

## WHEN THE NAGATA RING $D(X)$ IS A SHARP DOMAIN

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ABSTRACT. Let  $D$  be an integral domain,  $X$  be an indeterminate over  $D$ ,  $D[X]$  be the polynomial ring over  $D$ , and  $D(X)$  be the Nagata ring of  $D$ . Let  $[d]$  be the star operation on  $D[X]$ , which is an extension of the  $d$ -operation on  $D$  as in [5, Theorem 2.3]. In this paper, we show that  $D$  is a sharp domain if and only if  $D[X]$  is a  $[d]$ -sharp domain, if and only if  $D(X)$  is a sharp domain.

### 1. Introduction

Let  $D$  be an integral domain and  $*$  be a star operation on  $D$ . (The definitions related to star operations will be reviewed in Section 2.) As in [2], we say that  $D$  is a *\*-sharp domain* if whenever  $I \supseteq AB$  with  $I, A, B$  nonzero ideals of  $D$ , there exist nonzero ideals  $H$  and  $J$  of  $D$  such that  $I^* = (HJ)^*$ ,  $H^* \supseteq A$ , and  $J^* \supseteq B$ . Following [1], we say that a  $d$ -sharp domain is a *sharp domain*, i.e.,  $D$  is a sharp domain if whenever  $I \supseteq AB$  with  $I, A, B$  nonzero ideals of  $D$ , there exist ideals  $A_0 \supseteq A$  and  $B_0 \supseteq B$  of  $D$  such that  $I = A_0B_0$ . Assume that  $D$  is a *\*-sharp domain*. It is known that  $D$  is a *t*-sharp domain (and hence  $D$  is a PvMD whose prime *t*-ideals are maximal *t*-ideal); if  $*$  =  $*_w$ , then  $D$  is a P\*MD whose maximal *\**-ideals have height-one; and  $I_v$  is *\**-invertible for all nonzero fractional ideals  $I$  of  $D$  [2, Propositions 2.2, 3.1, 2.3 and

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2.4]. Also,  $D$  is a  $v$ -sharp domain if and only if  $D$  is completely integrally closed [2, Corollary 2.6].

In [2, Theorem 3.7(a)], the authors showed that  $D$  is a  $t$ -sharp domain if and only if  $D[X]$ , the polynomial ring over  $D$ , is a  $t$ -sharp domain. They then remarked that “we do not have a “ $d$ -analogue” of [2, Theorem 3.7(a)] because a sharp domain has Krull dimension  $\leq 1$ ”. In fact, if  $D$  is a sharp domain, then  $D$  is a Prüfer domain with  $\dim(D) \leq 1$  [1, Theorem 11]. Hence,  $D[X]$  is a sharp domain if and only if  $D$  is a field. However, in this paper, we use the star operation  $[d]$  on  $D[X]$  (see Lemma 2) to prove the  $d$ -operation analogue of [2, Theorem 3.7(a)]. Precisely, we prove that if  $*$  is a star operation on  $D$  such that  $*_w = *$ , then  $D$  is a  $*$ -sharp domain if and only if  $D[X]$  is a  $[*]$ -sharp domain, if and only if  $D[X]_{N_*}$  is a sharp domain, where  $N_* = \{f \in D[X] \mid c(f)^* = D\}$ . Let  $D(X)$  be the Nagata ring of  $D$ , i.e.,  $D(X) = \{\frac{f}{g} \mid f, g \in D[X] \text{ and } c(g) = D\}$ . As a corollary, we have that  $D$  is a sharp domain if and only if  $D[X]$  is a  $[d]$ -sharp domain, if and only if  $D(X)$  is a sharp domain. Finally, we study when  $D[X]$  is a  $\star$ -sharp domain if  $\star$  is a star operation on  $D[X]$  such that  $\star_w = \star$ .

## 2. Definitions related to star operations

Let  $D$  be an integral domain with quotient field  $K$ ,  $F(D)$  be the set of nonzero fractional ideals of  $D$ , and  $f(D)$  be the set of nonzero finitely generated fractional ideals of  $D$ ; so  $f(D) \subseteq F(D)$ , and equality holds if and only if  $D$  is Noetherian. We say that a mapping  $*$  :  $F(D) \rightarrow F(D)$ ,  $I \mapsto I^*$ , is a *star operation* on  $D$  if the following three conditions are satisfied for all  $0 \neq a \in K$  and  $I, J \in F(D)$ : (i)  $(aD)^* = aD$  and  $(aI)^* = aI^*$ , (ii)  $I \subseteq I^*$  and if  $I \subseteq J$ , then  $I^* \subseteq J^*$ , and (iii)  $(I^*)^* = I^*$ . Given a star operation  $*$  on  $D$ , two new star operations  $*_f$  and  $*_w$  on  $D$  can be constructed as follows for all  $I \in F(D)$ ;  $I^{*f} = \bigcup \{J^* \mid J \subseteq I \text{ and } J \in f(D)\}$  and  $I^{*w} = \{x \in K \mid xJ \subseteq I \text{ for some } J \in f(D) \text{ with } J^* = D\}$ . Clearly,  $(*_f)_f = *_f$  and  $(*_f)_w = (*_w)_f = *_w$ . An  $I \in F(D)$  is called a  *$*$ -ideal* if  $I^* = I$ , and a  $*$ -ideal is a *maximal  $*$ -ideal* if it is maximal among proper integral  $*$ -ideals. Let  $*\text{-Max}(D)$  be the set of maximal  $*$ -ideals of  $D$ . It is known that  $*_f\text{-Max}(D) \neq \emptyset$  when  $D$  is not a field;  $*_f\text{-Max}(D) = *_w\text{-Max}(D)$  [3, Theorem 2.16]; and  $I^{*w} = \bigcap_{P \in *_f\text{-Max}(D)} ID_P$  for all  $I \in F(D)$  [3, Corollary 2.10]. For  $I \in F(D)$ , let  $I^{-1} = \{x \in K \mid xI \subseteq D\}$ ; then  $I^{-1} \in F(D)$ . We say

that  $I \in F(D)$  is  $*$ -invertible if  $(II^{-1})^* = D$ , and  $D$  is a *Prüfer  $*$ -multiplication domain* ( $P^*MD$ ) if every nonzero finitely generated ideal of  $D$  is  $*_f$ -invertible. Examples of star operations include the  $d$ -,  $v$ -,  $t$ -, and  $w$ -operations. The  $d$ -operation is the identity function of  $F(D)$ , i.e.,  $I^d = I$  for all  $I \in F(D)$ , the  $v$ -operation is defined by  $I^v = (I^{-1})^{-1}$ , the  $t$ -operation is defined by  $t = v_f$ , and the  $w$ -operation is given by  $w = v_w$ . For more on basic properties of star operations, see [8, Sections 32 and 34].

Let  $X$  be an indeterminate over  $D$ ,  $D[X]$  be the polynomial ring over  $D$ , and  $c(f)$  be the ideal of  $D$  generated by the coefficients of  $f \in D[X]$ . The next lemma is nice characterizations of  $P^*MD$ s, which appear in [7, Theorem 3.1 and Proposition 3.15] in a more general setting of semistar operations.

LEMMA 1. *Let  $*$  be a star operation on  $D$  with  $*_f = *$ . Then the following statements are equivalent.*

1.  $D$  is a  $P^*MD$ .
2.  $D$  is a  $PvMD$  and  $* = t$ .
3.  $D$  is a  $PvMD$  and  $*_w = t$ .
4.  $D[X]_{N_*}$  is a Prüfer domain, where  $N_* = \{f \in D[X] \mid c(f)^* = D\}$ .

In this case,  $*_f = *_w = t = w$ .

Let  $*$  be a star operation on  $D$ . Then there is a star operation  $[*]$  on  $D[X]$ , which is an extension of  $*_w$  to  $D[X]$  in the sense that  $(I[X])^{[*]} \cap K = I^{*w}$  for each  $I \in F(D)$ . We recall this result for easy reference of the reader.

LEMMA 2. [5, Theorem 2.3] *Let  $X, Y$  be two indeterminates over  $D$ ,  $*$  be a star operation on  $D$ , and let*

$$\Delta = \{Q \in \text{Spec}(D[X]) \mid Q \cap D = (0) \text{ with } htQ = 1 \\ \text{or } Q = (Q \cap D)[X] \text{ and } (Q \cap D)^{*f} \subsetneq D\}.$$

Set  $\mathcal{S} = D[X][Y] \setminus (\cup\{Q[Y] \mid Q \in \Delta\})$  and define

$$A^{[*]} = A[Y]_{\mathcal{S}} \cap K(X) \quad \text{for all } A \in F(D[X]).$$

1. The mapping  $[*] : F(D[X]) \rightarrow F(D[X])$ , given by  $A \mapsto A^{[*]}$ , is a star operation on  $D[X]$  such that  $[*] = [*]_f = [*]_w$ .
2.  $[*] = [*]_f = [*]_w$ .
3.  $(ID[X])^{[*]} \cap K = I^{*w}$  for all  $I \in F(D)$ .
4.  $(ID[X])^{[*]} = I^{*w}D[X]$  for all  $I \in F(D)$ .

5.  $[\ast]\text{-Max}(D[X]) = \{Q \mid Q \in \text{Spec}(D[X]) \text{ such that } Q \cap D = (0), \text{ ht}Q = 1, \text{ and } (\sum_{g \in Q} c(g))^{\ast f} = D\} \cup \{P[X] \mid P \in \ast_f\text{-Max}(D)\}$ .
6.  $[v]$  is the  $w$ -operation on  $D[X]$ .

Let  $\ast$  be a star operation on  $D$ . It is known that  $D$  is a  $P\ast\text{MD}$  if and only if  $D[X]$  is a  $P[\ast]\text{MD}$  [5, Corollary 2.5]; hence  $D$  is a  $Pv\text{MD}$  if and only if  $D[X]$  is a  $Pv\text{MD}$ . Also, since a  $Pd\text{MD}$  is just the Prüfer domain,  $D$  is a Prüfer domain if and only if  $D[X]$  is a  $P[d]\text{MD}$ .

### 3. Main Results

Let  $D$  be an integral domain with quotient field  $K$ , and we assume that  $D \neq K$  in order to avoid the trivial case. Let  $X$  be an indeterminate over  $D$ ,  $D[X]$  be the polynomial ring over  $D$ , and  $N_v = \{f \in D[X] \mid c(f)^v = D\}$ .

LEMMA 3. Let  $N_v = \{f \in D[X] \mid c(f)^v = D\}$ ,  $I \in F(D)$ , and  $A \in F(D[X])$ .

1.  $ID[X]_{N_v} \cap K = I^w$ , and hence  $I^w D[X]_{N_v} = ID[X]_{N_v}$ .
2.  $A_{N_v} = (A^w)_{N_v}$ .

*Proof.* (1) [4, Lemma 2.1].

(2) It suffices to show that  $A^w \subseteq A_{N_v}$ . For this, let  $0 \neq f \in A^w$ . Then there is a nonzero finitely generated ideal  $J$  of  $D[X]$  such that  $J^v = D[X]$  and  $fJ \subseteq A$ . Since  $J^v = D[X]$ ,  $J \not\subseteq P[X]$  for all  $P \in t\text{-Max}(D)$ , and hence  $(\sum_{h \in J} c(h))^t = D$ . Hence, there is a  $0 \neq g \in J$  with  $c(g)^v = D$ ; so  $g \in N_v$  and  $fg \in A$ . Thus,  $f \in A_{N_v}$ .  $\square$

It is known that  $D$  is a  $t$ -sharp domain if and only if  $D[X]$  is a  $t$ -sharp domain, if and only if  $D[X]_{N_v}$  is a sharp domain [2, Theorem 3.7]. We next generalize this result to an arbitrary star operation  $\ast$  on  $D$  with  $\ast_w = \ast$ .

THEOREM 4. Let  $\ast$  be a star operation on  $D$  such that  $\ast_w = \ast$ . Then the following statements are equivalent.

1.  $D$  is a  $\ast$ -sharp domain.
2.  $D[X]$  is a  $[\ast]$ -sharp domain.
3.  $D[X]_{N_\ast}$  is a sharp domain, where  $N_\ast = \{f \in D[X] \mid c(f)^\ast = D\}$ .

In this case,  $\ast = t = w$  on  $D$  and  $[\ast] = t = w$  on  $D[X]$ .

*Proof.* (1)  $\Rightarrow$  (2) If  $D$  is a  $*$ -sharp domain, then  $D$  is a P\*MD [2, Proposition 2.3], and hence  $D$  is a PvMD with  $* = t = w$  on  $D$  by Lemma 1. Hence,  $D$  is a  $t$ -sharp domain, and thus  $D[X]$  is a  $t$ -sharp domain. However, note that  $D[X]$  is a PvMD and  $[t] = w = t$  on  $D[X]$  by Lemmas 1 and 2; so  $[*] = t$ . Thus,  $D[X]$  is a  $[*]$ -sharp domain.

(2)  $\Rightarrow$  (3) Assume that  $D[X]$  is a  $[*]$ -sharp domain. Then  $D[X]$  is a PvMD and  $[*] = t = w$  on  $D[X]$ ; hence  $I^* = (ID[X])^{[*]} \cap K = (ID[X])^t \cap K = I^t D[X] \cap K = I^t$  for all  $I \in F(D)$  by Lemma 2 and [9, Proposition 4.3]. Thus,  $D$  is a PvMD and  $* = t = w$  on  $D$ .

Let  $I_{N_*} \supseteq (A_{N_*})(B_{N_*})$  with  $I, A, B$  nonzero ideals of  $D[X]$ , and let  $C = ID[X]_{N_*} \cap D[X]$ . Then  $C \supseteq AB$ , and hence by assumption, there exist nonzero ideals  $A_0$  and  $B_0$  of  $D[X]$  such that  $C^w = (A_0 B_0)^w$ ,  $(A_0)^w \supseteq A$ , and  $(B_0)^w \supseteq B$ . Thus, by Lemma 3,  $I_{N_*} = C_{N_*} = (C^w)_{N_*} = ((A_0 B_0)^w)_{N_*} = (A_0 B_0)_{N_*} = ((A_0)_{N_*})((B_0)_{N_*})$ ,  $(A_0)_{N_*} = ((A_0)^w)_{N_*} \supseteq A_{N_*}$ , and  $(B_0)_{N_*} = ((B_0)^w)_{N_*} \supseteq B_{N_*}$ . Thus,  $D[X]_{N_*}$  is a sharp domain.

(3)  $\Rightarrow$  (1) Suppose that  $D[X]_{N_*}$  is a sharp domain. Then  $D[X]_{N_*}$  is a Prüfer domain [2, Theorem 11], and hence  $D$  is a P\*MD by Lemma 1 and every ideal of  $D[X]_{N_*}$  is extended from  $D$  [10, Theorem 3.1]. Let  $I \supseteq AB$  with  $I, A, B$  nonzero ideals of  $D$ . Then  $ID[X]_{N_*} \supseteq (AD[X]_{N_*})(BD[X]_{N_*})$ , and hence by assumption, there exist nonzero ideals  $H$  and  $J$  of  $D$  such that

$$ID[X]_{N_*} = (HD[X]_{N_*})(JD[X]_{N_*}) = (HJ)D[X]_{N_*},$$

$HD[X]_{N_*} \supseteq AD[X]_{N_*}$ , and  $JD[X]_{N_*} \supseteq BD[X]_{N_*}$ . Thus, by Lemma 3,  $I^* = ID[X]_{N_*} \cap K = (HJ)D[X]_{N_*} \cap K = (HJ)^*$ ,  $H^* = HD[X]_{N_*} \cap K \supseteq AD[X]_{N_*} \cap K \supseteq A$ , and  $J^* = JD[X]_{N_*} \cap K \supseteq BD[X]_{N_*} \cap K \supseteq B$ . Thus,  $D$  is a  $*$ -sharp domain.  $\square$

The next result is the  $d$ -operation analogue of [2, Theorem 3.7(a)] that  $D$  is a  $t$ -sharp domain if and only if  $D[X]$  is a  $t$ -sharp domain.

**COROLLARY 5.** *The following statements are equivalent for an integral domain  $D$ .*

1.  $D$  is a sharp domain.
2.  $D[X]$  is a  $[d]$ -sharp domain.
3.  $D(X)$  is a sharp domain.

*Proof.* This follows directly from Theorem 4 because  $D(X) = D[X]_{N_d}$ .  $\square$

Let  $\star$  be a star operation on  $D[X]$ , and let  $I^\star = (ID[X])^\star \cap K$  for all  $I \in F(D)$ . Then it is easy to see that  $\star$  is a star operation on  $D$  (cf. [6, Lemma 5]). A nonzero prime ideal  $Q$  of  $D[X]$  is said to be an *upper to zero* in  $D[X]$  if  $Q \cap D = (0)$ ; so each upper to zero in  $D[X]$  has height-one. Let  $\star = \star_f$ , and note that if every upper to zero in  $D[X]$  is a maximal  $\star$ -ideal, then  $\star\text{-Max}(D[X]) = t\text{-Max}(D[X])$  [11, Theorem 2.9], and hence  $\star_w = w$  on  $D[X]$ .

**COROLLARY 6.** *Let  $\star$  be a star operation on  $D[X]$  with  $\star_w = \star$ , and let  $\ast$  be the star operation on  $D$  defined by  $I^\ast = (ID[X])^\ast \cap K$  for all  $I \in F(D)$ . Suppose that every upper to zero in  $D[X]$  is a maximal  $\star$ -ideal. Then  $D[X]$  is a  $\star$ -sharp domain if and only if  $D$  is a  $\ast$ -sharp domain.*

*Proof.* ( $\Rightarrow$ ) Assume that  $D[X]$  is a  $\star$ -sharp domain. Then  $D[X]$  is a PvMD and  $\star = t = w$  on  $D[X]$  as in the proof of Theorem 4, and hence  $D$  is a PvMD and  $\ast = t = w$  on  $D$  [6, Lemma 5]. Note that  $[t] = w$  on  $D[X]$ ; so  $[\ast] = \star$ . Thus,  $D$  is a  $\ast$ -sharp domain by Theorem 4.

( $\Leftarrow$ ) Suppose that  $D$  is a  $\ast$ -sharp domain, and note that  $\star_w = \star$  [6, Lemma 5] because  $\star_w = \star$ . Hence,  $D[X]$  is a  $[\ast]$ -sharp domain and  $[\ast] = t = w$  on  $D[X]$  by Theorem 4. Since every upper to zero in  $D[X]$  is a maximal  $\star$ -ideal,  $\star = w = [\ast]$  on  $D[X]$ . Thus,  $D[X]$  is a  $\star$ -sharp domain.  $\square$

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