left(\left(1-\frac{1}{\delta}\right)\left(\frac{h_\xi(stz)}{stz}-\frac{1-\xi(1+st)}{(1-\xi)(1+st)^2}\right)\right.\right. \\ \left.\left.+\left(\frac{1}{\delta}\right)\left(h'_\xi(stz)-\frac{1-st-\xi(1+st)}{(1-\xi)(1+st)^3}\right)\right)s^{\delta/\mu-1}ds\right)dt\geq 0.$$

Changing the variable $ts=\eta$, in the above expression, integrating with respect to t and using (2.9) gives

$$\mathrm{Re}\int_0^1\Pi^\delta_{\mu,\nu}(t)\frac{d}{dt}\left(\int_0^t\left(\left(1-\frac{1}{\delta}\right)\left(\frac{h_\xi(\eta z)}{\eta z}-\frac{1-\xi(1+\eta)}{(1-\xi)(1+\eta)^2}\right)\right.\right. \\ \left.\left.+\left(\frac{1}{\delta}\right)\left(h'_\xi(\eta z)-\frac{1-\eta-\xi(1+\eta)}{(1-\xi)(1+\eta)^3}\right)\right)\eta^{\delta/\mu-1}d\eta\right)dt\geq 0$$

or equivalently,

$$\begin{aligned}\mathrm{Re}\int_0^1 \Pi^{\delta}_{\mu,\nu}(t)t^{\delta/\mu-1}\left[\left(1-\frac{1}{\delta}\right)\left(\frac{h_\xi(tz)}{tz}-\frac{1-\xi(1+t)}{(1-\xi)(1+t)^2}\right)\right.\\ \left.+\left(\frac{1}{\delta}\right)\left(h'_\xi(tz)-\frac{1-t-\xi(1+t)}{(1-\xi)(1+t)^3}\right)\right]dt\geq 0\end{aligned}$$

which clearly implies that the function $\mathcal{M}_{\Pi^{\delta}_{\mu,\nu}}(h_\xi)\geq 0$ and the proof is complete.

Now, to validate the condition of sharpness for the function $f(z)\in\mathcal{W}^{\delta}_{\beta}(\alpha,\gamma)$, satisfying the differential equation

$$\frac{\mu\nu}{\delta^2}\,z^{1-\delta/\mu}\left(z^{\delta/\mu-\delta/\nu+1}\left(z^{\delta/\nu}\left(\frac{f}{z}\right)^{\delta}\right)'\right)'=\beta+(1-\beta)\frac{1+z}{1-z} \tag{3.14}$$

with the parameter $\beta<1$ defined in (3.1). From (3.14), a simple calculation gives

$$\left(\frac{f}{z}\right)^{\delta}=1+2(1-\beta)\sum_{n=1}^{\infty}\frac{\delta^2 z^n}{(\delta+n\nu)(\delta+n\mu)}. \tag{3.15}$$

Substituting (3.15) in (3.7) will give

$$z\left(\frac{f}{z}\right)^{\delta}\left(\frac{zf'}{f}\right)=z+2(1-\beta)\sum_{n=1}^{\infty}\frac{(n+\delta)\delta z^{n+1}}{(\delta+n\nu)(\delta+n\mu)}. \tag{3.16}$$

Further, substituting (3.16) in the expression involving the logarithmic derivative of (1.1) leads to

$$\begin{aligned} z\left(\frac{F_\delta}{z}\right)^{\delta}\left(\frac{z(F_\delta)'}{F_\delta}\right) &= \int_0^1\frac{\lambda(t)}{t}tz\left(\frac{f(tz)}{tz}\right)^{\delta}\left(\frac{tzf'(tz)}{f(tz)}\right)dt\\ &= z+2(1-\beta)\sum_{n=1}^{\infty}\frac{(n+\delta)\delta\tau_n z^{n+1}}{(\delta+n\nu)(\delta+n\mu)}\end{aligned} \tag{3.17}$$

where $\tau_n=\int_0^1 t^n\lambda(t)dt$. Differentiating (3.17) will give

$$\left(z\left(\frac{F_\delta}{z}\right)^{\delta}\left(\frac{z(F_\delta)'}{F_\delta}\right)\right)'=1+2(1-\beta)\sum_{n=1}^{\infty}\frac{(n+1)(n+\delta)\delta\tau_n z^n}{(\delta+n\nu)(\delta+n\mu)}.$$

which clearly implies

$$z\left(z\left(\frac{F_\delta}{z}\right)^\delta\left(\frac{z(F_\delta)'}{F_\delta}\right)\right)'\Bigg|_{z=-1}$$

$$= -1-2(1-\beta)\sum_{n=1}^{\infty}\frac{(-1)^n(n+1-\xi)(n+\delta)\delta\tau_n}{(\delta+n\nu)(\delta+n\mu)}$$

$$+2(1-\beta)\xi\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n+\delta)\delta\tau_n}{(\delta+n\nu)(\delta+n\mu)}. \tag{3.18}$$

The series expansion of the function $q^\delta_{\mu,\nu}(t)$ defined in (2.7) is

$$q^\delta_{\mu,\nu}(t)=1+\frac{\delta}{(1-\xi)}\sum_{n=1}^{\infty}\frac{(n+\delta)(n+1-\xi)(-1)^n t^n}{(\delta+n\nu)(\delta+n\mu)}. \tag{3.19}$$

whose representation in the form of generalized hypergeometric function is given as

(3.20)

$$q^\delta_{\mu,\nu}(t)= {}_5F_4\left(1,(1+\delta),(2-\xi),\frac{\delta}{\mu},\frac{\delta}{\nu};\,\delta,(1-\xi),\left(1+\frac{\delta}{\mu}\right),\left(1+\frac{\delta}{\nu}\right);\,-t\right).$$

Using (3.19) in (3.1) gives

$$\frac{\left(\beta-\frac{1}{2}\right)}{(1-\beta)}=-1-\frac{\delta}{(1-\xi)}\sum_{n=1}^{\infty}\frac{(n+\delta)(n+1-\xi)(-1)^n\tau_n}{(\delta+n\nu)(\delta+n\mu)}. \tag{3.21}$$

From (3.17) and (3.21), the expression (3.18) is equivalent to

$$z\left(z\left(\frac{F_\delta}{z}\right)^\delta\left(\frac{z(F_\delta)'}{F_\delta}\right)\right)'\Bigg|_{z=-1}=\xi\ z\left(\frac{F_\delta}{z}\right)^\delta\left(\frac{z(F_\delta)'}{F_\delta}\right)\Bigg|_{z=-1},$$

which means that the result is sharp. □

REMARK 3.1.

1. For $\delta=1$ and $\xi=0$, Theorem 3.1 is similar to [2, Theorem 3.1].
2. For $\delta=1$, Theorem 3.1 reduces to [17, Theorem 3.1].

The condition $\mathcal{M}_{\Pi^\delta_{\mu,\nu}}(h_\xi)\geq 0$ derived in Theorem 3.1 is difficult to use, therefore a simpler sufficient condition is presented in the next result.

Theorem 3.2. *Let $\mu \in [1/2, 1]$, $\nu \geq 1$ and $\left(1-\frac{1}{\delta}\right) \leq \zeta \leq \left(1-\frac{1}{2\delta}\right)$, where $\delta \geq 1$. Let $\beta < 1$ satisfy* (3.1) *and*

$$\frac{t^{1/\mu(\delta-1)}\left(\delta\left(1-\frac{1}{\mu}\right)\Pi^{\delta}_{\mu,\nu}(t) - t\left(\Pi^{\delta}_{\mu,\nu}(t)\right)'\right)}{(\log(1/t))^{3-2\delta(1-\zeta)}} \tag{3.22}$$

is decreasing on $(0,1)$. *Then the function $\mathcal{M}_{\Pi^{\delta}_{\mu,\nu}}(h_\xi)(z) \geq 0$, where $\xi = 1-\delta(1-\zeta)$ and $0 \leq \xi \leq 1/2$.*

Proof. Since the function

$$\begin{aligned}\mathcal{M}_{\Pi^{\delta}_{\mu,\nu}}(h_\xi)(z) = \int_0^1 t^{\delta/\mu-1}\Pi^{\delta}_{\mu,\nu}(t)\Bigg(\left(1-\frac{1}{\delta}\right)\left(\mathrm{Re}\frac{h_\xi(tz)}{tz} + \frac{1-\xi(1+t)}{(1-\xi)(1+t)^2}\right)\\ + \left(\frac{1}{\delta}\right)\left(\mathrm{Re}\, h'_\xi(tz) - \frac{1-t-\xi(1+t)}{(1-\xi)(1+t)^3}\right)\Bigg)\,dt,\end{aligned}$$

where $\xi = 1-\delta(1-\zeta)$ and $0 \leq \xi \leq 1/2$. Equivalently, it can also be written as

$$\begin{aligned}\mathcal{M}_{\Pi^{\delta}_{\mu,\nu}}(h_\xi)(z) = \int_0^1 t^{\delta/\mu-1}\Pi^{\delta}_{\mu,\nu}(t)\Bigg(\left(1-\frac{1}{\delta}\right)\left(\mathrm{Re}\frac{h_\xi(tz)}{tz} + \frac{1-\xi(1+t)}{(1-\xi)(1+t)^2}\right)\\ +\frac{d}{dt}\left(\frac{1}{\delta}\left(\mathrm{Re}\frac{h_\xi(tz)}{z} - \frac{t(1-\xi(1+t))}{(1-\xi)(1+t)^2}\right)\right)\Bigg)\,dt,\end{aligned}$$

which on further simplification gives

(3.23)

$$\begin{aligned}\mathcal{M}_{\Pi^{\delta}_{\mu,\nu}}(h_\xi)(z) =& \left(1-\frac{1}{\delta}\right)\int_0^1 t^{\delta/\mu-1}\Pi^{\delta}_{\mu,\nu}(t)\left(\mathrm{Re}\frac{h_\xi(tz)}{tz} + \frac{1-\xi(1+t)}{(1-\xi)(1+t)^2}\right)dt\\ &+\int_0^1 t^{\delta/\mu-1}\left(\frac{1}{\delta}\right)\left(\left(1-\frac{\delta}{\mu}\right)\Pi^{\delta}_{\mu,\nu}(t) - t\left(\Pi^{\delta}_{\mu,\nu}(t)\right)'\right)\left(\mathrm{Re}\frac{h_\xi(tz)}{tz} + \frac{1-\xi(1+t)}{(1-\xi)(1+t)^2}\right)dt\\ =& \int_0^1 t^{\delta/\mu-1}\left(\left(1-\frac{1}{\mu}\right)\Pi^{\delta}_{\mu,\nu}(t) - \left(\frac{1}{\delta}\right)t\left(\Pi^{\delta}_{\mu,\nu}(t)\right)'\right)\left(\mathrm{Re}\frac{h_\xi(tz)}{tz} - \frac{1-\xi(1+t)}{(1-\xi)(1+t)^2}\right)dt.\end{aligned}$$

The right side of (3.23) is bounded from below. So, due to the existence of lower bound, the minimum principle states that, the minimum value of (3.23) lies on the boundary i.e., on $|z| = 1$, where $z \neq 1$. Now, minimizing $\mathrm{Re}(h_\xi(tz)/(tz))$ with respect to ϵ will give

$$\mathrm{Re}\frac{h_\xi(tz)}{tz} \geq \frac{1}{2(1-\xi)}\left(\mathrm{Re}\frac{2(1-\xi)+(2\xi-1)tz}{(1-tz)^2} - \frac{t}{|1-tz|^2}\right).$$

Hence, (3.23) is equivalent of obtaining

$$\int_0^1 t^{\delta/\mu-1}\left(\delta\left(1-\frac{1}{\mu}\right)\Pi^{\delta}_{\mu,\nu}(t) - t\left(\Pi^{\delta}_{\mu,\nu}(t)\right)'\right)$$
$$\left(\mathrm{Re}\frac{2(1-\xi)+(2\xi-1)tz}{(1-tz)^2} - \frac{t}{|1-tz|^2} - \frac{2(1-\xi(1+t))}{(1+t)^2}\right)dt \geq 0.$$

The equality of the above integral exist at $z=-1$. Since $|z|=1$ and $z\neq 1$, now letting $\mathrm{Re}z = y$ will reduce it to considering

$$H^{(\xi)}_{\Pi}(y) = \int_0^1 t^{\delta/\mu-1}\left(\delta\left(1-\frac{1}{\mu}\right)\Pi^{\delta}_{\mu,\nu}(t) - t\left(\Pi^{\delta}_{\mu,\nu}(t)\right)'\right)$$
$$\left(\frac{t(3-4(1+y)t+2(4y-1)t^2+4(y-1)t^3-t^4)}{(1-2yt+t^2)^2(1+t)^2} - \frac{2\xi(1-t)}{(1-2yt+t^2)(1+t)}\right)dt \geq 0$$

where $|z|=1$ and $z\neq 1$, gives $-1\leq y<1$. Since the term $(1+y)\geq 0$, $H^{(\xi)}_{\Pi}(y)$ can be written in the series form as

$$H^{(\xi)}_{\Pi}(y) = \sum_{j=0}^{\infty} H^{(\xi)}_{j,\Pi}(1+y)^j, \qquad |1+y|<2.$$

An easy computation shows that the jth term of $H^{(\xi)}_{j,\Pi}$ is a positive multiple of

$$\tilde{H}^{(\xi)}_{j,\Pi} = \int_0^1 t^{\delta/\mu-1}\left(\delta\left(1-\frac{1}{\mu}\right)\Pi^{\delta}_{\mu,\nu}(t) - t\left(\Pi^{\delta}_{\mu,\nu}(t)\right)'\right)(s_j(t)-2\xi u_j(t))dt,$$

where

$$s_j(t) := \frac{(j+3)t^{j+1}}{(1+t)^{2j+4}}\left(1-2t+\frac{j-1}{j+3}t^2\right) \quad \text{and} \quad u_j(t) := \frac{t^{j+1}}{(1+t)^{2j+4}}(1-t^2).$$

to give

$$s_j(t) - 2\xi u_j(t) = \frac{t^{j+1}}{(1+t)^{2j+4}}\,v(t),$$

with $v(t) := ((j+3)(1-2t)+(j-1)t^2 - 2\xi(1-t^2))$.

The function $v(t)$ is decreasing on $t \in (0,1)$. At $t=0$, $v(t)$ is positive and at $t=1$, $v(t)$ is negative, which clearly implies that the function $(s_j(t)-2\xi u_j(t))$ has exactly one zero for $t \in (0,1)$. Set this zero by $t_j^{(\xi)}$. Therefore, $(s_j(t)-2\xi u_j(t))>0$, for $0 \leq t < t_j^{(\xi)}$ and $(s_j(t)-2\xi u_j(t))<0$, for $t_j^{(\xi)} < t < 1$.

Now, define the functions

$$\tilde{H}_j^{(\xi)} = \int_0^1 t^{1/\mu-1}(s_j(t)-2\xi u_j(t))\left(\log\left(\frac{1}{t}\right)\right)^{1+2\xi} dt \tag{3.24}$$

and

$$\begin{aligned}\tilde{\Pi}_{\mu,\nu}^{\delta,\xi}(t) &= t^{\frac{1}{\mu}(\delta-1)}\left(\delta\left(1-\frac{1}{\mu}\right)\Pi_{\mu,\nu}^{\delta}(t) - t\left(\Pi_{\mu,\nu}^{\delta}(t)\right)'\right)\\ &\quad - \frac{(t_j^{(\xi)})^{\frac{1}{\mu}(\delta-1)}\left(\delta\left(1-\frac{1}{\mu}\right)\Pi_{\mu,\nu}^{\delta}(t_j^{(\xi)}) - t_j^{(\xi)}\left(\Pi_{\mu,\nu}^{\delta}(t_j^{(\xi)})\right)'\right)}{(\log(1/t_j^{(\xi)}))^{1+2\xi}}(\log(1/t))^{1+2\xi}.\end{aligned}$$

Since the hypothesis (3.22) of the theorem implies that the function

$$\frac{t^{\frac{1}{\mu}(\delta-1)}\left(\delta\left(1-\frac{1}{\mu}\right)\Pi_{\mu,\nu}^{\delta}(t) - t\left(\Pi_{\mu,\nu}^{\delta}(t)\right)'\right)}{(\log(1/t))^{1+2\xi}}$$

is decreasing, where $\xi = 1-\delta(1-\zeta)$ and $0 \leq \xi \leq 1/2$, thus it is easy to observe that the condition on $(s_j(t)-2\xi u_j(t))$ and the function $\tilde{\Pi}_{\mu,\nu}^{\delta,\xi}(t)$ have same sign for $t \in (0,1)$. Hence

$$\begin{aligned}&0 \leq \int_0^1 t^{\frac{1}{\mu}-1}\tilde{\Pi}_{\mu,\nu}^{\delta,\xi}(t)(s_j(t)-2\xi u_j(t))dt\\ &\quad = \tilde{H}_{j,\Pi}^{(\xi)} - \frac{(t_j^{(\xi)})^{\frac{1}{\mu}(\delta-1)}\left(\delta\left(1-\frac{1}{\mu}\right)\Pi_{\mu,\nu}^{\delta}(t_j^{(\xi)}) - t_j^{(\xi)}\left(\Pi_{\mu,\nu}^{\delta}(t_j^{(\xi)})\right)'\right)}{(\log(1/t_j^{(\xi)}))^{1+2\xi}}\tilde{H}_j^{(\xi)}.\end{aligned} \tag{3.25}$$

Using (2.8) and (2.9), we have

$$\left(\Lambda_\nu^\delta(t)\right)' = -\lambda(t)t^{-\delta/\nu} \quad \text{and} \quad \left(\Pi_{\mu,\nu}^{\delta}(t)\right)' = -\Lambda_\nu^\delta(t)t^{-\delta/\mu+\delta/\nu-1}$$

which clearly shows that

$$\begin{aligned}
&\frac{d}{dt}\left(\delta\left(1-\frac{1}{\mu}\right)\Pi^{\delta}_{\mu,\,\nu}(t)-t\left(\Pi^{\delta}_{\mu,\,\nu}(t)\right)'\right)\\
&\quad=\frac{d}{dt}\left(\delta\left(1-\frac{1}{\mu}\right)\Pi^{\delta}_{\mu,\,\nu}(t)+t^{\delta/\nu-\delta/\mu}\Lambda^{\delta}_{\nu}(t)\right)\\
&\quad=-\delta\left(1-\frac{1}{\nu}\right)t^{\delta/\nu-\delta/\mu-1}\Lambda^{\delta}_{\nu}(t)-t^{-\delta/\mu}\lambda(t)<0.
\end{aligned}$$

for $\nu \geq 1$ and $t \in (0,1)$. Thus, the above condition implies

$$\delta\left(1-\frac{1}{\mu}\right)\Pi^{\delta}_{\mu,\,\nu}(t)-t\left(\Pi^{\delta}_{\mu,\,\nu}(t)\right)'>0.$$

Using similar arguments as in [9, Page 280] for the positivity of $\tilde{H}_j^{(\xi)}$ defined by (3.24), from (3.25), it follows that $\tilde{H}_{j,\Pi}^{(\xi)} \geq 0$ and this completes the proof. □

4. Applications of theorem 3.2

To apply Theorem 3.2, for the case $\gamma > 0$ ($\mu > 0, \nu > 0$), it is required to show that the function

$$\frac{t^{(\delta-1)/\mu}\left(\delta\left(1-1/\mu\right)\Pi^{\delta}_{\mu,\nu}(t)-t\left(\Pi^{\delta}_{\mu,\nu}(t)\right)'\right)}{(\log(1/t))^{3-2\delta(1-\zeta)}}$$

is decreasing in the range $t \in (0,1)$, where $\mu \in [1/2, 1]$, $\nu \geq 1$, $\delta \geq 1$ and $\left(1-\frac{1}{\delta}\right) \leq \zeta \leq \left(1-\frac{1}{2\delta}\right)$. Since $\xi = (1-\delta(1-\zeta))$, thus using (2.9), the above expression can be rewritten as

$$g(t) := \frac{\delta\left(1-\frac{1}{\mu}\right)t^{\delta/\mu-1/\mu}\,\Pi^{\delta}_{\mu,\nu}(t)+t^{\delta/\nu-1/\mu}\,\Lambda^{\delta}_{\nu}(t)}{(\log(1/t))^{1+2\xi}},$$

where $\xi \in [0, 1/2]$. Note that the chosen function $\lambda(t)$ satisfy the condition $\lambda(1) = 0$. Therefore, in the overall discussion, the assumed conditions hold.

Taking the derivative of $g(t)$ and using (2.8) and (2.9) will give

$$\begin{aligned} g'(t) =& \frac{t^{\delta/\mu-1/\mu-1}h(t)}{\left(\log\frac{1}{t}\right)^{2(1+\xi)}}\left[\delta\left(1-\frac{1}{\mu}\right)\Pi_{\mu,\nu}^{\delta}(t)\right.\\ &\left.+\left(1+\delta\left(\frac{1}{\nu}-1\right)\frac{\log\frac{1}{t}}{h(t)}\right)t^{\delta/\nu-\delta/\mu}\Lambda_{\nu}^{\delta}(t) - t^{1-\delta/\mu}\frac{\log\frac{1}{t}}{h(t)}\lambda(t)\right], \end{aligned}$$

where the function $h(t) := \frac{1}{\mu}(\delta-1)\log\frac{1}{t}+(1+2\xi)$, which by simple computation for $0<t<1$, $\delta\geq 1$ and $0\leq\xi\leq 1/2$ gives $h(t)\geq 1$.

Therefore, proving $g'(t)\leq 0$ is equivalent of getting $k(t)\leq 0$, where

$$\begin{aligned} k(t) :=& \delta\left(1-\frac{1}{\mu}\right)\Pi_{\mu,\nu}^{\delta}(t)+\left(1+\delta\left(\frac{1}{\nu}-1\right)\frac{\log\frac{1}{t}}{h(t)}\right)t^{\delta/\nu-\delta/\mu}\,\Lambda_{\nu}^{\delta}(t)\\ &- t^{1-\delta/\mu}\frac{\log\frac{1}{t}}{h(t)}\lambda(t). \end{aligned}$$

Clearly $k(1)=0$ implies that if $k(t)$ is increasing function of $t\in(0,1)$ then $g'(t)\leq 0$. Hence, it is required to show that

$$k'(t) = t^{\delta/\nu-\delta/\mu-1}\frac{l(t)}{h(t)},$$

where

$$\begin{aligned} l(t) :=& \left(\frac{\delta}{\nu}-\delta\right)\Lambda_{\nu}^{\delta}(t)\left[\left(\frac{\delta}{\nu}-\frac{1}{\mu}\right)\log\frac{1}{t}+1+2\xi-\frac{(1+2\xi)}{h(t)}\right]\\ &+t^{1-\delta/\nu}\lambda(t)\left[\left(\frac{1}{\mu}-\frac{\delta}{\nu}+\delta-1\right)\log\frac{1}{t}-1-2\xi+\frac{(1+2\xi)}{h(t)}\right]\\ &-t^{2-\delta/\nu}\log\frac{1}{t}\lambda'(t)\geq 0. \end{aligned}$$

Now, using the hypothesis $\lambda(1)=0$ implies that $l(1)=0$. Therefore $l(t)$ is decreasing function of $t\in(0,1)$, i.e., if $l'(t)\leq 0$, clearly means that the function $g(t)$ is decreasing. Now, we calculate

$$\begin{aligned} l'(t) =& \delta\left(1-\frac{1}{\nu}\right)\frac{\Lambda_\nu^\delta(t)}{t}\left[\left(\frac{\delta}{\nu}-\frac{1}{\mu}\right)+\left(\frac{\delta}{\mu}-\frac{1}{\mu}\right)\frac{(1+2\xi)}{(h(t))^2}\right] \\ &+t^{-\delta/\nu}\lambda(t)\left[(\delta-1)\left(1-\frac{1}{\mu}\right)\log\frac{1}{t}\right. \\ &\left.+\left(\frac{\delta}{\nu}-\frac{1}{\mu}+2\xi(\delta-1)\right)-\frac{(\delta-1)(1+2\xi)}{h(t)}+\left(\frac{\delta}{\mu}-\frac{1}{\mu}\right)\frac{(1+2\xi)}{(h(t))^2}\right] \\ &+t^{1-\delta/\nu}\lambda'(t)\left[\left(\frac{1}{\mu}+\delta-3\right)\log\frac{1}{t}-2\xi+\frac{(1+2\xi)}{h(t)}\right]-\log\frac{1}{t}t^{2-\delta/\nu}\lambda''(t). \end{aligned}$$

Thus, the function $g'(t)\leq 0$ is counterpart of the following inequalities:

$$\Lambda_\nu^\delta(t)\left[\left(\frac{1}{\mu}-\frac{\delta}{\nu}\right)(h(t))^2-(1+2\xi)\left(\frac{\delta}{\mu}-\frac{1}{\mu}\right)\right]\geq 0 \tag{4.1}$$

and

$$\begin{aligned} &\lambda(t)\left[\left(\frac{\delta}{\nu}-\frac{1}{\mu}+2\xi(\delta-1)\right)+(\delta-1)\left(1-\frac{1}{\mu}\right)\log\frac{1}{t}\right. \\ &\left.-\frac{(\delta-1)(1+2\xi)}{h(t)}+\left(\frac{\delta}{\mu}-\frac{1}{\mu}\right)\frac{(1+2\xi)}{(h(t))^2}\right] \\ &+t\lambda'(t)\left[\left(\frac{1}{\mu}+\delta-3\right)\log\frac{1}{t}-2\xi+\frac{(1+2\xi)}{h(t)}\right]-\log\frac{1}{t}t^2\lambda''(t)\leq 0, \end{aligned} \tag{4.2}$$

for $\nu\geq 1$ and $t\in(0,1)$. Letting $(2-\delta)/\mu\geq\delta/\nu$ implies that the inequality (4.1) is true, which clearly means that the function $g(t)$ is decreasing, if the inequality (4.2) holds along with the condition $(2-\delta)/\mu\geq\delta/\nu$, for $1\leq\delta<2$, $\mu\in[1/2,1]$ and $\nu\geq 1$.

The function $h(t)\geq 1$ and $(1-\delta/\mu)+(\delta/\mu-1/\mu)/h(t)\leq 0$, for $1/2\leq\mu\leq 1$ and $\delta\geq 1$. Thus the inequality (4.2) is true when

$$\begin{aligned} &\lambda(t)\left[\left(\frac{1}{\mu}-\frac{\delta}{\nu}-2\xi(\delta-1)\right)+(\delta-1)\left(\frac{1}{\mu}-1\right)\log\frac{1}{t}+\left(\delta-\frac{\delta}{\mu}\right)\frac{(1+2\xi)}{h(t)}\right] \\ &+t\lambda'(t)\left[\left(3-\delta-\frac{1}{\mu}\right)\log\frac{1}{t}+2\xi-\frac{(1+2\xi)}{h(t)}\right]+\log\frac{1}{t}t^2\lambda''(t)\geq 0. \end{aligned} \tag{4.3}$$

In order to use the above condition for the application purposes, we consider the following. For the parameters $A, B, C>0$, set

$$\lambda(t)=Kt^{B-1}(1-t)^{C-A-B}\omega(1-t), \tag{4.4}$$

where the function

$$\omega(1-t) = 1 + \sum_{n=1}^{\infty} x_n(1-t)^n, \quad \text{with} \quad x_n \geq 0, \quad t \in (0,1).$$

The constant K is chosen such that it satisfies normalization condition $\int_0^1 \lambda(t)dt = 1$ and $(C-A-B) > 0$ which clearly implies that the function $\lambda(t)$ is zero at $t = 1$.

By an easy calculation, we get

(4.5)

$$\lambda'(t) = Kt^{B-2}(1-t)^{C-A-B-1}\Big[\Big((B-1)(1-t) - (C-A-B)t\Big)\omega(1-t) - t(1-t)\omega'(1-t)\Big],$$

and

(4.6)

$$\lambda''(t) = Kt^{B-3}(1-t)^{C-A-B-2}\Big[\Big((B-1)(B-2)(1-t)^2 - 2(B-1)(C-A-B)t(1-t) + (C-A-B)(C-A-B-1)t^2\Big)\omega(1-t) + \Big(2(C-A-B)t - 2(B-1)(1-t)\Big)t(1-t)\omega'(1-t) + t^2(1-t)^2\omega''(1-t)\Big].$$

Now, substituting the values of $\lambda(t)$, $\lambda'(t)$ and $\lambda''(t)$ given in (4.4), (4.5) and (4.6), respectively in inequality (4.3) will give the corresponding condition as

(4.7)

$$t^2(1-t)^2 \log\frac{1}{t}\omega''(1-t) + t(1-t)X_1(t)\omega'(1-t) + X_2(t)\omega(1-t) \geq 0$$

where

$$X_1(t) := \log\frac{1}{t}\left[(1-t)\left(\frac{1}{\mu} + \delta - 2B - 1\right) + 2(C-A-B)t\right] + (1-t)\left(-2\xi + \frac{(1+2\xi)}{h(t)}\right).$$

and

$$X_2(t) :=$$
$$\log\frac{1}{t}\left[(1-t)^2\left[(\delta-1)\left(\frac{1}{\mu}-1\right)+(1-B)\left(\frac{1}{\mu}+\delta-B-1\right)\right]+(C-A-B)t\times\right.$$
$$\left.\left[(1-t)\left(\frac{1}{\mu}+\delta-2B-1\right)+(C-A-B-1)t\right]\right]+(1-t)\left[(1-t)\left[\left(\frac{1}{\mu}-\frac{\delta}{\nu}-2\xi(\delta-B)\right)\right.\right.$$
$$\left.\left.+\left(\delta+1-B-\frac{\delta}{\mu}\right)\frac{(1+2\xi)}{h(t)}\right]+(C-A-B)t\left[-2\xi+\frac{(1+2\xi)}{h(t)}\right]\right].$$

Since the function $\omega(1-t) = 1+\sum_{n=1}^{\infty} x_n(1-t)^n$, with the condition $x_n \geq 0$, which clearly means that the function $\omega(1-t)$, $\omega'(1-t)$ and $\omega''(1-t)$ are non-negative for all values of $t \in (0,1)$. Therefore, proving inequality (4.7), it suffice to show

$$X_1(t) \geq 0 \quad \text{and} \quad X_2(t) \geq 0.$$

Now, in this respect the following two cases are examined:

Case (i) Let $0 < B \leq \delta$. By a simple adjustment, it can be easily obtained that the inequality $X_1(t) \geq 0$ holds true if

$$\log\frac{1}{t}\left[(1-t)\left(\frac{1}{\mu}+\delta-2B-1\right)+2(C-A-B)t\right] \geq 2\xi(1-t),$$

where $\xi = 1-\delta(1-\zeta)$, for $\left(1-\frac{1}{\delta}\right) \leq \zeta \leq \left(1-\frac{1}{2\delta}\right)$. Since the right side of the above inequality is positive for $\xi \in [0,1/2]$ and $t \in (0,1)$, hence on using the condition

$$(1-t) \leq \frac{(1+t)}{2}\log\frac{1}{t}, \quad t \in (0,1), \tag{4.8}$$

it is enough to get

$$\left(\frac{1}{\mu}+\delta-1-2B-\xi\right)(1-t)+2(C-A-B-\xi)t \geq 0. \tag{4.9}$$

Further, the equivalent condition for $X_2(t) \geq 0$ is obtained. By the assumed hypothesis $(2-\delta)/\mu \geq \delta/\nu$ directly implies $1/\mu \geq \delta/\nu$. Now using this condition, the function $X_2(t) \geq 0$ is valid if

$$\log\frac{1}{t}\left((C-A-B)t\left[(1-t)\left(\frac{1}{\mu}+\delta-2B-1\right)+(C-A-B-1)t\right]\right.$$
$$\left.+(1-t)^2\left[(\delta-1)\left(\frac{1}{\mu}-1\right)+(1-B)\left(\frac{1}{\mu}+\delta-B-1\right)\right]\right)+(1-t)\Big((1-t)\times$$
$$\left[-2\xi(\delta-B)-\left(\frac{1}{\mu}+B-1\right)\frac{(1+2\xi)}{h(t)}\right]-2\xi(C-A-B)t\Big)\geq 0.$$

or equivalently,

$$\log\frac{1}{t}\left((C-A-B)t\left[(1-t)\left(\frac{1}{\mu}+\delta-2B-1\right)+(C-A-B-1)t\right]\right.$$
$$\left.+(1-t)^2\left[(\delta-1)\left(\frac{1}{\mu}-1\right)+(1-B)\left(\frac{1}{\mu}+\delta-B-1\right)\right]\right)$$
$$\geq(1-t)\left(2\xi(C-A-B)t+(1-t)\left[2\xi(\delta-B)+\left(\frac{1}{\mu}+B-1\right)\frac{(1+2\xi)}{h(t)}\right]\right). \tag{4.10}$$

As $0\leq B\leq\delta$, therefore using the conditions $0\leq\xi\leq 1/2$, $1/2\leq\mu\leq 1$, and $(C-A-B)>0$, it is easy to check that the coefficient of $(1-t)$ on right side of the above expression is positive. Therefore, in view of the inequality (4.8), the condition (4.10) holds true for $t\in(0,1)$ if

$$2\left((C-A-B)t\left[(1-t)\left(\frac{1}{\mu}+\delta-2B-1\right)+(C-A-B-1)t\right]\right.$$
$$\left.+(1-t)^2\left[(\delta-1)\left(\frac{1}{\mu}-1\right)+(1-B)\left(\frac{1}{\mu}+\delta-B-1\right)\right]\right)$$
$$\geq(1+t)\left(2\xi(C-A-B)t+(1-t)\left[2\xi(\delta-B)+\left(\frac{1}{\mu}+B-1\right)\frac{(1+2\xi)}{h(t)}\right]\right)$$

or equivalently,

$$(1-t)^2\left[2(\delta-1)\left(\frac{1}{\mu}-1\right)+2(1-B)\left(\frac{1}{\mu}+\delta-B-1\right)+R(t)\right]$$
$$+2t(1-t)\left[(C-A-B)\left(\frac{1}{\mu}+\delta-2B-1-\xi\right)+R(t)\right]$$
$$+2t^2(C-A-B)(C-A-B-1-2\xi)\geq 0, \tag{4.11}$$

where

$$R(t) := \left(-\frac{1}{\mu}+1-B\right)\frac{(1+2\xi)}{h(t)} - 2\xi(\delta-B).$$

Consequently, the condition (4.11) holds good if the coefficients of t^2, $t(1-t)$, and $(1-t)^2$ are positive. Now it remains to prove the following inequalities:

$$(C-A-B)(C-A-B-1-2\xi) \geq 0, \tag{4.12}$$

$$(C-A-B)\left(\frac{1}{\mu}+\delta-2B-1-\xi\right)+R(t) \geq 0, \tag{4.13}$$

and

$$2(\delta-1)\left(\frac{1}{\mu}-1\right)+2(1-B)\left(\frac{1}{\mu}+\delta-B-1\right)+R(t) \geq 0, \tag{4.14}$$

where $\xi = 1-\delta+\delta\zeta$, $\left(1-\frac{1}{\delta}\right) \leq \zeta \leq \left(1-\frac{1}{2\delta}\right)$ and $\delta \geq 1$.

Case (ii) Consider the case when $B \geq \delta$. It is easy to observe that the condition (4.9) is true when

$$\left(\frac{1}{\mu}+\delta-1-\xi\right) \geq 2B,$$

which clearly implies when $B \leq \delta$. Hence this case is not valid.

With the availability of the conditions given above we prove the result for the case $\gamma > 0$ ($\mu > 0, \nu > 0$) and $\lambda(t)$ defined in (4.4).

Theorem 4.1. *Let $A, B, C > 0$, $1/2 \leq \mu \leq 1 \leq \nu$ and $1-\frac{1}{\delta} \leq \zeta \leq 1-\frac{1}{2\delta}$, for $1 \leq \delta \leq 2$. Let $\beta < 1$ satisfy*

$$\frac{\beta-\frac{1}{2}}{1-\beta} = -K\int_0^1 t^{B-1}(1-t)^{C-A-B}\omega(1-t)q^{\delta}_{\mu,\nu}(t)dt,$$

where $q^{\delta}_{\mu,\nu}(t)$ is defined by the differential equation (2.6), the constant K and the function $\omega(1-t)$ is given in (4.4). Then for $f(z) \in \mathcal{W}^{\delta}_{\beta}(\alpha,\gamma)$, the function

$$H^{\delta}_{A,B,C}(f)(z) = \left(K\int_0^1 t^{B-1}(1-t)^{C-A-B}\omega(1-t)\left(\frac{f(tz)}{t}\right)^{\delta} dt\right)^{1/\delta}$$

belongs to $\mathcal{C}_{\delta}(\zeta)$ for the condition $(2-\delta)/\mu \geq \delta/\nu$, if

$$C \geq A+B+2 \text{ and } B \leq \min\left\{\frac{1}{4}\left(\frac{1}{\mu}-3+\delta(3-2\zeta)\right), \frac{2}{(\delta+1/\mu)}\left(\frac{(2\delta-1)}{\mu}-\delta+1\right)\right\}.$$

Proof. In order to prove the result, it is enough to get the inequalities (4.9), (4.12), (4.13) and (4.14) by using the above hypothesis.

The inequalities (4.9) and (4.12) are true if $(C-A-B)\geq 1+2\xi$ and $2B\leq (1/\mu+\delta-1-\xi)$, where $\xi=1-\delta(1-\zeta)$. Since the parameters $(C-A-B)>2$ and $4B\leq (1/\mu+\delta-1-2\xi)$, directly implies that these two inequalities hold. Moreover, to show the existence of inequality (4.13) under the given hypothesis, it is enough to prove

$$(C-A-B)\left(\frac{1}{\mu}+\delta-2B-1-\xi\right)\geq\left(\frac{1}{\mu}+\delta-1\right)$$

or equivalently,

$$(C-A-B-2)\left(\frac{1}{\mu}+\delta-2B-1-\xi\right)+\left(\frac{1}{\mu}+\delta-1-2\xi-4B\right)\geq 0,$$

that can be shown easily. Finally, to prove inequality (4.14), it is sufficient to get

$$2(\delta-1)\left(\frac{1}{\mu}-1\right)+2(1-B)\left(\frac{1}{\mu}+\delta-B-1\right)\geq\left(\frac{1}{\mu}+\delta-1\right).$$

By simple computation, the above condition is true if

$$\left(\frac{2\delta}{\mu}-\frac{1}{\mu}-\delta+1\right)-2B\left(\delta+\frac{1}{\mu}\right)\geq 0,$$

which is clearly true. Hence by the given hypothesis and Theorem 3.2, the result directly follows. □

So far, the case $\gamma>0$ was discussed in detail. Now, to apply Theorem 3.1 for the case $\gamma=0$ ($\mu=0$, $\nu=\alpha\geq 0$), it is required to show that the function

$$a(t):=\frac{\delta\left(1-\frac{1}{\alpha}\right)t^{(\delta-1)/\alpha}\Lambda_{\alpha}^{\delta}(t)+t^{1-1/\alpha}\lambda(t)}{(\log(1/t))^{1+2\xi}}$$

is decreasing on $t\in(0,1)$, where $\xi=1-\delta(1-\zeta)$, for $1/2\leq\alpha\leq 1$, $0\leq\xi\leq 1/2$ and $\delta\geq 1$. Now, differentiating $a(t)$ and on using (2.8) will give

$$a'(t)=\frac{p(t)\,t^{\delta/\alpha-1/\alpha-1}}{(\log(1/t))^{2+2\xi}}\;b(t),$$

where

$$b(t) := \delta\left(1-\frac{1}{\alpha}\right)\Lambda_\alpha^\delta(t) + \left(1-\frac{(\delta-1)\log(1/t)}{p(t)}\right)t^{1-\delta/\alpha}\lambda(t) + \frac{\log(1/t)}{p(t)}t^{2-\delta/\alpha}\lambda'(t)$$

and

$$p(t) := \frac{1}{\alpha}(\delta-1)\log\frac{1}{t} + (1+2\xi).$$

When $\delta \geq 1$, $\alpha \in [1/2, 1]$ and $\xi \in [0, 1/2]$, it can be easily seen that the function $p(t) \geq 1$, for $t \in (0,1)$. Hence, proving $a'(t) \leq 0$ is equivalent of getting $b(t) \leq 0$. Assuming $\lambda(1) = 0$ will give $b(1) = 0$. Hence, if $b(t)$ is increasing function of $t \in (0,1)$, then $a'(t) \leq 0$ and this completes the proof. Now

$$b'(t) = \frac{t^{-\delta/\alpha}}{p(t)}\left[(\delta-1)\lambda(t)\left(\left(\frac{1}{\alpha}-1\right)\log\frac{1}{t} - 1 - 2\xi + \frac{(1+2\xi)}{p(t)}\right) + t\lambda'(t)\left(\left(3-\delta-\frac{1}{\alpha}\right)\log\frac{1}{t} + 1 + 2\xi - \frac{(1+2\xi)}{p(t)}\right) + \log\frac{1}{t}\; t^2\lambda''(t)\right].$$

Therefore, $b'(t) \geq 0$, if

$$(\delta-1)\lambda(t)\left(\left(\frac{1}{\alpha}-1\right)\log\frac{1}{t} - 1 - 2\xi + \frac{(1+2\xi)}{p(t)}\right) + t\lambda'(t)\left(\left(3-\delta-\frac{1}{\alpha}\right)\log\frac{1}{t} + 1 + 2\xi - \frac{(1+2\xi)}{p(t)}\right) + \log\frac{1}{t}\; t^2\lambda''(t) \geq 0. \tag{4.15}$$

Now, using $\lambda(t)$, $\lambda'(t)$ and $\lambda''(t)$ given in (4.4), (4.5) and (4.6), respectively in inequality (4.15), will give the corresponding condition as

$$t^2(1-t)^2\log\frac{1}{t}\;\omega''(1-t) + t(1-t)\;X_3(t)\;\omega'(1-t) + X_4(t)\,\omega(1-t) \geq 0 \tag{4.16}$$

where

$$X_3(t) := \log\frac{1}{t}\left[(1-t)\left(\frac{1}{\alpha}+\delta-2B-1\right) + 2(C-A-B)t\right] - (1+2\xi)(1-t)\left[1-\frac{1}{p(t)}\right]$$

and

$$X_4(t) := \log\frac{1}{t}\left[(1-t)^2\left[(\delta-1)\left(\frac{1}{\alpha}-1\right)+(1-B)\left(\frac{1}{\alpha}+\delta-B-1\right)\right]\right.$$
$$\left.+(C-A-B)t\times\left[(1-t)\left(\frac{1}{\alpha}+\delta-2B-1\right)+(C-A-B-1)t\right]\right]$$
$$+(1+2\xi)(1-t)\Big((B-\delta)(1-t)-(C-A-B)t\Big)\left[1-\frac{1}{p(t)}\right].$$

As, the functions $\omega(1-t)$, $\omega'(1-t)$ and $\omega''(1-t)$ are non-negative for all values of $t\in(0,1)$, therefore to prove inequality (4.16), it is enough to show

$$X_3(t)\geq 0 \quad \text{and} \quad X_4(t)\geq 0.$$

Now, we divide the proof into two cases:
Case (i) Let $0<B\leq\delta$. Since the function $p(t)$ defined before is non-negative, therefore by a small adjustment, the inequality $X_3(t)\geq 0$ is valid, if

$$\log\frac{1}{t}\left[(1-t)\left(\frac{1}{\alpha}+\delta-2B-1\right)+2(C-A-B)t\right]\geq(1+2\xi)(1-t),$$

where the parameter ξ is defined above. It is easy to see that the right side of the above inequality is positive, hence applying the condition (4.8), the inequality is true when

(4.17)
$$(1-t)\left[2\left(\frac{1}{\alpha}+\delta-2B-\xi\right)-3\right]+2t\Big[2(C-A-B-\xi)-1\Big]\geq 0.$$

By the assumptions $B\leq\delta$ and $(C-A-B)>0$, the condition $X_4\geq 0$, holds good if

$$\log\frac{1}{t}\left((1-t)^2\left[(\delta-1)\left(\frac{1}{\alpha}-1\right)+(1-B)\left(\frac{1}{\alpha}+\delta-B-1\right)\right]\right.$$
$$\left.+(C-A-B)t\times\left[(1-t)\left(\frac{1}{\alpha}+\delta-2B-1\right)+(C-A-B-1)t\right]\right)$$
$$\geq(1+2\xi)(1-t)\Big((\delta-B)(1-t)+(C-A-B)t\Big).$$

For $t\in(0,1)$, the right side term of the above inequality is positive, hence in view of the condition (4.8), the above inequality can be obtained if

$$
\begin{aligned}
&(1-t)^2\left[2(\delta-1)\left(\frac{1}{\alpha}-1\right)+2(1-B)\left(\frac{1}{\alpha}+\delta-B-1\right)-(1+2\xi)(\delta-B)\right]\\
&+t(1-t)\left[(C-A-B)\left(2\left(\frac{1}{\alpha}+\delta-2B-1\right)-(1+2\xi)\right)-2(1+2\xi)(\delta-B)\right]\\
&\qquad +2t^2(C-A-B)(C-A-B-2-2\xi)\geq 0. \qquad (4.18)
\end{aligned}
$$

Thus, the condition (4.18) is true, if the coefficients of t^2, $t(1-t)$, and $(1-t)^2$ are positive. Now, it remains to prove the following inequalities:

(4.19)
$$2(\delta-1)\left(\frac{1}{\alpha}-1\right)+2(1-B)\left(\frac{1}{\alpha}+\delta-B-1\right)-(1+2\xi)(\delta-B)\geq 0,$$

(4.20)
$$(C-A-B)\left(2\left(\frac{1}{\alpha}+\delta-2B-1\right)-(1+2\xi)\right)-2(1+2\xi)(\delta-B)\geq 0,$$

and

$$(C-A-B)(C-A-B-2-2\xi)\geq 0 \qquad (4.21)$$

where $\xi=1-\delta(1-\zeta)$, for $\left(1-\frac{1}{\delta}\right)\leq\zeta\leq\left(1-\frac{1}{2\delta}\right)$ and $\delta\geq 1$.

Case (ii) $B\geq\delta$. It is easy to note that the condition (4.17) is true when

$$4B\leq 2\left(\frac{1}{\alpha}+\delta-\xi\right)-3,$$

which clearly means that $B\leq\delta$. Therefore this case is not valid.

Now, for the case $\gamma=0$ ($\mu=0,\nu=\alpha>0$) and $\lambda(t)$ defined in (4.4), the following result is stated as under.

Theorem 4.2. *Let $A,B,C>0$, $1/2\leq\alpha\leq 1$ and $1-\frac{1}{\delta}\leq\zeta\leq 1-\frac{1}{2\delta}$, for $\delta\geq 3$. Let $\beta<1$ satisfy*

$$\frac{\beta-\frac{1}{2}}{1-\beta}=-K\int_0^1 t^{B-1}(1-t)^{C-A-B}\omega(1-t)q_{0,\alpha}^{\delta}(t)dt,$$

where $q_{0,\alpha}^{\delta}(t)$ is defined by the differential equation (2.5), the constant K and the function $\omega(1-t)$ is given in (4.4). Then for $f(z)\in\mathcal{W}_{\beta}^{\delta}(\alpha,0)$, the function

$$H^{\delta}_{A,\,B,\,C}(f)(z) = \left(K\int_0^1 t^{B-1}(1-t)^{C-A-B}\omega(1-t)\left(\frac{f(tz)}{t}\right)^{\delta}dt\right)^{1/\delta}$$

belongs to $\mathcal{C}_\delta(\zeta)$, *if*

$$C \geq A+B+3 \text{ and } B \leq \min\left\{\frac{1}{2}\left(\frac{1}{\alpha}+\delta-2\right), \frac{\delta\left(\frac{1}{\alpha}-1\right)}{\left(\frac{1}{\alpha}+\delta-1\right)}, \frac{1}{4}\left(\frac{3}{\alpha}+\delta-6\right)\right\}.$$

Proof. In order to prove the result, it is enough to show the inequalities (4.17), (4.19), (4.20) and (4.21) by using the above hypothesis. The inequality (4.17) is valid if $(C-A-B)\geq 1$ and $2B \leq (1/\alpha+\delta-2)$, and (4.19) is true when $\delta\,(1/\alpha-1) \geq (1/\alpha+\delta-1)\,B$. Since the parameters $(C-A-B)\geq 3$ and conditions on B holds, which directly implies that these two inequalities along with the condition (4.21) are true.

Further, to prove inequality (4.20), it is sufficient to get the condition

$$(C-A-B)\left(2\left(\frac{1}{\alpha}+\delta-2B-1\right)-(1+2\xi)\right) \geq 2(1+2\xi)(\delta-B),$$

By simple computation, the above expression is holds, if

$$(C-A-B-3)\left(\frac{1}{\alpha}+\delta-2B-2\right)+\left(\frac{3}{\alpha}+\delta-4B-6\right) \geq 0,$$

which is clearly true. Hence by the given hypothesis and Theorem 3.2 the result directly follows. □

Let

$$\lambda(t) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}t^{b-1}(1-t)^{c-a-b}\,{}_2F_1\left(\begin{array}{cc} c-a, & 1-a \\ c-a-b+1 & \end{array}; 1-t\right),$$

then the integral operator (1.1) defined by the above weight function $\lambda(t)$ is the known as generalized Hohlov operator denoted by $\mathcal{H}^{\delta}_{a,\,b,\,c}$. This integral operator was considered in the work of A. Ebadian [9] (see also [7]). When $\delta=1$, the reduced integral transform was introduced by Y. C. Kim and F. Ronning [11] and studied by several authors later. The operator $\mathcal{H}^{\delta}_{a,\,b,\,c}$, with $a=1$ is the generalized Carlson-Shaffer operator $(\mathcal{L}^{\delta}_{b,\,c})$ [4].

Using the above operators the following results are obtained.

THEOREM 4.3. *Let* $a, b, c > 0$, $\gamma \geq 0$ $(\mu \geq, \nu \geq 0)$ *and* $1 - \frac{1}{\delta} \leq \zeta \leq 1 - \frac{1}{2\delta}$. *Let* $\beta < 1$ *satisfy*

(4.22)
$$\frac{\beta}{1-\beta} = -\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} \int_0^1 t^{b-1}(1-t)^{c-a-b}\, {}_2F_1\left(\begin{matrix} c-a,\, 1-a \\ c-a-b+1 \end{matrix} ; 1-t\right) q^{\delta}_{\mu,\nu}(t)dt,$$

where $q^{\delta}_{\mu,\nu}(t)$ *is defined by the differential equation* (2.6) *for* $\gamma > 0$, *and* (2.5) *for* $\gamma = 0$. *Then for* $f(z) \in \mathcal{W}^{\delta}_{\beta}(\alpha, \gamma)$, *the function* $\mathcal{H}^{\delta}_{a,b,c}(f)(z)$ *belongs to the class* $\mathcal{C}_{\delta}(\zeta)$, *whenever*

(i)
$$b \leq \min\left\{\frac{1}{4}\left(\frac{1}{\mu} - 3 + \delta(3 - 2\zeta)\right), \frac{2}{(\delta + 1/\mu)}\left(\frac{(2\delta - 1)}{\mu} - \delta + 1\right)\right\} \quad \text{and}$$
$c \geq a + b + 2$ for $\gamma > 0$ $(1/2 \leq \mu \leq 1 \leq \nu)$ and $1 \leq \delta \leq 2$,

(ii)
$$b \leq \min\left\{\frac{1}{2}\left(\frac{1}{\alpha} + \delta - 2\right), \frac{\delta\left(\frac{1}{\alpha} - 1\right)}{\left(\frac{1}{\alpha} + \delta - 1\right)}, \frac{1}{4}\left(\frac{3}{\alpha} + \delta - 6\right)\right\} \quad \text{and}$$
$c \geq a + b + 3$ for $1/2 \leq \alpha \leq 1$, $\gamma = 0$ and $\delta \geq 3$.

Proof. Choosing
$$K = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} \text{ and } \omega(1-t) = {}_2F_1\left(\begin{matrix} c-a,\, 1-a \\ c-a-b+1 \end{matrix} ; 1-t\right),$$
in Theorem 4.1 and 4.2 for the case $\gamma > 0$ and $\gamma = 0$, respectively to get the required result. □

For $a = 1$, Theorem 4.3 lead to the following particular cases which are of independent interest.

COROLLARY 4.1. *Let* $b, c > 0$, $\gamma \geq 0$ $(\mu \geq 0, \nu \geq 0)$ *and* $1 - \frac{1}{\delta} \leq \zeta \leq 1 - \frac{1}{2\delta}$. *Let* $\beta < 1$ *satisfy*

$$\frac{\beta}{(1-\beta)} = -\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} q^{\delta}_{\mu,\nu}(t)dt,$$

where $q^{\delta}_{\mu,\nu}(t)$ *is defined by the differential equation* (2.6) *for* $\gamma > 0$, *and* (2.5) *for* $\gamma = 0$. *Then for* $f(z) \in \mathcal{W}^{\delta}_{\beta}(\alpha, \gamma)$, *the function* $\mathcal{L}^{\delta}_{b,c}(f)(z)$ *belongs to the class* $\mathcal{C}_{\delta}(\zeta)$, *whenever*

(i)

$$b \leq \min\left\{\frac{1}{4}\left(\frac{1}{\mu} - 3 + \delta(3-2\zeta)\right), \frac{2}{(\delta+1/\mu)}\left(\frac{(2\delta-1)}{\mu} - \delta + 1\right)\right\} \quad \text{and}$$

$c \geq b+3$ for $\gamma > 0\,(1/2 \leq \mu \leq 1 \leq \nu)$ *and* $1 \leq \delta \leq 2$

(ii)

$$b \leq \min\left\{\frac{1}{2}\left(\frac{1}{\alpha} + \delta - 2\right), \frac{\delta\left(\frac{1}{\alpha}-1\right)}{\left(\frac{1}{\alpha}+\delta-1\right)}, \frac{1}{4}\left(\frac{3}{\alpha} + \delta - 6\right)\right\} \quad \text{and}$$

$c \geq b+4$ for $1/2 \leq \alpha \leq 1,\ \gamma = 0$ and $\delta \geq 3$

Corollary 4.2. *Let* $b, c > 0$, $\gamma \geq 0$ $(\mu \geq 0, \nu \geq 0)$ *and* $1 - \frac{1}{\delta} \leq \zeta \leq 1 - \frac{1}{2\delta}$. *Let* $\beta_0 < \beta < 1$, *where*

$$\beta_0 = 1 - \frac{1}{\left(1 - {}_6F_5\left(\begin{array}{c} 1, b, (1+\delta), (2-\xi), \frac{\delta}{\mu}, \frac{\delta}{\nu}, \\ c, \delta, (1-\xi), \left(1+\frac{\delta}{\mu}\right), \left(1+\frac{\delta}{\nu}\right) \end{array}; -1\right)\right)}.$$

Then, for $f \in \mathcal{W}_\beta^\delta(\alpha,\gamma)$, *the function* $\mathcal{L}_{b,c}^\delta(f)(z) \in \mathcal{C}_\delta(\zeta)$, *whenever*

(i)

$$b \leq \min\left\{\frac{1}{4}\left(\frac{1}{\mu} - 3 + \delta(3-2\zeta)\right), \frac{2}{(\delta+1/\mu)}\left(\frac{(2\delta-1)}{\mu} - \delta + 1\right)\right\} \quad \text{and}$$

$c \geq b+3$ for $\gamma > 0\,(1/2 \leq \mu \leq 1 \leq \nu)$ *and* $1 \leq \delta \leq 2$

(ii)

$$b \leq \min\left\{\frac{1}{2}\left(\frac{1}{\alpha} + \delta - 2\right), \frac{\delta\left(\frac{1}{\alpha}-1\right)}{\left(\frac{1}{\alpha}+\delta-1\right)}, \frac{1}{4}\left(\frac{3}{\alpha} + \delta - 6\right)\right\} \quad \text{and}$$

$c \geq b+4$ for $1/2 \leq \alpha \leq 1,\ \gamma = 0$ and $\delta \geq 3$

Proof. Putting $a = 1$ in (4.22) and on further using (3.20) will give

$$\frac{\beta}{1-\beta} = -\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\,{}_5F_4\left(\begin{array}{c} 1, (1+\delta), (2-\xi), \frac{\delta}{\mu}, \frac{\delta}{\nu} \\ \delta, (1-\xi), \left(1+\frac{\delta}{\mu}\right), \left(1+\frac{\delta}{\nu}\right) \end{array}; -t\right) dt$$

or equivalently,

$$\frac{\beta}{1-\beta} = -\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \left(\sum_{n=0}^{\infty} \frac{(1+\delta)_n (2-\xi)_n \left(\frac{\delta}{\mu}\right)_n \left(\frac{\delta}{\nu}\right)_n (-1)^n}{(\delta)_n (1-\xi)_n \left(n+\frac{\delta}{\nu}\right)_n \left(n+\frac{\delta}{\mu}\right)_n} t^n \right) dt.$$

Now a simple computation leads to

$$\frac{\beta}{1-\beta} = -\,{}_6F_5 \left(\begin{array}{c} 1, b, (1+\delta), (2-\xi), \frac{\delta}{\mu}, \frac{\delta}{\nu} \\ c, \delta, (1-\xi), \left(1+\frac{\delta}{\mu}\right), \left(1+\frac{\delta}{\nu}\right) \end{array} ; \; -1 \right).$$

Thus, applying Theorem 4.3 will give the required result. □

Consider

$$\lambda(t) = \frac{(1+k)^p}{\Gamma(p)} t^k \left(\log\frac{1}{t}\right)^{p-1}, \quad p \geq 0 \quad k > -1. \tag{4.23}$$

Then the integral operator (1.1) defined by the above weight function $\lambda(t)$ is the known as generalized Komatu operator denoted by $(F^{\delta}_{k,p})$. This integral operator was considered in the work of A. Ebadian [9]. When $\delta = 1$, the operator is reduced to the one introduced by Y. Komatu [12].

Now, we state the following result.

THEOREM 4.4. *Let $\gamma \geq 0$ ($\mu \geq, \nu \geq 0$), $k > -1$, $p \geq 1$ and $1 - \frac{1}{\delta} \leq \zeta \leq 1 - \frac{1}{2\delta}$. Let $\beta < 1$ satisfy* (3.1)*, where $\lambda(t)$ is given in* (4.23)*. Then for $f(z) \in \mathcal{W}^{\delta}_{\beta}(\alpha,\gamma)$, the function $F^{\delta}_{k,p}(f)(z) \in \mathcal{C}_{\delta}(\zeta)$, whenever*

(i)

$$-1 < k \leq \min\left\{ \frac{1}{4}\left(\frac{1}{\mu} - 3 + \delta(3-2\zeta)\right) - 1\,,\ \frac{2}{(\delta+1/\mu)}\left(\frac{(2\delta-1)}{\mu} - \delta + 1\right) - 1 \right\}$$

and $p \geq 1$ for $\gamma > 0\,(1/2 \leq \mu \leq 1 \leq \nu)$ and $1 \leq \delta \leq 2$,

(ii)

$$-1 < k \leq \min\left\{ \frac{1}{2}\left(\frac{1}{\alpha} + \delta - 4\right),\ \frac{\delta\left(\frac{1}{\alpha}-1\right)}{\left(\frac{1}{\alpha}+\delta-1\right)} - 1\,,\ \frac{1}{4}\left(\frac{3}{\alpha} + \delta - 10\right) \right\}$$

and $p \geq 2$ for $1/2 \leq \alpha \leq 1,\ \gamma = 0$ and $\delta \geq 3$.

Proof. Letting $(C - A - B) = p - 1$, $B = k + 1$ and $\omega(1-t) = \left(\frac{\log(1/t)}{(1-t)}\right)^{p-1}$. Therefore $\lambda(t)$ given in (4.4) can be represented as

$$\lambda(t) = Kt^k(1-t)^{p-1}\omega(1-t), \quad \text{where} \quad K = \frac{(1+k)^p}{\Gamma(p)}.$$

Now, by the given hypothesis the result directly follows from Theorem 4.1 and 4.2 for the case $\gamma > 0$ and $\gamma = 0$, respectively. □

References

[1] R. M. Ali, A. O. Badghaish, V. Ravichandran and A. Swaminathan, *Starlikeness of integral transforms and duality*, J. Math. Anal. Appl. **385** (2012), no. 2, 808–822.

[2] R. M. Ali, M. M. Nargesi and V. Ravichandran, *Convexity of integral transforms and duality*, Complex Var. Elliptic Equ. **58** (2013), no. 11, 1569–1590.

[3] R. M. Ali and V. Singh, *Convexity and starlikeness of functions defined by a class of integral operators*, Complex Variables Theory Appl. **26** (1995), no. 4, 299–309.

[4] B. C. Carlson and D. B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal. **15** (1984), no. 4, 737–745.

[5] J. H. Choi, Y. C. Kim and M. Saigo, *Geometric properties of convolution operators defined by Gaussian hypergeometric functions*, Integral Transforms Spec. Funct. **13** (2002), no. 2, 117–130.

[6] S. Devi and A. Swaminathan, *Integral transforms of functions to be in a class of analytic functions using duality techniques*, J. Complex Anal. **2014**, Art. ID 473069, 10 pp.

[7] S. Devi and A. Swaminathan, *Starlikeness of the Generalized Integral Transform using Duality Techniques*, 24 pages, Submitted for publication, (http://arxiv.org/abs/1411.5217).

[8] P. L. Duren, *Univalent functions*, *Grundlehren der Mathematischen Wissenschaften*, 259, Springer, New York, 1983.

[9] A. Ebadian, R. Aghalary and S. Shams, *Application of duality techniques to starlikeness of weighted integral transforms*, Bull. Belg. Math. Soc. Simon Stevin **17** (2010), no. 2, 275–285.

[10] R. Fournier and S. Ruscheweyh, *On two extremal problems related to univalent functions*, Rocky Mountain J. Math. **24** (1994), no. 2, 529–538.

[11] Y. C. Kim and F. Rønning, *Integral transforms of certain subclasses of analytic functions*, J. Math. Anal. Appl. **258** (2001), no. 2, 466–489.

[12] Y. Komatu, *On analytic prolongation of a family of operators*, Mathematica (Cluj) **32(55)** (1990), no. 2, 141–145.

[13] P. T. Mocanu, *Une propriété de convexité généralisée dans la théorie de la représentation conforme*, Mathematica (Cluj) **11 (34)** (1969), 127–133.

[14] R. Omar, S. A. Halim and R. W. Ibrahim, *Convexity and Starlikeness of certain integral transforms using duality*, Pre-print.

[15] S. Ruscheweyh, *Convolutions in geometric function theory*, Séminaire de Mathématiques Supérieures, 83, Presses Univ. Montréal, Montreal, QC, 1982.

[16] R. Singh, *On Bazilevič functions*, Proc. Amer. Math. Soc. **38** (1973), 261–271.

[17] S. Verma, S. Gupta and S. Singh, *Order of convexity of integral transforms and duality*, Acta Univ. Apulensis Math. Inform. No. 38 (2014), 279–296.

Satwanti Devi
Department of Applied Sciences
The NorthCap University
Sector 23-A Gurgaon - 122017, Haryana, India
E-mail: ssatwanti@gmail.com, satwantidevi@ncuindia.edu

A. Swaminathan
Department of Mathematics
Indian Institute of Technology
Roorkee-247 667 Uttarakhand, India
E-mail: swamifma@iitr.ac.in, mathswami@gmail.com