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GENERALIZED CONDITIONAL YEH-WIENER INTEGRALS FOR THE SAMPLE PATH-VALUED CONDITIONING FUNCTION

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Abstract. The purpose of this paper is to treat the generalized conditional Yeh-Wiener integral for the sample path-valued conditioning function. As a special case of our results, we obtain the results in [6].

1. Introduction

Let $t = g(s)$ be a monotonically decreasing and continuous function on $[0, S]$ with $g(S) > 0$ and let $\Omega = \{(s,t) \mid 0 \le s \le S,\ 0 \le t \le g(s)\}$. Let $C(\Omega)$ be a space of all real continuous functions x on Ω such that $x(s,t) = 0$ for all (s,t) in Ω satisfying $st = 0$.

In [3], the authors treated the generalized conditional Yeh-Wiener integral which includes the conditional Yeh-Wiener integral in [5] and the modified conditional Yeh-Wiener integral in [1]. In [5–8], Park and Skoug treated the conditional Yeh-Wiener integral for various kinds of conditioning functions including the sample path-valued conditioning function.

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The purpose of this paper is to treat the generalized conditional Yeh-Wiener integral for the sample path-valued conditioning function. We obtain the formula for the generalized conditional Yeh-Wiener integral and then evaluate it for two kinds of functionals. As a special case of our results, we obtain the results in [6].

2. Generalized Conditional Yeh-Wiener integrals for sample path-valued conditioning function

For a functional F of x in $C(\Omega)$, $E(F) = \int_{C(\Omega)} F(x)dm(x)$ is called a generalized Yeh-Wiener integral of F if it exists ([3]). As a stochastic process, $\{(x(s,t)|(s,t) \in \Omega\}$ has a mean $E(x(s,t)) = 0$ and $E(x(s,t)x(u,v)) = min\{s,u\}min\{t,v\}$. Let $C[0, g(S)]$ denote the standard Wiener space with the Wiener measure and assume that ψ is in $C[0, g(S)]$.

For a generalized Yeh-Wiener integrable function F of x in $C(\Omega)$, consider the generalized conditional Yeh-Wiener integral of the form

$$E(F(x)|x(S,(\cdot) \wedge T) = \psi((\cdot) \wedge T)) \tag{2.1}$$

with $g(S) = T$ and $a \wedge b = min\{a, b\}$. Here, $(\cdot)$ belongs to $[0, g(s)]$ for $0 \le s \le S$. Since two processes $x(S, t \wedge T)$ and $\{x(s,t) - (s/S)x(S, t \wedge T)|(s,t) \in \Omega\}$ are (stochastically) independent, we have

$$\begin{aligned} E(F(x)|x(S,(\cdot) \wedge T) &= \psi((\cdot) \wedge T)) \\ &= E(F(x(\star, \cdot) - \frac{\star}{S}x(S,(\cdot) \wedge T) + \frac{\star}{S}\psi((\cdot) \wedge T))) \end{aligned} \tag{2.2}$$

for almost all ψ in $C[0, T]$. Here, for the notational convenience, we denote $\cdot = (\cdot)$.

Especially, if $g(s) = T$ for all $0 \le s \le S$, then $(\cdot) \wedge T = (\cdot)$, which agrees with (2.2) in [6]. This means that our result (2.2) is a slight generalization of the result in [6].

Let $y(\cdot)$ be a tied-down Brownian motion, that is,

$$\{y(t) \mid 0 \le t \le T\} = \{w \in C[0, T] \mid w(T) = \xi\}.$$

Then, as is well known([6]), $y(\cdot)$ can be expressed in terms of the standard Wiener process,

$$y(t) = w(t) - \frac{t}{T}w(T) + \frac{t}{T}\xi.$$

The following theorem is one of our main results, which is slightly different from Theorem 1 in [6].

THEOREM 2.1. *If $F \in L_1(C(\Omega), m)$, then we have*

$$E_w(E(F(x) \mid x(S,(\cdot)\wedge T) = \sqrt{S}w((\cdot)\wedge T)) = E(F(x)), \tag{2.3}$$

$$E(F(x)|x(S,T) = \sqrt{S}\ \xi) \tag{2.4}$$

$$= E_w\left\{E\left(F(x)\ \middle|\ x(S,(\cdot)\wedge T) = \sqrt{S}\ (w(\cdot) - \tfrac{(\cdot)\wedge T}{T}w((\cdot)\wedge T) + \tfrac{(\cdot)\wedge T}{T}\xi)\right)\right\}.$$

Proof. (1) Using (2.2), we write

$$\begin{aligned} E_w(E(F(x) \mid x(S,(\cdot)\wedge T) = \sqrt{S}\ w((\cdot)\wedge T)) \\ = E_w\left\{E\left(F(x(\star,\cdot) - \frac{\star}{S}\ x(S,(\cdot)\wedge T) + \frac{\star}{\sqrt{S}}\psi((\cdot)\wedge T)\right)\right\}. \end{aligned} \tag{2.5}$$

Let $y(s,t) = x(s,t) - (s/S)x(S,t\wedge T) + (s/\sqrt{S})\psi(t\wedge T)$ for all (s,t) in Ω. Then we have $E(y(s,t)) = 0$ and $E(y(s,t)y(u,v)) = min\{s,u\}min\{t,v\}$. This means that $\{(y(s,t)|(s,t)\in\Omega\}$ is a generalized Yeh-Wiener process, and the right hand side of (2.5) becomes $\int_{C(\Omega)} F(y)dm(y) = E(F(x))$. Thus, we obtain the formula (2.3).

(2) We use Theorem 2 in [5] to have

$$\begin{aligned} E(F(x) \mid x(S,T) = \sqrt{S}\ \xi\ ) \\ = E\left\{F\left(x(\star,\cdot) - \frac{\star}{S}\frac{(\cdot)\wedge T}{T}x(S,T) + \frac{\star}{\sqrt{S}}\frac{(\cdot)\wedge T}{T}\xi\right)\right\}. \end{aligned} \tag{2.6}$$

We can rewrite the right-hand side of (2.6) as the following form:

$$\begin{aligned} E\{F(x(\star,\cdot) - \frac{\star}{S}x(S,(\cdot)\wedge T) \\ + \frac{\star}{S}[x(S,(\cdot)\wedge T) - \frac{(\cdot)\wedge T}{T}x(S,T) + \frac{(\cdot)\wedge T}{T}\sqrt{S}\ \xi])\}. \end{aligned} \tag{2.7}$$

We use $E(x(s,t)x(u,v)) = min\{s,u\}min\{t,v\}$ to show that two processes $x(s,\cdot) - (s/S)x(S,(\cdot)\wedge T)$ and $x(S,(\cdot)\wedge T) - ((\cdot)\wedge T)(x(S,T)/T)$ are stochastically independent. Furthermore, $\sqrt{S}\ (w(\cdot) - ((\cdot)\wedge T)(w(T)/T))$ and $x(S,(\cdot)\wedge T) - ((\cdot)\wedge T)(x(S,T)/T)$ are equivalent processes, where $w(\cdot)$ is the standard Wiener process. Thus, (2.7) becomes

$$
\begin{aligned}
(2.8)\qquad & E_w\{E\{F(x(\star,\cdot)-\frac{\star}{S}x(S,(\cdot)\wedge T)\\
&\quad+\frac{\star}{\sqrt{S}}[\ (w(\cdot)-((\cdot)\wedge T)(w(T)/T))+((\cdot)\wedge T)\ \frac{\xi}{T}])\}\}\\
=&E_w\Bigg\{E\Bigg(F(x)\ \Bigg|\ x(S,(\cdot)\wedge T)\\
&\qquad=\sqrt{S}\ (w(\cdot)-\frac{(\cdot)\wedge T}{T}w((\cdot)\wedge T)+\frac{(\cdot)\wedge T}{T}\xi)\Bigg)\Bigg\}.
\end{aligned}
$$

Therefore. we get the formula (2.4). □

For the special case $g(s)=T$ for $0\le s\le S$, we have the same result of Theorem 1 in [6]. In a certain sense, our result is a slight generalization of the result in [6].

In [6], Park and Skoug treated the rectangle Q, but we treat the more general region Ω. Let Ω be the region given by

$$\Omega=\{(s,t)\mid 0\le s\le S,\ 0\le t\le g(s)\ \}$$

where $t=g(s)$ is a monotonically decreasing and continuous function on $[0,\ S]$ with $g(S)=T>0$. In the following two theorems we evaluate the generalized conditional Yeh-Wiener integral for the sample path-valued conditioning function.

THEOREM 2.2. *Let F be a functional on $C(\Omega)$ given by $F(x)=\int_\Omega x(s,t)dsdt$. Then we have*

$$
\begin{aligned}
(2.9)\qquad & E(F(x)\mid x(S,(\cdot)\wedge T)=\psi((\cdot)\wedge T))\\
&\qquad=\frac{S}{2}\int_0^T\psi(t)dt+\frac{\psi(T)}{S}\int_0^S\int_T^{g(s)}sdtds.
\end{aligned}
$$

Proof. By (2.2) and Fubini theorem, we have

$$
\begin{aligned}
(2.10)\quad & E(F(x)\mid x(S,(\cdot)\wedge T)=\psi((\cdot)\wedge T))\\
&=\int_\Omega E\left(x(s,t)-\frac{s}{S}x(S,(\cdot)\wedge T)+\frac{s}{S}\psi((\cdot)\wedge T)\right)dsdt.\\
&=\int_\Omega\frac{s}{S}\psi((\cdot)\wedge T)dsdt.
\end{aligned}
$$

The right hand side of the last equality in (2.10) comes from the fact that $E(x(s,t)) = 0$ and $m(C(\Omega)) = 1$. By the straightward calculation, we have

$$(2.11)\quad \begin{aligned} &E(F(x) \mid x(S,(\cdot)\wedge T) = \psi((\cdot)\wedge T)) \\ &= \int_0^S \int_0^{g(s)} \frac{s}{S}\psi((\cdot)\wedge T)dtds \\ &= \frac{S}{2}\int_0^T \psi(t)dt + \frac{\psi(T)}{S}\int_0^S \int_T^{g(s)} sdtds, \end{aligned}$$

which is our desired result. □

THEOREM 2.3. *Let F be a functional on $C(\Omega)$ given by $F(x) = \int_\Omega x^2(s,t)dsdt$ and $g(S) = T$. Then we have*

$$(2.12)\quad \begin{aligned} &E(F(x) \mid x(S,(\cdot)\wedge T) = \psi((\cdot)\wedge T)) \\ &= \frac{S^2T^2}{12} + \frac{S}{3}\int_0^T \psi^2(t)dt + \int_0^S \int_T^{g(s)} \left(st - \frac{s^2}{S}T + \frac{s^2}{S^2}\psi^2(T)\right) dtds. \end{aligned}$$

Proof. By (2.2) and Fubini theorem, we have

$$(2.13)\quad \begin{aligned} &E(F(x) \mid x(S,(\cdot)\wedge T) = \psi((\cdot)\wedge T)) \\ &= \int_\Omega E\left(\left\{x(s,t) - \frac{s}{S}x(S,(\cdot)\wedge T) + \frac{s}{S}\psi((\cdot)\wedge T)\right\}^2\right) dsdt \\ &= \int_\Omega \left\{st - \frac{s^2}{S}(t\wedge T) + \frac{s^2}{S^2}\psi^2(t\wedge T)\right\} dtds. \end{aligned}$$

The right hand side of the last equality in (2.13) comes from the fact that $E(x(s,t)) = 0$, $E(x(s,t)x(u,v)) = min\{s,u\}min\{t,v\}$ and $m(C(\Omega)) = 1$. By the straightward calculation, the right hand side of the last equality in (2.13) becomes

$$(2.14)\quad \begin{aligned} &\frac{S^2T^2}{12} + \frac{S}{3}\int_0^T \psi^2(t)dt \\ &+ \int_0^S \int_T^{g(s)} \left(st - \frac{s^2}{S}T + \frac{s^2}{S^2}\psi^2(T)\right) dtds, \end{aligned}$$

which is our desired result.

□

Remark 2.4. In Theorem 2.2 and Theorem 2.3, we have the extra terms which does not appear in Example 1 and Example 2 of [6]. This means that Park and Skoug's examples in [6] are the special case of our results for the rectangle Ω.

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