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NONLINEAR ALGORITHMS FOR A COMMON SOLUTION OF A SYSTEM OF VARIATIONAL INEQUALITIES, A SPLIT EQUILIBRIUM PROBLEM AND FIXED POINT PROBLEMS

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Abstract. In this paper, we propose an iterative algorithm for finding a common solution of a system of generalized equilibrium problems, a split equilibrium problem and a hierarchical fixed point problem over the common fixed points set of a finite family of nonexpansive mappings in Hilbert spaces. Furthermore, we prove that the proposed iterative method has strong convergence under some mild conditions imposed on algorithm parameters. The results presented in this paper improve and extend the corresponding results reported by some authors recently.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let $T : C \to H$ be a nonlinear mapping. We use $\mathrm{Fix}(T)$ to denote the set of fixed points of T, *i.e.*, $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$. A mapping T is called nonexpansive if the following inequality holds:

$$\|Tx - Ty\| \leq \|x - y\|$$

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for all $x, y \in C$. Moreover, we also denote by $\mathbb{R}$ the set of all real numbers.

Recently, Ceng and Yao [10] considered the following system of generalized equilibrium problems, which involves finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} F_1(x^*, x) + \langle B_1 y^*, x - x^* \rangle + \frac{1}{\mu_1}\langle x^* - y^*, x - x^* \rangle \geq 0, \forall x \in C, \\ F_2(y^*, y) + \langle B_2 x^*, y - y^* \rangle + \frac{1}{\mu_2}\langle y^* - x^*, y - y^* \rangle \geq 0, \forall y \in C, \end{cases} \tag{1.1}$$

where $F_1, F_2 : C \times C \to \mathbb{R}$ are two bifunctions, $B_1, B_2 : C \to H$ are two nonlinear mappings and $\mu_1, \mu_2 > 0$ are two constants. The solution set of (1.1) is denoted by Ω.

If $F_1 = F_2 = 0$, then problem (1.1) reduces to the following general system of variational inequalities: Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, y - y^* \rangle \geq 0, \quad \forall y \in C, \end{cases} \tag{1.2}$$

which is introduced and considered by Ceng et al. [9].

If $B_1 = B_2 = B$ in (1.2), then problem (1.2) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \mu_2 B x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.3}$$

which has been introduced and studied by Verma [23,24].

If $x^* = y^*$ and $\mu_1 = \mu_2$, then problem (1.3) collapses to the classical variational inequality: Find $x^* \in C$ such that

$$\langle B x^*, x - x^* \rangle \geq 0, \quad \forall x \in C.$$

The theory of variational inequality emerged as rapidly growing area of research because of its applications in nonlinear analysis, optimization, economics and game theory (see [1,2,4,5] and the references cited therein).

The equilibrium problem is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

The solution set of (1.4) is denoted by $EP(F)$. Numerous problems in physics, optimization and economics reduce to finding a solution of (1.4) (see [12,19]).

In 1994, Censor and Elfving [11] introduced and studied the following split feasibility problem:

Let C and K be nonempty closed convex subsets of the infinite-dimensional real Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. Then the split feasibility problem is to find $x^* \in C$ such that $Ax^* \in K$.

In this paper, we study the following split equilibrium problem:

Let $\Theta : K \times K \to \mathbb{R}$ be a nonlinear bifunction and $A : H_1 \to H_2$ be a bounded linear operator. Then the split equilibrium problem (SEP) is to find $x^* \in C$ such that

$$y^* = Ax^* \in K \quad \text{solves} \quad \Theta(y^*, y) \geq 0, \quad \forall y \in K. \tag{1.5}$$

The solution set of SEP (1.5) is denoted by $\Lambda = \{p \in C : Ap \in EP(\Theta)\}$.

Let $S : C \to H$ be a nonexpansive mapping. The following problem is called a hierarchical fixed point problem: find $x \in F(T)$ such that

$$\langle x - Sx, y - x \rangle \geq 0, \quad \forall y \in F(T). \tag{1.6}$$

It is well known that the iterative methods for finding hierarchical fixed points of nonexpansive mappings can be used to solve a convex minimization problem (see [25,26] and the references therein).

In 2001, Yamada [26] considered the following hybrid steepest-decent iterative method:

$$x_{n+1} = Tx_n - \mu\lambda_n F(Tx_n),$$

where F is κ-Lipschitzian continuous and η-strongly monotone operator with $\kappa > 0$, $\eta > 0$ and $0 < \mu < \frac{2\eta}{\kappa^2}$. Under some appropriate conditions, the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T). \tag{1.7}$$

Zhou and Wang [28] proposed a simple explicit iterative algorithm for finding a solution of variational inequality over the set of common fixed points of a finite family nonexpansive mappings. They introduced an explicit scheme as follows:

Theorem 1.1. *Let H be a real Hilbert space and $F : H \to H$ be an κ-Lipschitzian continuous and η-strongly monotone mapping with $\kappa > 0$ and $\eta > 0$. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-mappings of H such that*

$C = \cap_{i=1}^N Fix(T_i) \neq \phi$. *For any point* $x_0 \in H$, *define a sequence* $\{x_n\}$ *as follows:*

$$x_{n+1} = (1 - \alpha_n \mu F) T_N^n T_{N-1}^n \cdots T_1^n x_n, \quad n \geq 0,$$

where $\mu \in (0, \frac{2\eta}{\kappa^2})$ *and* $T_i^n = (1 - \sigma_n^i)I + \sigma_n^i T_i$ *for* $i = 1, 2, \cdots, N$. *When the parameters satisfy appropriate conditions, the sequence* $\{x_n\}$ *converges strongly to the unique solution of the variational inequality* (1.7).

Recently, Zhang and Yang [27] proposed an explicit iterative algorithm based on the viscosity method for finding a solution for a class of variational inequalities over the common fixed points set of a finite family of nonexpansive mappings as follows:

THEOREM 1.2. *Let* H *be a real Hilbert space and* $F : H \to H$ *be an* κ*-Lipschitzian continuous and* η*-strongly monotone mapping with* $\kappa > 0$ *and* $\eta > 0$. *Let* $\{T_i\}_{i=1}^N$ *be* N *nonexpansive mappings of* H *such that* $C = \cap_{i=1}^N Fix(T_i) \neq \phi$ *and* V *be an* ρ*-Lipschitzian continuous on* H *with* $\rho > 0$. *For any point* $x_0 \in H$, *define a sequence* $\{x_n\}$ *as follows:*

$$x_{n+1} = \alpha_n \gamma V(x_n) + (I - \alpha_n \mu F) T_N^n T_{N-1}^n \cdots T_1^n x_n, \quad n \geq 0,$$

where $0 < \gamma\rho < \tau$ *with* $\tau = \mu(2\eta - \mu\kappa^2)$, $0 < \mu < \frac{2\eta}{\kappa^2}$, $T_i^n = (1 - \sigma_n^i)I + \sigma_n^i T_i$ *for* $i = 1, 2, \cdots, N$ *and* $\sigma_n^i \in (\zeta_1, \zeta_2)$ *for some* $\zeta_1, \zeta_2 \in (0, 1)$. *When the parameters satisfy appropriate conditions, the sequence* $\{x_n\}$ *converges strongly to the unique solution* $x^* \in C$ *of the variational inequality:*

$$\langle (\mu F - \gamma V)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \cap_{i=1}^N Fix(T_i). \tag{1.8}$$

In this paper, motivated by the above works, we introduce a new iterative algorithm for finding the approximate element of the common set of solutions of (1.1), (1.5) and (1.8) in real Hilbert spaces. Strong convergence theorems for common elements are established. Our result improves and extends many known results for solving a system of variational inequality problems, split equilibrium problems and hierarchical fixed point theorems (see [6,8,15,18,22,27,28] and the references cited therein).

2. Preliminaries

Let C and K be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H_1$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. For every point $x \in H_1$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \forall y \in C.$$

Then P_C is called the metric projection of H_1 onto C. It is well known that P_C is nonexpansive and satisfies the following property:

$$\langle x - P_C(x), P_C(x) - y\rangle \geq 0, \quad \forall x \in H_1, y \in C. \tag{2.1}$$

Definition 2.1. A mapping $T : H_1 \to H_1$ is said to be

(1) η-strongly monotone if there exists $\eta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in H_1;$$

(2) δ-inverse strongly monotone if there exists $\delta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \delta\|Tx - Ty\|^2, \quad \forall x, y \in H_1;$$

(3) κ-Lipschitzian continuous if there exists $\kappa > 0$ such that

$$\|Tx - Ty\| \leq \kappa\|x - y\|, \quad \forall x, y \in H_1;$$

(4) σ-averaged if there exists $\sigma \in (0, 1)$ such that $T = (1 - \sigma)I + \sigma S$, where $I : H_1 \to H_1$ is the identity mapping and $S : H_1 \to H_1$ is nonexpansive.

In order to prove our main results in the next section, we need the following lemmas.

Lemma 2.1. *For all $x, y \in H_1$, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

Assumption 2.1 [3]. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:

(A1) $F(x, x) = 0$, $\forall x \in C$;

(A2) F is monotone, *i.e.*, $F(x, y) + F(y, x) \leq 0$, $\forall x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\to 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.2 [14]. *Let C be a nonempty closed convex subset of H_1. Let $F_1 : C \times C \to \mathbb{R}$ satisfy (A1)-(A4). Assume that for $r > 0$, define a mapping $T_r^{F_1} : H_1 \to C$ as follows:*

$$T_r^{F_1}(x) = \{z \in C : F_1(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}, \forall x \in H_1.$$

Then the following hold:
(i) $T_r^{F_1}$ *is nonempty and single-valued;*
(ii) $T_r^{F_1}$ *is firmly nonexpansive, i.e.,*

$$\|T_r^{F_1}(x) - T_r^{F_1}(y)\|^2 \leq \langle T_r^{F_1}(x) - T_r^{F_1}(y), x - y\rangle, \forall x, y \in H_1;$$

(iii) $Fix(T_r^{F_1}) = EP(F_1)$*;*
(iv) $EP(F_1)$ *is closed and convex.*

Lemma 2.3. [13] *Assume that $F_1 : C \times C \to \mathbb{R}$ satisfies Assumption 2.1 and let $T_r^{F_1}$ be defined as in Lemma 2.2. Let $x, y \in H_1$ and $r_1, r_2 > 0$. Then*

$$\|T_{r_2}^{F_1}(y) - T_{r_1}^{F_1}(x)\| \leq \|y - x\| + |\frac{r_2 - r_1}{r_2}|\|T_{r_2}^{F_1}(y) - y\|.$$

Lemma 2.4. [7] *Let $F_1, F_2 : C \times C \to \mathbb{R}$ be two bifunctions satisfying (A1)-(A4). For any $(x^*, y^*) \in C \times C$, (x^*, y^*) is a solution of (1.1) if and only if x^* is a fixed point of the mapping $Q : C \to C$ defined by*

$$Q(x) = T_{\mu_1}^{F_1}[T_{\mu_2}^{F_2}(x - \mu_2 B_2 x) - \mu_1 B_1 T_{\mu_2}^{F_2}(x - \mu_2 B_2 x)], \forall x \in C,$$

where $y^* = T_{\mu_2}^{F_2}(x^* - \mu_2 B_2 x^*)$, $\mu_i \in (0, 2\theta_i)$ and $B_i : C \to C$ is a θ_i-inverse strongly monotone mapping for each $i = 1, 2$.

Lemma 2.5. [21] *Suppose that $\lambda \in (0, 1)$ and $\mu > 0$. Let $F : C \to C$ be a κ-Lipschitzian continuous and η-strongly monotone mapping with $\kappa > 0$ and $\eta > 0$. In association with a nonexpansive mapping $T : C \to C$, define the mapping $T^\lambda : C \to H_1$ by*

$$T^\lambda x = Tx - \lambda\mu F(Tx), \quad \forall x \in C.$$

Then T^λ is a contraction provided $\mu < \frac{2\eta}{\kappa^2}$, i.e.,

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in C,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$.

Lemma 2.6. [17]
(i) *The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the*

composite $T_1 \cdots T_N$. *In particular, if* T_1 *is* α_1*-averaged and* T_2 *is* α_2*-averaged, where* $\alpha_1, \alpha_2 \in (0,1)$*, then both* T_1T_2 *and* T_2T_1 *are* α*-averaged, where* $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.

(ii) *If the mappings* $\{T_i\}_{i=1}^N$ *are averaged and have a common fixed point, then* $\cap_{i=1}^N Fix(T_i) = Fix(T_1 \cdots T_N)$. *In particular, if* $N = 2$*, then we have* $Fix(T_1) \cap Fix(T_2) = Fix(T_1T_2) = Fix(T_2T_1)$.

LEMMA 2.7. [16] *Let* H_1 *be a Hilbert space,* C *be a closed convex subset of* H_1 *and* $T : C \to C$ *be a nonexpansive mapping with* $Fix(T) \neq \phi$. *If* $\{x_n\}$ *is a sequence in* C *weakly converging to* $x \in C$ *and* $\{(I-T)x_n\}$ *converges strongly to* $y \in C$*, then* $(I - T)x = y$. *In particular, if* $y = 0$*, then* $x \in Fix(T)$.

LEMMA 2.8. [20] *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* E *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose* $x_{n+1} = \beta_n x_n + (1-\beta_n)y_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

LEMMA 2.9. [25]. *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* δ_n *is a sequence such that*
(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

THEOREM 3.1. *Let* H_1 *and* H_2 *be two real spaces, and* $C \subseteq H_1$ *and* $K \subseteq H_2$ *be two nonempty closed convex subsets. Let* $A : H_1 \to H_2$ *be a bounded linear operator with its adjoint* A^*. *Assume that* $F_1, F_2 : C \times C \to \mathbb{R}$ *and* $\Theta : K \times K \to \mathbb{R}$ *are the bifunctions satisfying Assumption 2.1 and* Θ *is upper semicontinuous in the first argument. Let* $B_i : C \to H_1$ *be a* γ_i*-inverse strongly monotone mapping for each* $i = 1, 2$ *and* $T_i : C \to C$ *be a nonexpansive mapping for each* $i = 1, 2, \cdots, N$ *such that* $\Gamma = \cap_{i=1}^N Fix(T_i) \cap \Omega \cap \Lambda \neq \phi$. *Let* $F : C \to C$ *be a* κ*-Lipschitzian continuous and* η*-strongly monotone mapping with* $\kappa > 0$ *and* $\eta > 0$*, and* $V : C \to C$ *be a* σ*-Lipschitzian continuous mapping with* $\sigma > 0$. *Let*

$0 < \mu < \frac{2\eta}{\kappa^2}$ *and* $0 < \rho\sigma < \tau$, *where* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. *Suppose* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are two sequences in* $(0,1)$. *Given* $x_1 \in C$, *let* $\{x_n\}$ *be defined by*

$$\begin{cases} z_n = P_C(x_n + \delta A^*(T_{r_n}^{\Theta} - I)Ax_n), \\ y_n = T_{\mu_1}^{F_1}[T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n)], \\ x_{n+1} = \alpha_n \rho V(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n \mu F)T_N^n T_{N-1}^n \cdots T_1^n y_n, \forall n \geq 1, \end{cases} \tag{3.1}$$

where $T_i^n = (1-\sigma_n^i)I + \sigma_n^i T_i$ *for* $i = 1, 2, \cdots, N$, $\{r_n\} \subset (0, 2\zeta)$, $\zeta > 0$, $\mu_i \in (0, 2\gamma_i)$ *for each* $i = 1, 2$, $\delta \in (0, \frac{1}{L})$, L *is the spectral radius of the operator* A^*A, A^* *is the adjoint of* A *and* $\sigma_n^i \in (\zeta_1, \zeta_2)$ *for some* $\zeta_1, \zeta_2 \in (0,1)$. *If the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\zeta$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;

(iv) $\lim_{n\to\infty} |\sigma_{n+1}^i - \sigma_n^i| = 0$ *for* $i = 1, 2, \cdots, N$.

Then the sequence $\{x_n\}$ *converges strongly to* $x^* \in \Gamma = \cap_{i=1}^N Fix(T_i) \cap \Omega \cap \Lambda$, *where* $x^* = P_\Gamma(I - \mu F + \rho V)x^*$ *is the unique solution of the variational inequality:*

$$\langle (\mu F - \rho V)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Gamma. \tag{3.2}$$

Proof. Since our methods easily deduce the general case, we prove Theorem 3.1 for $N = 2$. Now we divide the proof into several steps.

Step 1. $\{x_n\}$ is bounded.

Indeed, take $x^* \in \Gamma = \cap_{i=1}^N \mathrm{Fix}(T_i) \cap \Omega \cap \Lambda$ arbitrarily. Since $Ax^* = T_{r_n}^{\Theta}(Ax^*)$, we have

$$\begin{aligned} \|Ax_n - Ax^*\|^2 &= \|(T_{r_n}^{\Theta} - I)Ax_n - (T_{r_n}^{\Theta}Ax_n - Ax^*)\|^2 \\ &= \|(T_{r_n}^{\Theta} - I)Ax_n\|^2 - 2\langle (T_{r_n}^{\Theta} - I)Ax_n, T_{r_n}^{\Theta}Ax_n - Ax^* \rangle \\ &\quad + \|T_{r_n}^{\Theta}Ax_n - Ax^*\|^2 \\ &\leq \|(T_{r_n}^{\Theta} - I)Ax_n\|^2 - 2\langle (T_{r_n}^{\Theta} - I)Ax_n, T_{r_n}^{\Theta}Ax_n - Ax^* \rangle \\ &\quad + \|Ax_n - Ax^*\|^2. \end{aligned}$$

It follows that

$$\langle (T_{r_n}^{\Theta} - I)Ax_n, T_{r_n}^{\Theta}Ax_n - Ax^*\rangle \le \frac{1}{2}\|(T_{r_n}^{\Theta} - I)Ax_n\|^2. \tag{3.3}$$

Note that $x^* = P_C(x^*)$. From (3.1) and (3.3), it follows that

$$\begin{aligned}
&\|z_n - x^*\|^2 \\
&= \|P_C(x_n + \delta A^*(T_{r_n}^{\Theta} - I)Ax_n) - P_C(x^*)\|^2 \\
&\le \|x_n + \delta A^*(T_{r_n}^{\Theta} - I)Ax_n - x^*\|^2 \\
&= \|x_n - x^*\|^2 + 2\delta\langle x_n - x^*, A^*(T_{r_n}^{\Theta} - I)Ax_n\rangle \\
&\quad + \delta^2\|A^*(T_{r_n}^{\Theta} - I)Ax_n\|^2 \\
&= \|x_n - x^*\|^2 + 2\delta\langle A(x_n - x^*), (T_{r_n}^{\Theta} - I)Ax_n\rangle \\
&\quad + \delta^2\langle (T_{r_n}^{\Theta} - I)Ax_n, AA^*(T_{r_n}^{\Theta} - I)Ax_n\rangle \\
&= \|x_n - x^*\|^2 + 2\delta(\langle T_{r_n}^{\Theta}(Ax_n) - Ax^*, (T_{r_n}^{\Theta} - I)Ax_n\rangle - \|(T_{r_n}^{\Theta} - I)Ax_n\|^2) \\
&\quad + \delta^2\langle (T_{r_n}^{\Theta} - I)Ax_n, AA^*(T_{r_n}^{\Theta} - I)Ax_n\rangle \\
&\le \|x_n - x^*\|^2 + 2\delta(\frac{1}{2}\|(T_{r_n}^{\Theta} - I)Ax_n\|^2 - \|(T_{r_n}^{\Theta} - I)Ax_n\|^2) \\
&\quad + \delta^2\|AA^*\|\|(T_{r_n}^{\Theta} - I)Ax_n\|^2 \\
&= \|x_n - x^*\| - \delta(1 - L\delta)\|(T_{r_n}^{\Theta} - I)Ax_n\|^2 \\
&\le \|x_n - x^*\|^2.
\end{aligned} \tag{3.4}$$

Since B_i is a γ_i-inverse strongly monotone mapping for each $i = 1, 2$, $x^* = T_{\mu_1}^{F_1}(y^* - \mu_1 B_1 y^*)$ and $y^* = T_{\mu_2}^{F_2}(x^* - \mu_2 B_2 x^*)$, we obtain from (3.4) that

$$\begin{aligned}
&\|y_n - y^*\|^2 \\
&= \|T_{\mu_1}^{F_1}[T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n)] \\
&\quad - T_{\mu_1}^{F_1}[T_{\mu_2}^{F_2}(x^* - \mu_2 B_2 x^*) - \mu_1 B_1 T_{\mu_2}^{F_2}(x^* - \mu_2 B_2 x^*)]\|^2 \\
&\le \|T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - T_{\mu_2}^{F_2}(x^* - \mu_2 B_2 x^*) \\
&\quad - \mu_1(B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - B_1 T_{\mu_2}^{F_2}(x^* - \mu_2 B_2 x^*))\|^2
\end{aligned}$$

$$\begin{aligned}
&\leq \|T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - T_{\mu_2}^{F_2}(x^* - \mu_2 B_2 x^*)\|^2 \\
&\quad - 2\mu_1 \langle B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - B_1 T_{\mu_2}^{F_2}(x^* - \mu_2 B_2 x^*), \\
&\qquad T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - T_{\mu_2}^{F_2}(x^* - \mu_2 B_2 x^*)\rangle \\
&\quad + \mu_1^2 \|B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - B_1 T_{\mu_2}^{F_2}(x^* - \mu_2 B_2 x^*)\|^2 \\
&\leq \|T_{\mu_1}^{F_2}(z_n - \mu_2 B_2 z_n) - T_{\mu_2}^{F_2}(x^* - \mu_2 B_2 x^*)\|^2 \\
&\quad - \mu_1(2\gamma_1 - \mu_1)\|B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - B_1 T_{\mu_2}^{F_2}(x^* - \mu_2 B_2 x^*)\|^2 \\
&\leq \|(z_n - x^*) - \mu_2(B_2 z_n - B_2 x^*)\|^2 \\
&\quad - \mu_1(2\gamma_1 - \mu_1)\|B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - B_1 T_{\mu_2}^{F_2}(x^* - \mu_2 B_2 x^*)\|^2 \\
&= \|z_n - x^*\|^2 - 2\mu_2 \langle B_2 z_n - B_2 x^*, z_n - x^*\rangle + \mu_2^2 \|B_2 z_n - B_2 x^*\|^2 \\
&\quad - \mu_1(2\gamma_1 - \mu_1)\|B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - B_1 T_{\mu_2}^{F_2}(x^* - \mu_2 B_2 x^*)\|^2 \\
&\leq \|z_n - x^*\|^2 - \mu_2(2\gamma_2 - \mu_2)\|B_2 z_n - B_2 x^*\|^2 \\
&\quad - \mu_1(2\gamma_1 - \mu_1)\|B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - B_1 T_{\mu_2}^{F_2}(x^* - \mu_2 B_2 x^*)\|^2 \\
&\leq \|z_n - x^*\|^2 \\
&\leq \|x_n - x^*\|^2. \qquad (3.5)
\end{aligned}$$

From (3.1), (3.5) and Lemma 2.5, we have

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&= \|\alpha_n \rho V(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n \mu F)T_2^n T_1^n y_n - x^*\| \\
&\leq \alpha_n \|\rho V(x_n) - \mu F(x^*)\| + \beta_n \|x_n - x^*\| \\
&\quad + \|((1-\beta_n)I - \alpha_n \mu F)T_2^n T_1^n y_n - ((1-\beta_n)I - \alpha_n \mu F)T_2^n T_1^n x^*\| \\
&= \alpha_n \|\rho V(x_n) - \mu F(x^*)\| + \beta_n \|x_n - x^*\| \\
&\quad + (1-\beta_n)\|(I - \frac{\alpha_n}{1-\beta_n}\mu F)T_2^n T_1^n y_n - (I - \frac{\alpha_n}{1-\beta_n}\mu F)T_2^n T_1^n x^*\| \\
&\leq (1-\beta_n)(1 - \frac{\alpha_n \tau}{1-\beta_n})\|y_n - x^*\| + \beta_n \|x^* - x^*\| + \alpha_n \|\rho V(x_n) - \mu F(x^*)\| \\
&\leq (1 - \alpha_n \tau)\|x_n - x^*\| + \alpha_n \rho \|V(x_n) - V(x^*)\| + \alpha_n \|\rho V(x^*) - \mu F(x^*)\| \\
&\leq (1 - \alpha_n(\tau - \rho\sigma))\|x_n - x^*\| + \alpha_n(\tau - \rho\sigma)\frac{\|\rho V(x^*) - \mu F(x^*)\|}{\tau - \rho\sigma} \\
&\leq \max\left\{\|x_n - x^*\|, \frac{\|\rho V(x^*) - \mu F(x^*)\|}{\tau - \rho\sigma}\right\}. \qquad (3.6)
\end{aligned}$$

It follows from (3.6) and induction that

$$\|x_n - x^*\| \le \max\left\{\|x_0 - x^*\|, \frac{\|\rho V(x^*) - \mu F(x^*)\|}{\tau - \rho\sigma}\right\}, \quad n = 1, 2, \cdots .$$

Therefore $\{x_n\}$ is bounded. We also obtain that $\{y_n\}$ and $\{z_n\}$ are all bounded.

Step 2. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Indeed, set $x_{n+1} = \beta_n x_n + (1 - \beta_n)w_n$ for all $n \ge 1$. Then we obtain

$$\begin{aligned} w_{n+1} - w_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}\rho V(x_{n+1}) + ((1 - \beta_{n+1})I - \alpha_{n+1}\mu F)T_2^{n+1}T_1^{n+1}y_{n+1}}{1 - \beta_{n+1}} \\ &\quad - \frac{\alpha_n \rho V(x_n) + ((1 - \beta_n)I - \alpha_n \mu F)T_2^n T_1^n y_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(\rho V(x_{n+1}) - \mu F(T_2^{n+1}T_1^{n+1}y_{n+1})) \\ &\quad + \frac{\alpha_n}{1 - \beta_n}(\mu F(T_2^n T_1^n y_n) - \rho V(x_n)) \\ &\quad + T_2^{n+1}T_1^{n+1}y_{n+1} - T_2^{n+1}T_1^{n+1}y_n + T_2^{n+1}T_1^{n+1}y_n - T_2^n T_1^n y_n. \end{aligned}$$

It follows that

$$\begin{aligned} &\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\| \\ &\le \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(\rho\|V(x_{n+1})\| + \mu\|F(T_2^{n+1}T_1^{n+1}y_{n+1})\|) \\ &\quad + \frac{\alpha_n}{1 - \beta_n}(\mu\|F(T_2^n T_1^n y_n)\| + \rho\|V(x_n)\|) \\ &\quad + \|T_2^{n+1}T_1^{n+1}y_n - T_2^n T_1^n y_n\| + \|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|. \end{aligned} \tag{3.7}$$

Utilizing the nonexpansivity of T_2^{n+1}, we obtain

$$\begin{aligned} &\|T_2^{n+1}T_1^{n+1}y_n - T_2^n T_1^n y_n\| \\ &= \|T_2^{n+1}T_1^{n+1}y_n - T_2^{n+1}T_1^n y_n\| + \|T_2^{n+1}T_1^n y_n - T_2^n T_1^n y_n\| \\ &\le \|T_1^{n+1}y_n - T_1^n y_n\| + \|T_2^{n+1}T_1^n y_n - T_2^n T_1^n y_n\|. \end{aligned} \tag{3.8}$$

Note that

$$\begin{aligned} \|T_1^{n+1}y_n - T_1^n y_n\| &= \|(1 - \sigma_{n+1}^1)y_n + \sigma_{n+1}^1 T_1 y_n - (1 - \sigma_n^1)y_n - \sigma_n^1 T_1 y_n\| \\ &\le |\sigma_{n+1}^1 - \sigma_n^1|(\|y_n\| + \|T_1 y_n\|). \end{aligned}$$

Since $\lim_{n\to\infty} |\sigma^i_{n+1} - \sigma^i_n| = 0$ for $i = 1, 2$ and $\{y_n\}$, $\{T_1 y_n\}$ are bounded, we easily obtain

$$\lim_{n\to\infty} \|T_1^{n+1} y_n - T_1^n y_n\| = 0. \tag{3.9}$$

Similarly, we get

$$\|T_2^{n+1} T_1^n y_n - T_2^n T_1^n y_n\| \le |\sigma^2_{n+1} - \sigma^2_n|(\|T_1^n y_n\| + \|T_2 T_1^n y_n\|,$$

which implies that

$$\lim_{n\to\infty} \|T_2^{n+1} T_1^n y_n - T_2^n T_1^n y_n\| = 0. \tag{3.10}$$

Since $z_n = P_C(x_n + \delta A^*(T^\Theta_{r_n} - I)Ax_n)$, it follows from Lemma 2.3 that

$$\begin{aligned}
&\|z_{n+1} - z_n\| \\
&= \|P_C(x_{n+1} + \delta A^*(T^\Theta_{r_{n+1}} - I)Ax_{n+1}) - P_C(x_n + \delta A^*(T^\Theta_{r_n} - I)Ax_n)\| \\
&\le \|x_{n+1} - x_n - \delta A^* A(x_{n+1} - x_n)\| + \delta\|A\|\|T^\Theta_{r_{n+1}}(Ax_{n+1}) - T^\Theta_{r_n}(Ax_n)\| \\
&\quad + |1 - \frac{r_n}{r_{n+1}}|\|P_C(x_{n+1} + \delta A^*(T^\Theta_{r_{n+1}} - I)Ax_{n+1}) \\
&\qquad\qquad - (x_{n+1} + \delta A^*(T^\Theta_{r_{n+1}} - I)Ax_{n+1})\| \\
&\le (\|x_{n+1} - x_n\|^2 - 2\delta\|A(x_{n+1} - x_n)\|^2 + \delta^2\|A\|^4\|x_{n+1} - x_n\|^2)^{\frac{1}{2}} \\
&\quad + \delta\|A\|(\|A(x_{n+1} - x_n)\| + |1 - \frac{r_n}{r_{n+1}}|\|T^\Theta_{r_{n+1}}(Ax_{n+1}) - Ax_{n+1}\|) \\
&\quad + |1 - \frac{r_n}{r_{n+1}}|\|P_C(x_{n+1} + \delta A^*(T^\Theta_{r_{n+1}} - I)Ax_{n+1}) \\
&\qquad\qquad - (x_{n+1} + \delta A^*(T^\Theta_{r_{n+1}} - I)Ax_{n+1})\| \\
&= (1 - \delta\|A\|^2)\|x_{n+1} - x_n\| + \delta\|A\|^2\|x_{n+1} - x_n\| \\
&\quad + \delta\|A\||1 - \frac{r_n}{r_{n+1}}|\|T^\Theta_{r_{n+1}}(Ax_{n+1}) - Ax_{n+1}\| \\
&\quad + |1 - \frac{r_n}{r_{n+1}}|\|P_C(x_{n+1} + \delta A^*(T^\Theta_{r_{n+1}} - I)Ax_{n+1}) \\
&\qquad\qquad - (x_{n+1} + \delta A^*(T^\Theta_{r_{n+1}} - I)Ax_{n+1})\| \\
&= \|x_{n+1} - x_n\| + |\frac{r_{n+1} - r_n}{r_{n+1}}|(\delta\|A\|\psi_{n+1} + \chi_{n+1}),
\end{aligned}$$

where $\psi_n = \|T^\Theta_{r_n}(Ax_n) - Ax_n\|$ and $\chi_n = \|P_C(x_n + \delta A^*(T^\Theta_{r_n} - I)Ax_n) - (x_n + \delta A^*(T^\Theta_{r_n} - I)Ax_n)\|$. Without loss of generality, let us assume that there exists a real number $\mu > 0$ such that $r_n > \mu > 0$ for all positive integers n. Then we get

$$\|z_{n+1} - z_n\| \le \|x_{n+1} - x_n\| + \frac{1}{\mu}|r_{n+1} - r_n|(\delta\|A\|\psi_{n+1} + \chi_{n+1}). \quad (3.11)$$

Next, we estimate that

$$\begin{aligned}
&\|y_{n+1} - y_n\|^2 \\
&= \|T_{\mu_1}^{F_1}[T_{\mu_2}^{F_2}(z_{n+1} - \mu_2 B_2 z_{n+1}) - \mu_1 B_1 T_{\mu_2}^{F_2}(z_{n+1} - \mu_2 B_2 z_{n+1})] \\
&\quad - T_{\mu_1}^{F_1}[T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n)]\|^2 \\
&\le \|T_{\mu_2}^{F_2}(z_{n+1} - \mu_2 B_2 z_{n+1}) - T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) \\
&\quad - \mu_1(B_1 T_{\mu_2}^{F_2}(z_{n+1} - \mu_2 B_2 z_{n+1}) - B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n))\|^2 \\
&\le \|T_{\mu_2}^{F_2}(z_{n+1} - \mu_2 B_2 z_{n+1}) - T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n)\|^2 \\
&\quad - \mu_1(2\gamma_1 - \mu_1)\|B_1 T_{\mu_2}^{F_2}(z_{n+1} - \mu_2 B_2 z_{n+1}) - B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n)\|^2 \\
&\le \|z_{n+1} - z_n - \mu_2(B_2 z_{n+1} - B_2 z_n)\|^2 \\
&\le \|z_{n+1} - z_n\|^2 - \mu_2(2\gamma_2 - \mu_2)\|B_2 z_{n+1} - B_2 z_n\|^2 \\
&\le \|z_{n+1} - z_n\|^2. \qquad (3.12)
\end{aligned}$$

It follows from (3.11) and (3.12) that

$$\begin{aligned}
\|y_{n+1} - y_n\| &\le \|z_{n+1} - z_n\| \\
&\le \|x_{n+1} - x_n\| + \frac{1}{\mu}|r_{n+1} - r_n|(\delta\|A\|\psi_{n+1} + \chi_{n+1}). \quad (3.13)
\end{aligned}$$

Using (3.8), (3.13) in (3.7), we get

$$\begin{aligned}
&\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\| \\
&\qquad \le \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(\rho\|V(x_{n+1})\| + \mu\|F(T_2^{n+1}T_1^{n+1}y_{n+1})\|) \\
&\qquad + \frac{\alpha_n}{1 - \beta_n}(\mu\|F(T_2^n T_1^n y_n)\| + \rho\|V(x_n)\|) \\
&\qquad + \|T_1^{n+1}y_n - T_1^n y_n\| + \|T_2^{n+1}T_1^n y_n - T_2^n T_1^n y_n\| \\
&\qquad + \frac{1}{\mu}|r_{n+1} - r_n|(\delta\|A\|\psi_{n+1} + \chi_{n+1}).
\end{aligned}$$

Consequently, it follows from (3.9), (3.10) and conditions (i)-(iii) that

$$\limsup_{n\to\infty}(\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Hence by Lemma 2.8, we have

$$\lim_{n\to\infty}\|w_n - x_n\| = 0.$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|w_n - x_n\| = 0. \tag{3.14}$$

Step 3. $\lim_{n\to\infty}\|x_n - y_n\| = 0$ and $\lim_{n\to\infty}\|z_n - x_n\| = 0$.

Indeed, from (3.13), (3.14) and condition (iii), we have

$$\begin{aligned}\lim_{n\to\infty}\|y_{n+1} - y_n\| &\le \lim_{n\to\infty}\|x_{n+1} - x_n\| + \frac{1}{\mu}\lim_{n\to\infty}|r_{n+1} - r_n|(\delta\|A\|\psi_{n+1} + \chi_{n+1})\\ &= 0.\end{aligned}$$

Since $x_{n+1} = \alpha_n\rho V(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n\mu F)T_2^nT_1^ny_n$, we obtain

$$\begin{aligned}\|x_n - T_2^nT_1^ny_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_2^nT_1^ny_n\|\\ &\le \|x_n - x_{n+1}\| + \alpha_n\|\rho V(x_n) - \mu F(T_2^nT_1^ny_n)\|\\ &\quad + \beta_n\|x_n - T_2^nT_1^ny_n\|,\end{aligned}$$

that is,

$$\|x_n - T_2^nT_1^ny_n\| \le \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|\rho V(x_n) - \mu F(T_2^nT_1^ny_n)\|.$$

It follows from (3.14) and condition (i) that

$$\lim_{n\to\infty}\|x_n - T_2^nT_1^ny_n\| = 0. \tag{3.15}$$

From (3.4), (3.5) and Lemma 2.5, we get

$$\begin{aligned}&\|x_{n+1} - x^*\|^2\\ &= \langle\alpha_n\rho V(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n\mu F)T_2^nT_1^ny_n - x^*, x_{n+1} - x^*\rangle\\ &= \langle\alpha_n\rho(V(x_n) - V(x^*)), x_{n+1} - x^*\rangle + \alpha_n\langle\rho V(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle\\ &\quad + \beta_n\langle x_n - T_2^nT_1^ny_n, x_{n+1} - x^*\rangle\\ &\quad + \langle(I - \alpha_n\mu F)T_2^nT_1^ny_n - (I - \alpha_n\mu F)T_2^nT_1^nx^*, x_{n+1} - x^*\rangle\\ &\le \alpha_n\rho\sigma\|x_n - x^*\|\|x_{n+1} - x^*\| + \alpha_n\langle\rho V(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle\\ &\quad + \beta_n\|x_n - T_2^nT_1^ny_n\|\|x_{n+1} - x^*\| + (1 - \alpha_n\tau)\|y_n - x^*\|\|x_{n+1} - x^*\|\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\alpha_n\rho\sigma}{2}(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + \alpha_n\langle \rho V(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle \\
&\quad + \beta_n\|x_n - T_2^nT_1^ny_n\|\|x_{n+1} - x^*\| + \frac{1-\alpha_n\tau}{2}(\|y_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\
&\le \frac{1-\alpha_n(\tau-\rho\sigma)}{2}\|x_{n+1} - x^*\|^2 + \frac{\alpha_n\rho\sigma}{2}\|x_n - x^*\|^2 \\
&\quad + \alpha_n\langle \rho V(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle + \beta_n\|x_n - T_2^nT_1^ny_n\|\|x_{n+1} - x^*\| \\
&\quad + \frac{1-\alpha_n\tau}{2}\|y_n - x^*\|^2 \\
&\le \frac{1-\alpha_n(\tau-\rho\sigma)}{2}\|x_{n+1} - x^*\|^2 + \frac{\alpha_n\rho\sigma}{2}\|x_n - x^*\|^2 \\
&\quad + \alpha_n\langle \rho V(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle + \beta_n\|x_n - T_2^nT_1^ny_n\|\|x_{n+1} - x^*\| \\
&\quad + \frac{1-\alpha_n\tau}{2}(\|z_n - x^*\|^2 - \mu_2(2\gamma_2 - \mu_2)\|B_2z_n - B_2x^*\|^2 \\
&\quad - \mu_1(2\gamma_1 - \mu_1)\|B_1T_{\mu_2}^{F_2}(z_n - \mu_2B_2z_n) - B_1T_{\mu_2}^{F_2}(x^* - \mu_2B_2x^*)\|^2) \\
&\le \frac{1-\alpha_n(\tau-\rho\sigma)}{2}\|x_{n+1} - x^*\|^2 + \frac{\alpha_n\rho\sigma}{2}\|x_n - x^*\|^2 \\
&\quad + \alpha_n\langle \rho V(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle + \beta_n\|x_n - T_2^nT_1^ny_n\|\|x_{n+1} - x^*\| \\
&\quad + \frac{1-\alpha_n\tau}{2}(\|x_n - x^*\| - \delta(1-L\delta)\|(T_{r_n}^{\Theta} - I)Ax_n\|^2 \\
&\quad - \mu_2(2\gamma_2 - \mu_2)\|B_2z_n - B_2x^*\|^2 \\
&\quad - \mu_1(2\gamma_1 - \mu_1)\|B_1T_{\mu_2}^{F_2}(z_n - \mu_2B_2z_n) - B_1T_{\mu_2}^{F_2}(x^* - \mu_2B_2x^*)\|^2),
\end{aligned} \tag{3.16}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1} - x^*\| \le{}& \frac{\alpha_n\rho\sigma}{1+\alpha_n(\tau-\rho\alpha)}\|x_n - x^*\|^2 \\
&+ \frac{2\alpha_n}{1+\alpha_n(\tau-\rho\sigma)}\langle \rho V(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle \\
&+ \frac{2\beta_n}{1+\alpha_n(\tau-\rho\sigma)}\|x_n - T_2^nT_1^ny_n\|\|x_{n+1} - x^*\| \\
&+ \|x_n - x^*\|^2 - \frac{1-\alpha_n\tau}{1+\alpha_n(\tau-\rho\sigma)}\{\delta(1-L\delta)\|(T_{r_n}^{\Theta} - I)Ax_n\|^2 \\
&+ \mu_2(2\gamma_2 - \mu_2)\|B_2z_n - B_2x^*\|^2 \\
&+ \mu_1(2\gamma_1 - \mu_1)\|B_1T_{\mu_2}^{F_2}(z_n - \mu_2B_2z_n) - B_1T_{\mu_2}^{F_2}(x^* - \mu_2B_2x^*)\|^2\}.
\end{aligned}
$$

Then from the above inequality, we get

$$
\begin{aligned}
&\frac{1-\alpha_n\tau}{1+\alpha_n(\tau-\rho\sigma)}\{\delta(1-L\delta)\|(T_{r_n}^{\Theta}-I)Ax_n\|^2+\mu_2(2\gamma_2-\mu_2)\|B_2z_n-B_2x^*\|^2\\
&\quad+\mu_1(2\gamma_1-\mu_1)\|B_1T_{\mu_2}^{F_2}(z_n-\mu_2B_2z_n)-B_1T_{\mu_2}^{F_2}(x^*-\mu_2B_2x^*)\|^2\}\\
&\le\frac{\alpha_n\rho\sigma}{1+\alpha_n(\tau-\rho\sigma)}\|x_n-x^*\|^2+\frac{2\alpha_n}{1+\alpha_n(\tau-\rho\sigma)}\langle\rho V(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle\\
&\quad+\frac{2\beta_n}{1+\alpha_n(\tau-\rho\sigma)}\|x_n-T_2^nT_1^ny_n\|\|x_{n+1}-x^*\|+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
&\le\frac{\alpha_n\rho\sigma}{1+\alpha_n(\tau-\rho\sigma)}\|x_n-x^*\|^2+\frac{2\alpha_n}{1+\alpha_n(\tau-\rho\sigma)}\langle\rho V(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle\\
&\quad+\frac{2\beta_n}{1+\alpha_n(\tau-\rho\sigma)}\|x_n-T_2^nT_1^ny_n\|\|x_{n+1}-x^*\|\\
&\quad+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|.
\end{aligned}
$$

From $\delta(1-L\delta)>0$, $2\gamma_i-\mu_i>0$ for $i=1,2$, (3.14) and (3.15), we obtain

$$
\lim_{n\to\infty}\|(T_{r_n}^{\Theta}-I)Ax_n\|=0,\quad \lim_{n\to\infty}\|B_2z_n-B_2x^*\|=0 \tag{3.17}
$$

and

$$
\lim_{n\to\infty}\|B_1T_{\mu_2}^{F_2}(z_n-\mu_2B_2z_n)-B_1T_{\mu_2}^{F_2}(x^*-\mu_2B_2x^*)\|=0. \tag{3.18}
$$

Since P_C is firmly nonexpansive, we have

$$
\begin{aligned}
&\|z_n-x^*\|^2\\
&=\|P_C(x_n+\delta A^*(T_{r_n}^{\Theta}-I)Ax_n)-P_C(x^*)\|^2\\
&\le\langle z_n-x^*,x_n+\delta A^*(T_{r_n}^{\Theta}-I)Ax_n-x^*\rangle\\
&=\frac{1}{2}\{\|z_n-x^*\|^2+\|x_n+\delta A^*(T_{r_n}^{\Theta}-I)Ax_n-x^*\|^2\\
&\quad-\|z_n-x^*-(x_n+\delta A^*(T_{r_n}^{\Theta}-I)Ax_n-x^*)\|^2\}\\
&\le\frac{1}{2}\{\|z_n-x^*\|^2+\|x_n-x^*\|^2-\|z_n-x_n-\delta A^*(T_{r_n}^{\Theta}-I)Ax_n\|^2\}\\
&\le\frac{1}{2}\{\|z_n-x^*\|^2+\|x_n-x^*\|^2-(\|z_n-x_n\|^2+\delta^2\|A^*(T_{r_n}^{\Theta}-I)Ax_n\|^2\\
&\quad-2\delta\langle z_n-x_n,A^*(T_{r_n}^{\Theta}-I)Ax_n\rangle)\}.
\end{aligned}
$$

Hence we get

$$\|z_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|z_n - x_n\|^2 + 2\delta\|Az_n - Ax_n\|\|(T^{\Theta}_{r_n} - I)Ax_n\|. \tag{3.19}$$

Utilizing the firm nonexpansivity of $T^{F_2}_{\mu_2}$, we have

$$\begin{aligned}
&\|T^{F_2}_{\mu_2}(z_n - \mu_2 B_2 z_n) - y^*\|^2 \\
&= \|T^{F_2}_{\mu_2}(z_n - \mu_2 B_2 z_n) - T^{F_2}_{\mu_2}(x^* - \mu_2 B_2 x^*)\|^2 \\
&\le \langle T^{F_2}_{\mu}(z_n - \mu_2 B_2 z_n) - y^*, z_n - \mu_2 B_2 z_n - (x^* - \mu_2 B_2 x^*)\rangle \\
&= \frac{1}{2}\{\|T^{F_2}_{\mu_2}(z_n - \mu_2 B_2 z_n) - y^*\|^2 + \|z_n - x^* - \mu_2(B_2 z_n - B_2 x^*)\|^2 \\
&\quad - \|z_n - x^* - \mu_2(B_2 z_n - B_2 x^*) - (T^{F_2}_{\mu_2}(z_n - \mu_2 B_2 z_n) - y^*)\|^2\} \\
&\le \frac{1}{2}\{\|T^{F_2}_{\mu_2}(z_n - \mu_2 B_2 z_n) - y^*\|^2 + \|z_n - x^*\|^2 - \mu_2(2\gamma_2 - \mu_2)\|B_2 z_n - B_2 x^*\|^2 \\
&\quad - \|z_n - T^{F_2}_{\mu_2}(z_n - \mu_2 B_2 z_n) - (x^* - y^*) - \mu_2(B_2 z_n - B_2 x^*)\|^2\} \\
&\le \frac{1}{2}\{\|T^{F_2}_{\mu_2}(z_n - \mu_2 B_2 z_n) - y^*\|^2 + \|z_n - x^*\|^2 \\
&\quad - \|z_n - T^{F_2}_{\mu_2}(z_n - \mu_2 B_2 z_n) - (x^* - y^*)\|^2 \\
&\quad + 2\mu_2\langle z_n - T^{F_2}_{\mu_2}(z_n - \mu_2 B_2 z_n) - (x^* - y^*), B_2 z_n - B_2 x^*\rangle \\
&\quad - \mu_2^2\|B_2 z_n - B_2 x^*\|^2\} \\
&\le \frac{1}{2}\{\|T^{F_2}_{\mu_2}(z_n - \mu_2 B_2 z_n) - y^*\|^2 + \|z_n - x^*\|^2 \\
&\quad - \|z_n - T^{F_2}_{\mu_2}(z_n - \mu_2 B_2 z_n) - (x^* - y^*)\|^2 \\
&\quad + 2\mu_2\|z_n - T^{F_2}_{\mu_2}(z_n - \mu_2 B_2 z_n) - (x^* - y^*)\|\|B_2 z_n - B_2 x^*\|\}.
\end{aligned}$$

It follows from (3.19) that

$$\begin{aligned}
&\|T^{F_2}_{\mu_2}(z_n - \mu_2 B_2 z_n) - y^*\|^2 \\
&\le \|z_n - x^*\|^2 - \|z_n - T^{F_2}_{\mu_2}(z_n - \mu_2 B_2 z_n) - (x^* - y^*)\|^2 \\
&\quad + 2\mu_2\|z_n - T^{F_2}_{\mu_2}(z_n - \mu_2 B_2 z_n) - (x^* - y^*)\|\|B_2 z_n - B_2 x^*\| \\
&\le \|x_n - x^*\|^2 - \|z_n - x_n\|^2 + 2\delta\|Az_n - Ax_n\|\|(T^{\Theta}_{r_n} - I)Ax_n\| \\
&\quad - \|z_n - T^{F_2}_{\mu_2}(z_n - \mu_2 B_2 z_n) - (x^* - y^*)\|^2 \\
&\quad + 2\mu_2\|z_n - T^{F_2}_{\mu_2}(z_n - \mu_2 B_2 z_n) - (x^* - y^*)\|\|B_2 z_n - B_2 x^*\|.
\end{aligned} \tag{3.20}$$

And we have

$$\begin{aligned}
&\|y_n - x^*\|^2 \\
&= \|T_{\mu_1}^{F_1}(T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n)) - T_{\mu_1}^{F_1}(y^* - \mu_1 B_1 y^*)\|^2 \\
&\le \langle y_n - x^*, T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - (y^* - \mu_1 B_1 y^*)\rangle \\
&= \frac{1}{2}\{\|y_n - x^*\|^2 + \|T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - y^* - \mu_1(B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - B_1 y^*)\|^2 \\
&\quad - \|T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - y^* - \mu_1(B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - B_1 y^*) - (y_n - x^*)\|^2\} \\
&= \frac{1}{2}\{\|y_n - x^*\|^2 + \|T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - y^*\|^2 \\
&\quad - 2\mu_1\langle T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - y^*, B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - B_1 y^*\rangle \\
&\quad + \mu_1^2\|B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - B_1 y^*\|^2 \\
&\quad - \|T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - y_n - \mu_1(B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - B_1 y^*) + x^* - y^*\|^2\} \\
&\le \frac{1}{2}\{\|y_n - x^*\|^2 + \|T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - y^*\|^2 \\
&\quad - \mu_1(2\gamma_1 - \mu_1)\|B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - B_1 y^*\|^2 \\
&\quad - \|T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - y_n - \mu_1(B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - B_1 y^*) + x^* - y^*\|^2\} \\
&\le \frac{1}{2}\{\|y_n - x^*\|^2 + \|T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - y^*\|^2 \\
&\quad - \|T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - y_n + x^* - y^*\|^2 \\
&\quad + 2\mu_1\langle T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - y_n + x^* - y^*, B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - B_1 y^*\rangle\}.
\end{aligned}$$

So, we obtain from (3.20) that

$$\begin{aligned}
&\|y_n - x^*\|^2 \\
&\le \|T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - y^*\|^2 - \|T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - y_n + x^* - y^*\|^2 \\
&\quad + 2\mu_1\|T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - y_n + x^* - y^*\|\|B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - B_1 y^*\| \\
&\le \|x_n - x^*\|^2 - \|z_n - x_n\|^2 + 2\delta\|Az_n - Ax_n\|\|(T_{r_n}^{\Theta} - I)Ax_n\| \\
&\quad - \|z_n - T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - (x^* - y^*)\|^2 \\
&\quad + 2\mu_2\|z_n - T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - (x^* - y^*)\|\|B_2 z_n - B_2 x^*\| \\
&\quad - \|T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - y_n + x^* - y^*\|^2 \\
&\quad + 2\mu_1\|T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - y_n + x^* - y^*\|\|B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - B_1 y^*\|.
\end{aligned} \tag{3.21}$$

From (3.16) and (3.21), we have

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\le \frac{1-\alpha_n(\tau-\rho\sigma)}{2}\|x_{n+1}-x^*\|^2+\frac{\alpha_n\rho\sigma}{2}\|x_n-x^*\|^2 \\
&\quad+\alpha_n\langle\rho V(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle+\beta_n\|x_n-T_2^nT_1^ny_n\|\|x_{n+1}-x^*\| \\
&\quad+\frac{1-\alpha_n\tau}{2}\|y_n-x^*\|^2 \\
&\le \frac{1-\alpha_n(\tau-\rho\sigma)}{2}\|x_{n+1}-x^*\|^2+\frac{\alpha_n\rho\sigma}{2}\|x_n-x^*\|^2 \\
&\quad+\alpha_n\langle\rho V(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle+\beta_n\|x_n-T_2^nT_1^ny_n\|\|x_{n+1}-x^*\| \\
&\quad+\frac{1-\alpha_n\tau}{2}\{\|x_n-x^*\|^2-\|z_n-x_n\|^2+2\delta\|Az_n-Ax_n\|\|(T_{r_n}^{\Theta}-I)Ax_n\| \\
&\quad-\|z_n-T_{\mu_2}^{F_2}(z_n-\mu_2B_2z_n)-(x^*-y^*)\|^2 \\
&\quad+2\mu_2\|z_n-T_{\mu_2}^{F_2}(z_n-\mu_2B_2z_n)-(x^*-y^*)\|\|B_2z_n-B_2x^*\| \\
&\quad-\|T_{\mu_2}^{F_2}(z_n-\mu_2B_2z_n)-y_n+x^*-y^*\|^2 \\
&\quad+2\mu_1\|T_{\mu_1}^{F_2}(z_n-\mu_2B_2z_n)-y_n+x^*-y^*\|\|B_1T_{\mu_2}^{F_2}(z_n-\mu_2B_2z_n)-B_1y^*\|\},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\le \frac{\alpha_n\rho\sigma}{1+\alpha_n(\tau-\rho\sigma)}\|x_n-x^*\|^2+\frac{2\alpha_n}{1+\alpha_n(\tau-\rho\sigma)}\langle\rho V(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle \\
&\quad+\frac{2\beta_n}{1+\alpha_n(\tau-\rho\sigma)}\|x_n-T_2^nT_1^ny_n\|\|x_{n+1}-x^*\| \\
&\quad+\frac{1-\alpha_n\tau}{1+\alpha_n(\tau-\rho\sigma)}\{\|x_n-x^*\|^2-\|z_n-x_n\|^2+2\delta\|Az_n \\
&\quad-Ax_n\|\|(T_{r_n}^{\Theta}-I)Ax_n\| \\
&\quad-\|z_n-T_{\mu_2}^{F_2}(z_n-\mu_2B_2z_n)-(x^*-y^*)\|^2 \\
&\quad+2\mu_2\|z_n-T_{\mu_2}^{F_2}(z_n-\mu_2B_2z_n)-(x^*-y^*)\|\|B_2z_n-B_2x^*\|\} \\
&\quad-\|T_{\mu_2}^{F_2}(z_n-\mu_2B_2z_n)-y_n+x^*-y^*\|^2 \\
&\quad+2\mu_1\|T_{\mu_2}^{F_2}(z_n-\mu_2B_2z_n)-y_n+x^*-y^*\|\|B_1T_{\mu_2}^{F_2}(z_n-\mu_2B_2z_n)-B_1y^*\|\}
\end{aligned}
$$

$$\leq \frac{\alpha_n \rho\sigma}{1+\alpha_n(\tau-\rho\sigma)}\|x_n - x^*\|^2 + \frac{2\alpha_n}{1+\alpha_n(\tau-\rho\sigma)}\langle \rho V(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle$$
$$+ \frac{2\beta_n}{1+\alpha_n(\tau-\rho\sigma)}\|x_n - T_2^n T_1^n y_n\|\|x_{n+1} - x^*\| + \|x_n - x^*\|^2$$
$$+ 2\delta\|Az_n - Ax_n\|\|(T_{r_n}^{\Theta} - I)Ax_n\|$$
$$+ 2\mu_2\|z_n - T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - (x^* - y^*)\|\|B_2 z_n - B_2 x^*\|$$
$$+ 2\mu_1\|T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - y_n + x^* - y^*\|\|B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - B_1 y^*\|$$
$$- \frac{1-\alpha_n\tau}{1+\alpha_n(\tau-\rho\sigma)}\{\|z_n - x_n\|^2 + \|z_n - T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - (x^* - y^*)\|^2$$
$$+ \|T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - y_n + x^* - y^*\|^2\}.$$

Therefore we obtain

$$\frac{1-\alpha_n\tau}{1+\alpha_n(\tau-\rho\sigma)}\{\|z_n - x_n\|^2 + \|z_n - T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - (x^* - y^*)\|^2$$
$$+ \|T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - y_n + x^* - y^*\|^2\}$$
$$\leq \frac{\alpha_n \rho\sigma}{1+\alpha_n(\tau-\rho\sigma)}\|x_n - x^*\|^2 + \frac{2\alpha_n}{1+\alpha_n(\tau-\rho\sigma)}\langle \rho V(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle$$
$$+ \frac{2\beta_n}{1+\alpha_n(\tau-\rho\sigma)}\|x_n - T_2^n T_1^n y_n\|\|x_{n+1} - x^*\|$$
$$+ (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\| + 2\delta\|Az_n - Ax_n\|\|(T_{r_n}^{\Theta} - I)Ax_n\|$$
$$+ 2\mu_2\|z_n - T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - (x^* - y^*)\|\|B_2 z_n - B_2 x^*\|$$
$$+ 2\mu_1\|T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - y_n + x^* - y^*\|\|B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - B_1 y^*\|.$$

Thus, from (3.14), (3.15), (3.17) and (3.18), we conclude that

$$\lim_{n\to\infty}\|z_n - T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - (x^* - y^*)\| = 0,$$

$$\lim_{n\to\infty}\|T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - y_n + x^* - y^*\| = 0$$

and

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.22}$$

Since

$$\|z_n - y_n\| \leq \|z_n - T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - (x^* - y^*)\|$$
$$+ \|T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - y_n + x^* - y^*\|,$$

we get

$$\lim_{n\to\infty} \|z_n - y_n\| = 0. \tag{3.23}$$

It follows from (3.22) and (3.23) that

$$\begin{aligned}\lim_{n\to\infty} \|x_n - y_n\| &\le \lim_{n\to\infty} \|x_n - z_n\| + \lim_{n\to\infty} \|z_n - y_n\| \\ &= 0. \end{aligned} \tag{3.24}$$

Step 4. $\lim_{n\to\infty} \|x_n - T_2^n T_1^n x_n\| = 0$.

Indeed, from (3.1), we get

$$\begin{aligned}\|x_n - T_2^n T_1^n x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_2^n T_1^n x_n\| \\ &\le \|x_n - x_{n+1}\| + \|\alpha_n(\rho V(x_n) - \mu F(T_2^n T_1^n y_n)) \\ &\quad + \beta_n(x_n - T_2^n T_1^n y_n) + T_2^n T_1^n y_n - T_2^n T_1^n x_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|\rho V(x_n) - \mu F(T_2^n T_1^n y_n)\| \\ &\quad + \beta_n\|x_n - T_2^n T_1^n y_n\| + \|y_n - x_n\|.\end{aligned}$$

So, from (3.14), (3.15) , (3.24) and condition (i), we have

$$\lim_{n\to\infty} \|x_n - T_2^n T_1^n x_n\| = 0. \tag{3.25}$$

Step 5. $\limsup_{n\to\infty} \langle(\mu F - \gamma V)x^*, x^* - x_n\rangle \le 0$, where $x^* = P_\Gamma(I - \mu F + \gamma V)x^*$.

Indeed, since $\{\sigma_k^i\}$ is bounded for $i = 1, 2$, we can assume that $\sigma_{k_j}^i \to \sigma_\infty^i$ as $j \to \infty$, where $0 < \zeta_1 \le \sigma_\infty^i \le \zeta_2 < 1$ for $i = 1, 2$. Define $T_i^\infty = (1 - \sigma_\infty^i)I + \sigma_\infty^i T_i$ for $i = 1, 2$. Then we have $\mathrm{Fix}(T_i^\infty) = \mathrm{Fix}(T_i)$ for $i = 1, 2$. Note that

$$\begin{aligned}\|T_i^{k_j} x - T_i^\infty x\| &= \|(1 - \sigma_{k_j}^i)x + \sigma_{k_j}^i T_i x - (1 - \sigma_\infty^i)x - \sigma_\infty^i T_i x\| \\ &\le |\sigma_{k_j}^i - \sigma_\infty^i|(\|x\| + \|T_i x\|).\end{aligned}$$

Hence we deduce that

$$\lim_{j\to\infty} \sup_{x\in D} \|T_i^{k_j} x - T_i^\infty x\| = 0, \tag{3.26}$$

where D is an arbitrary bounded subset of H_1. Since $\mathrm{Fix}(T_1^\infty) \cap \mathrm{Fix}(T_2^\infty) = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) \ne \phi$ and T_i^∞ is σ_∞^i-averaged for $i = 1, 2$, by Lemma 2.6, we know that $\mathrm{Fix}(T_2^\infty T_1^\infty) = \mathrm{Fix}(T_2^\infty) \cap \mathrm{Fix}(T_1^\infty)$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup w$ as $j \to \infty$. Note that

$$\begin{aligned}\|x_{n_j} - T_2^\infty T_1^\infty x_{n_j}\| &\le \|x_{n_j} - T_2^{n_j} T_1^{n_j} x_{n_j}\| + \|T_2^{n_j} T_1^{n_j} x_{n_j} - T_2^\infty T_1^{n_j} x_{n_j}\| \\ &\quad + \|T_2^\infty T_1^{n_j} x_{n_j} - T_2^\infty T_1^\infty x_{n_j}\| \\ &\le \|x_{n_j} - T_2^{n_j} T_1^{n_j} x_{n_j}\| + \|T_2^{n_j} T_1^{n_j} x_{n_j} - T_2^\infty T_1^{n_j} x_{n_j}\| \\ &\quad + \|T_1^{n_j} x_{n_j} - T_1^\infty x_{n_j}\| \\ &\le \|x_{n_j} - T_2^{n_j} T_1^{n_j} x_{n_j}\| + \sup_{x\in D'} \|T_2^{n_j} x - T_2^\infty x\| \\ &\quad + \sup_{x\in D''} \|T_1^{n_j} x - T_1^\infty x\|,\end{aligned}$$

where D' is a bounded subset including $\{T_1^{n_j} x_{n_j}\}$ and D'' is a bounded subset including $\{x_{n_j}\}$. According to (3.25) and (3.26), we obtain

$$\lim_{n\to\infty} \|x_{n_j} - T_2^\infty T_1^\infty x_{n_j}\| = 0.$$

First, it is clear from Lemma 2.7 that $w \in \mathrm{Fix}(T_2^\infty T_1^\infty)$.

Second, let us show that $Aw \in EP(\Theta)$. Since A is a bounded linear operator, $Ax_{n_j} \to Aw$. Now we set $\nu_{n_j} = Ax_{n_j} - T_{r_{n_j}}^\Theta Ax_{n_j}$. It follows from (3.17) that $\lim_{j\to\infty} \nu_{n_j} = 0$ and $Ax_{n_j} - \nu_{n_j} = T_{r_{n_j}}^\Theta Ax_{n_j}$. Therefore, from the definition of $T_{r_{n_j}}^\Theta$, we have

$$\Theta(Ax_{n_j} - \nu_{n_j}, y) + \frac{1}{r_{n_j}} \langle y - (Ax_{n_j} - \nu_{n_j}), (Ax_{n_j} - \nu_{n_j}) - Ax_{n_j}\rangle \ge 0, \forall y \in K.$$

Since Θ is upper semicontinuous in the first argument, taking $\limsup$ in the above inequality as $j \to \infty$, we obtain

$$\Theta(Aw, y) \ge 0, \quad \forall y \in K,$$

which implies that $Aw \in EP(\Theta)$ and $w \in \Lambda$.

Next, we show that $w \in \Omega$. From (3.22), it is easy to observe that $z_{n_j} \to w$. For any $x, y \in C$, we have

$$\begin{aligned}
\|Q(x) - Q(y)\|^2 &= \|T_{\mu_1}^{F_1}[T_{\mu_2}^{F_2}(x - \mu_2 B_2 x) - \mu_1 B_1 T_{\mu_2}^{F_2}(x - \mu_2 B_2 x)] \\
&\quad - T_{\mu_1}^{F_1}[T_{\mu_2}^{F_2}(y - \mu_2 B_2 y) - \mu_1 B_1 T_{\mu_2}^{F_2}(y - \mu_2 B_2 y)]\| \\
&\le \|(T_{\mu_2}^{F_2}(x - \mu_2 B_2 x) - T_{\mu_2}^{F_2}(y - \mu_2 B_2 y) \\
&\quad - \mu_1(B_1 T_{\mu_2}^{F_2}(x - \mu_2 B_2 x) - B_1 T_{\mu_2}^{F_2}(y - \mu_2 B_2 y))\|^2 \\
&\le \|T_{\mu_2}^{F_2}(x - \mu_2 B_2 x) - T_{\mu_2}^{F_2}(y - \mu_2 B_2 y)\|^2 \\
&\quad - \mu_1(2\gamma_1 - \mu_1)\|B_1 T_{\mu_2}^{F_2}(x - \mu_2 B_2 x) - B_1 T_{\mu_2}^{F_2}(y - \mu_2 B_2 x)\|^2 \\
&\le \|T_{\mu_2}^{F_2}(x - \mu_2 B_2 x) - T_{\mu_2}^{F_2}(y - \mu_2 B_2 y)\|^2 \\
&\le \|(x - \mu_2 B_2 x) - (y - \mu_2 B_2 y)\|^2 \\
&\le \|x - y\|^2 - \mu_2(2\gamma_2 - \mu_2)\|B_2 x - B_2 y\|^2 \\
&\le \|x - y\|^2.
\end{aligned}$$

This shows that $Q : C \to C$ is nonexpansive. Note that

$$\begin{aligned}
\|y_n - Q(y_n)\| &= \|Q(z_n) - Q(y_n)\| \\
&\le \|z_n - y_n\| \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

It follows from Lemma 2.7 that $w = Q(w)$. According to Lemma 2.4, we obtain $w \in \Omega$. Thus we have $w \in \Gamma = \cap_{i=1}^{N} \mathrm{Fix}(T_i) \cap \Omega \cap \Lambda$.

Finally, we claim that $\limsup_{n\to\infty} \langle (\mu F - \rho V)x^*, x^* - x_n \rangle \le 0$. By $x^* = P_\Gamma (I - \mu F + \rho V)x^*$ and (2.1), we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle (\mu F - \rho V)x^*, x^* - x_n \rangle &= \lim_{n\to\infty} \langle (\mu F - \rho V)x^*, x^* - x_{n_k} \rangle \\
&= \langle (\mu F - \rho V)x^*, x^* - w \rangle \\
&\le 0.
\end{aligned}$$

Step 6. $x_n \to x^*$ as $n \to \infty$.

Indeed, from (3.1), (3.5), Lemma 2.1 and 2.5, we have

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&=\|\alpha_n\rho V(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n\mu F)T_2^nT_1^ny_n-x^*\|^2\\
&=\|\alpha_n(\rho V(x_n)-\mu F(x^*))+\beta_n(x_n-x^*)+((1-\beta_n)I-\alpha_n\mu F)T_2^nT_1^ny_n\\
&\quad-((1-\beta_n)I-\alpha_n\mu F)T_2^nT_1^nx^*\|^2\\
&\le\|\beta_n(x_n-x^*)+((1-\beta_n)I-\alpha_n\mu F)T_2^nT_1^ny_n-((1-\beta_n)I-\alpha_n\mu F)T_2^nT_1^nx^*\|^2\\
&\quad+2\alpha_n\langle\rho V(x_n)-\mu F(x^*),x_{n+1}-x^*\rangle\\
&\le(\beta_n\|x_n-x^*\|+(1-\beta_n)\|(I-\frac{\alpha_n}{1-\beta_n}\mu F)T_2^nT_1^ny_n-(I-\frac{\alpha_n}{1-\beta_n}\mu F)T_2^nT_1^nx^*\|)^2\\
&\quad+2\alpha_n\rho\langle V(x_n)-V(x^*),x_{n+1}-x^*\rangle+2\alpha_n\langle\rho V(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle\\
&\le(\beta_n\|x_n-x^*\|+(1-\beta_n)(1-\frac{\alpha_n}{1-\beta_n}\tau)\|y_n-x^*\|)^2\\
&\quad+2\alpha_n\rho\sigma\|x_n-x^*\|\|x_{n+1}-x^*\|+2\alpha_n\langle\rho V(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle\\
&\le(\beta_n\|x_n-x^*\|+(1-\beta_n-\alpha_n\tau)\|x_n-x^*\|)^2\\
&\quad+\alpha_n\rho\sigma(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2)+2\alpha_n\langle\rho V(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle\\
&=[(1-\alpha_n\tau)^2+\alpha_n\rho\sigma]\|x_n-x^*\|^2+\alpha_n\rho\sigma\|x_{n+1}-x^*\|^2\\
&\quad+2\alpha_n\langle\rho V(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&\le\frac{(1-\alpha_n\tau)^2+\alpha_n\rho\sigma}{1-\alpha_n\rho\sigma}\|x_n-x^*\|^2\\
&\quad+\frac{2\alpha_n}{1-\alpha_n\rho\sigma}\langle\rho V(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle\\
&=\left(1-\frac{2(\tau-\rho\sigma)\alpha_n}{1-\alpha_n\rho\sigma}\right)\|x_n-x^*\|^2+\frac{\alpha_n^2\tau^2}{1-\alpha_n\rho\sigma}\|x_n-x^*\|^2\\
&\quad+\frac{2\alpha_n}{1-\alpha_n\rho\sigma}\langle\rho V(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle\\
&\le\left(1-\frac{2(\tau-\rho\sigma)\alpha_n}{1-\alpha_n\rho\sigma}\right)\|x_n-x^*\|^2+\frac{2(\tau-\rho\sigma)\alpha_n}{1-\alpha_n\rho\sigma}\Bigg\{\frac{\alpha_n\tau^2}{2(\tau-\rho\sigma)}M_1\\
&\quad+\frac{1}{\tau-\rho\sigma}\langle\rho V(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle\Bigg\}\\
&=(1-\delta_n)\|x_n-x^*\|^2+\delta_n\sigma_n,
\end{aligned}
$$

where $M_1 = \sup\{\|x_n - x^*\|^2 : n \geq 0\}$, $\delta_n = \frac{2(\tau-\rho\sigma)\alpha_n}{1-\alpha_n\rho\sigma}$ and $\sigma_n = \frac{\alpha_n\tau^2}{2(\tau-\rho\sigma)}M_1 + \frac{1}{\tau-\rho\sigma}\langle \rho V(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle$. It is easy to see that $\delta_n \to 0$, $\sum_{n=0}^{\infty}\delta_n = \infty$ and $\limsup_{n\to\infty}\sigma_n \leq 0$. Hence, applying Lemma 2.9 to the last inequality, we immediately obtain that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

COROLLARY 3.1. *Let C, K be nonempty closed convex subsets of real Hilbert spaces H_1, H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Assume $F_1, F_2 : C \times C \to \mathbb{R}$ and $\Theta : K \times K \to \mathbb{R}$ are the bifunctions satisfying Assumption 2.1. Let $B_i : C \to H_1$ be a γ_i-inverse strongly monotone mapping for each $i = 1, 2$ such that $\Gamma = \Omega \cap \Lambda \neq \phi$. Let $F : C \to C$ be a κ-Lipschitzian continuous and η-strongly monotone mapping with $\kappa > 0$ and $\eta > 0$, and $V : C \to C$ be a σ-Lipschitzian continuous mapping with $\sigma > 0$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 < \rho\sigma < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. Given $x_1 \in C$, let $\{x_n\}$ be defined by*

$$\begin{cases} z_n = P_C(x_n + \delta A^*(T_{r_n}^{\Theta} - I)Ax_n), \\ y_n = T_{\mu_1}^{F_1}[T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 T_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n)], \\ x_{n+1} = \alpha_n \rho V(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n \mu F)y_n, \quad \forall n \geq 1, \end{cases}$$

where $\{r_n\} \subset (0, 2\zeta)$, $\zeta > 0$, $\mu_i \in (0, 2\gamma_i)$ for each $i = 1, 2$, $\delta \in (0, \frac{1}{L})$, L is the spectral radius of the operator A^*A, A^* is the adjoint of A. If the following conditions are satisfied:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = 0$*;*

(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$*;*

(iii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\zeta$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

Then the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega \cap \Lambda$, where $x^* = P_{\Omega\cap\Lambda}(I - \mu F + \rho V)x^*$ is the unique solution of the variational inequality:

$$\langle (\mu F - \rho V)x^*, x - x^*\rangle \geq 0, \quad \forall x \in \Omega \cap \Lambda.$$

Proof. Put $T_i x = x$ for all $i = 1, 2, \cdots, N$ and $x \in C$, and take the finite family of sequences $\{\sigma_n^i\}_{i=1}^N$ in (ζ_1, ζ_2) for some $\zeta_1, \zeta_2 \in (0, 1)$ such that $\lim_{n\to\infty}|\sigma_{n+1}^i - \sigma_n^i| = 0$ for all $i = 1, 2, \cdots, N$. In this case, $T_N^n T_{N-1}^n \cdots T_1^n$ is the identity mapping I of C. It is easy to see that all conditions of Theorem 3.1 are satisfied. Thus, the desired result follows from Theorem 3.1. □

To verify the theoretical assertions, we consider the following example.

EXAMPLE 3.1. Let $H_1 = H_2 = \mathbb{R}$ with the inner product defined by $\langle x, y\rangle = xy$ for all $x, y \in \mathbb{R}$ and the standard norm $|\cdot|$. Let $C = [0, +\infty)$, $K = (-\infty, 0]$, $B_1 = B_2 = 0$ and $\mu_1 = \mu_2 = 1$. Define the mappings $A : \mathbb{R} \to \mathbb{R}$, $F_1, F_2 : C \times C \to \mathbb{R}$, $\Theta : K \times K \to \mathbb{R}$, $T_i : C \to C$ for each $i = 1, 2, \cdots, N$, $F : C \to C$ and $V : C \to C$ as follows:

$$F_1(x, y) = -3x^2 + xy + 2y^2, \quad F_2(x, y) = -5x^2 + xy + 4y^2,$$

$$T_i(x) = 0 \text{ for } i = 1, 2, \cdots, N,\ F(x) = x,\ V(x) = \frac{1}{2}x,\ \forall (x, y) \in C \times C,$$

$$\Theta(x, y) = -x(x - y), \quad \forall (x, y) \in K \times K$$

and

$$Ax = -2x, \quad \forall x \in \mathbb{R}.$$

It is easy to see that $\sigma = \frac{1}{2}$, $\eta = \kappa = 1$ and hence $0 < \mu < \frac{2\eta}{\kappa^2} = 2$. Put $\mu = 1$. Then $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} = 1$. From $0 < \rho\sigma < \tau$, we have $\rho \in (0, 2)$. Without loss of generality, we put $\rho = 1$. Let $\alpha_n = \frac{1}{3n}$, $\beta = \frac{2n-1}{3n}$, $r_n = 1$ and $\sigma_n^i = \frac{1}{2}$ for each $i = 1, 2, \cdots, N$. The sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$ and $\{\sigma_n^i\}$ satisfy conditions (i)-(iv). Since $T_i(x) = 0$ for $i = 1, 2, \cdots, N$ and $Ax = -2x$ for every $x \in \mathbb{R}$, we have $\cap_{i=1}^N \mathrm{Fix}(T_i) = \{0\}$ and A is a bounded linear operator with $A^* = A$ and $\|A\| = 2$. Obviously, $EP(\Theta) = \{0\}$, $\Lambda = \{p \in C : Ap \in EP(\Theta)\} = \{0\}$ and F_1, F_2, Θ satisfy Assumption 2.1. For $r_n = \mu_1 = \mu_2 = 1$, $u_n = T_{r_n}^{\Theta}(Ax_n)$ is equivalent to

$$\Theta(u_n, v) + \langle v - u_n, u_n - Ax_n\rangle \geq 0, \quad \forall v \in K, n \in \mathbb{N}.$$

Hence, we can easily find $u_n = -x_n \in K$. It is not hard to compute $A^*(T_{r_n}^{\Theta} - I)Ax_n = A^*(u_n - Ax_n) = -2(u_n - Ax_n) = -2x_n$ for all $n \in \mathbb{N}$. Hence, for $\delta = \frac{1}{8}$, we have

$$x_n + \frac{1}{8}A^*(T_{r_n}^{\Theta} - I)Ax_n = x_n + \frac{1}{8}(-2x_n) = \frac{3}{4}x_n \in C, \quad \forall n \in \mathbb{N}.$$

So, we have

$$z_n = \frac{3}{4}x_n. \tag{3.27}$$

From $v_n = T_{\mu_2}^{F_2} z_n$ and $y_n = T_{\mu_1}^{F_1} v_n$, we have

$$\begin{aligned} 0 &\le F_2(v_n, y) + \langle y - v_n, v_n - z_n \rangle \\ &= -5v_n^2 + v_n y + 4y^2 + (y - v_n)(v_n - z_n), \quad \forall y \in C, \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} 0 &\le F_1(y_n, y) + \langle y - y_n, y_n - v_n \rangle \\ &= -3y_n^2 + y_n y + 2y^2 + (y - y_n)(y_n - v_n), \quad \forall y \in C. \end{aligned} \tag{3.29}$$

From (3.28) and (3.29), we have

$$0 \le 4y^2 + (2v_n - z_n)y - 6v_n^2 + v_n z_n, \quad \forall y \in C,$$

and

$$0 \le 2y^2 + (2y_n - v_n)y - 4y_n^2 + y_n v_n, \quad \forall y \in C.$$

Let $A_1(y) = 4y^2 + (2v_n - z_n)y - 6v_n^2 + v_n z_n$ and $A_2(y) = 2y^2 + (2y_n - v_n)y - 4y_n^2 + y_n v_n$. Then we determine the discriminants $\triangle_1$ of A_1 and $\triangle_2$ of A_2 as follows:

$$\begin{aligned} \triangle_1 &= (2v_n^2 - z_n)^2 - 16(-6v_n^2 + v_n z_n) \\ &= z_n^2 - 20v_n z_n + 100v_n^2 \\ &= (z_n - 10v_n)^2 \end{aligned}$$

and

$$\begin{aligned} \triangle_2 &= (2y_n - v_n)^2 - 8(-4y_n^2 + y_n v_n) \\ &= v_n^2 - 12y_n v_n + 36y_n^2 \\ &= (v_n - 6y_n)^2. \end{aligned}$$

By $A_1(y) \ge 0$ and $A_2(y) \ge 0$, we have $\triangle_1 = \triangle_2 = 0$. So, we get

$$v_n = \frac{1}{10} z_n \quad \text{and} \quad y_n = \frac{1}{6} v_n. \tag{3.30}$$

For every $n \ge 1$, from (3.27) and (3.30), we can rewrite (3.1) as follows:

$$\begin{cases} z_n = \frac{3}{4} x_n, \\ y_n = \frac{1}{60} z_n, \\ x_{n+1} = \frac{1}{3n}\frac{1}{2} x_n + \frac{2n-1}{3n} x_n + ((1 - \frac{2n-1}{3n}) - \frac{1}{3n})\frac{1}{2^N} y_n, \end{cases}$$

that is,

$$\begin{aligned} x_{n+1} &= \frac{4n-1}{6n}x_n + \frac{1}{3\cdot 2^N}y_n \\ &= (\frac{2}{3} + \frac{1}{240\cdot 2^N} - \frac{1}{6n})x_n. \end{aligned}$$

Observe that for all $n \geq 1$,

$$\begin{aligned} |x_{n+1} - 0| &= |(\frac{2}{3} + \frac{1}{240\cdot 2^N} - \frac{1}{6n})x_n - 0| \\ &\leq \frac{161}{240}|x_n - 0|. \end{aligned}$$

Hence we have $|x_{n+1} - 0| \leq (\frac{161}{240})^n |x_1 - 0|$ for all $n \geq 1$. This implies that $\{x_n\}$ converges strongly to $0 \in \Gamma = \cap_{i=1}^N \mathrm{Fix}(T_i) \cap \Omega \cap \Lambda$. Observe that $\langle(\mu F - \rho V)0, x - 0\rangle\rangle \geq 0$, $x \in \Gamma$, that is, 0 is the solution of the variational inequality $\langle(\mu F - \rho V)x^*, x - x^*\rangle \geq 0$, $x \in \Gamma$.

Remark 3.1. Theorem 3.1 improves and extends many recent corresponding main results of other authors (see [15,21,27,28]) in the following ways:

(a) The explicit iterative methods in [15,27,28] have been extended to the new iterative method (3.1) in Theorem 3.1. So, their iterative methods are some special cases of our iterative method (3.1) and some of their main results have been included in Theorem 3.1.

(b) The iterative approximating point in Theorem 3.1 is also the unique solution of the variational inequality (3.2). In fact, (3.2) is a hierarchical fixed point problem which closely relates to a convex minimization problem. In hierarchical fixed point problem (1.6), if $S = I - (\rho V - \mu F)$, then we can get the variational inequality (3.2). In (3.2), if $V = 0$, then we get the variational inequality $\langle Fx^*, x - x^*\rangle \geq 0$, $\forall x \in \Gamma = \cap_{i=1}^N \mathrm{Fix}(T_i) \cap \Omega \cap \Lambda$, which just is the variational inequality studied by Suzuki [21] extending the common set of solutions of a system of variational inequalities (1.1), a split equilibrium problem (1.5), a hierarchical fixed point problem (3.2) and the common fixed points set of a finite family of avereged mappings. So, the result of Theorem 3.1 in this paper have many useful applications.

Competing interests

The author declares that there is no conflict of interest regarding the publication of this article.

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