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# CONDITIONAL INTEGRAL TRANSFORMS AND CONVOLUTIONS FOR A GENERAL VECTOR-VALUED CONDITIONING FUNCTIONS

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Abstract. We study the conditional integral transforms and conditional convolutions of functionals defined on $K[0,T]$. We consider a general vector-valued conditioning functions $X_k(x) = (\gamma_1(x), \dots, \gamma_k(x))$ where $\gamma_j(x)$ are Gaussian random variables on the Wiener space which need not depend upon the values of $x$ at only finitely many points in $(0,T]$. We then obtain several relationships and formulas for the conditioning functions that exist among conditional integral transform, conditional convolution and first variation of functionals in $E_\sigma$.

## 1. Preliminaries

Let $C_0[0,T]$ denote one-parameter Wiener space; that is, the space of all $\mathbb{R}$-valued continuous functions $x(t)$ on $[0,T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let $m$ denote Wiener measure. $(C_0[0,T], \mathcal{M}, m)$ is a complete measure space and we denote the Wiener integral of a Wiener integrable functional $F$ by

$$E_x[F(x)] = \int_{C_0[0,T]} F(x)\, m(dx). \tag{1.1}$$

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Let $K = K[0,T]$ be the space of all $\mathbb{C}$-valued continuous functions defined on $[0,T]$ which vanish at $t = 0$ and let $\alpha$ and $\beta$ be nonzero complex numbers. In [2], Cameron and Martin defined a Fourier-Wiener transform of functionals defined on $K$. In [3], Cameron and Storvick defined a Fourier-Feynman transform of functionals defined on $C_0[0,T]$. Furthermore, Lee defined an integral transform $\mathcal{F}_{\alpha,\beta}$ of analytic functionals on an abstract Wiener space [9]. For certain values of the parameters $\alpha$ and $\beta$ and for certain classes of functionals Fourier-Wiener transform[2] and Fourier-Feynman transform[3] are special cases of integral transform $\mathcal{F}_{\alpha,\beta}$[9].

In [4], Chung and Skoug introduced the concept of a conditional Feynman integral, while in [11], Park and Skoug introduced the concept of a conditional Fourier-Feynman transform and a conditional convolution for functionals defined on $C_0[0,T]$.

In this paper we study the conditional integral transforms(CIT) and conditional convolutions(CC) for a general vector-valued conditioning functions which need not depend upon the values of $x$ in $C_0[0,T]$ at only finitely many points in $(0,T]$. And then we obtain several relationships and formulas that exist among CIT, CC and the first variation(FV) for the conditioning functions and for a class of functionals defined on $K$.

We finish this section by stating definitions of integral transform $\mathcal{F}_{\alpha,\beta}$, convolution $(F*G)_\alpha$ and first variation $\delta F$ for functionals defined on $K$. The main results of [7] were to establish various relationships holding among $\mathcal{F}_{\alpha,\beta}F$, $\mathcal{F}_{\alpha,\beta}G$, $(F*G)_\alpha$, $\delta F$ and $\delta G$.

Definition 1.1. Let $F$ and $G$ be functionals defined on $K$. Then the integral transform, convolution and first variation are defined by followings,

$$\mathcal{F}_{\alpha,\beta}F(y) \equiv E_x[F(\alpha x + \beta y)], \tag{1.2}$$

$$(F*G)_\alpha(y) \equiv E_x\Big[F\Big(\frac{y+\alpha x}{\sqrt{2}}\Big)G\Big(\frac{y-\alpha x}{\sqrt{2}}\Big)\Big], \tag{1.3}$$

$$\delta F(y|w) \equiv \frac{\partial}{\partial t}F(y+tw)|_{t=0}, \tag{1.4}$$

where $w, y \in K$, if they exist [1, 7, 14].

## 2. Existence theorems

Let $\mathcal{H}$ be an infinite dimensional subspace of $L_2[0,T]$ with a complete orthonormal basis $\{\alpha_j\}$. From [13], we see that the corresponding Paley-Wiener-Zygmund stochastic integrals

$$\gamma_j(x) = \int_0^T \alpha_j(t)dx(t), j = 1, 2, \dots \tag{2.1}$$

form a set of independent standard Gaussian random variables on $C_0[0,T]$ with

$$E_x[x(t)\gamma_j(x)] = \int_0^t \alpha_j(s)ds \equiv \beta_j(t). \tag{2.2}$$

For each $k \in \mathbb{N}$ let $\mathcal{H}_k$ be a subspace of $\mathcal{H}$ spanned by $\{\alpha_1, \dots, \alpha_k\}$ and let $X_k : C_0[0,T] \to \mathbb{R}^k$ be the conditioning function defined by

$$X_k(x) = (\gamma_1(x), \dots, \gamma_k(x)). \tag{2.3}$$

Further, for $h \in L_2[0,T]$, let

$$\mathcal{P}_k h(t) = \sum_{j=1}^{k} (h, \alpha_j)\alpha_j(t) \tag{2.4}$$

be the orthogonal projection from $L_2[0,T]$ onto the subspace generated by $\{\alpha_1, \dots, \alpha_k\}$ where $(\cdot,\cdot)$ denotes the inner product on the real Hilbert space $L_2[0,T]$, and let $w_k(h) = h - \mathcal{P}_k h$ which is orthogonal to each $\alpha_j$, $j = 1, \dots k$. For $x \in C_0[0,T]$ and $\vec{\xi} = (\xi_1, \dots, \xi_k) \in \mathbb{R}^k$, let

$$x_k(t) = \int_0^T \mathcal{P}_k I_{[0,t]}(s)dx(s) = \sum_{j=1}^{k} \gamma_j(x)\beta_j(t) \tag{2.5}$$

and

$$\vec{\xi_k}(t) = \sum_{j=1}^{k} \xi_j(\alpha_j, I_{[0,t]}) = \sum_{j=1}^{k} \xi_j\beta_j(t) \tag{2.6}$$

where $I_{[0,t]}$ is the indicator function of the interval $[0,t]$.

Let $F : C_0[0,T] \to \mathbb{C}$ be integrable functional and let $X_k$ be a random vector on $C_0[0,T]$. Then we have the conditional Wiener integral $E_x[F\|X_k]$ given $X_k$ from a well-known probability theory. For a more detailed survey of the conditional Wiener integrals see [11, 12, 13]. In [13], Park and Skoug gave a useful simple formula for expressing conditional Wiener integrals in terms of ordinary(i.e., non-conditional) Wiener integrals; namely that for the conditioning function $X_k(x) = (\gamma_1(x), \dots, \gamma_k(x))$,

$$E_x[F(x)\|X_k(x)](\vec{\xi}) = E_x[F(x - x_k + \vec{\xi}_k)] \tag{2.7}$$

for a.e. $\vec{\xi} \in \mathbb{R}^k$.

In this paper we will always condition by $X_k(x)$ which is given by (2.3).

DEFINITION 2.1. Let $F$ and $G$ be functionals defined on $K$. Then we define the CIT, $\mathcal{F}_{\alpha,\beta}(F\|X_k)(y,\vec{\xi})$ of $F$ and the CC, $((F*G)_\alpha\|X_k)(y,\vec{\xi})$ of $(F*G)_\alpha$ given $X_k$, respectively, by the formulas

$$\mathcal{F}_{\alpha,\beta}(F\|X_k)(y,\vec{\xi}) = E_x[F(\alpha x + \beta y)\|X_k(x)](\vec{\xi}) \tag{2.8}$$

$$((F*G)_\alpha\|X_k)(y,\vec{\xi}) = E_x\Big[F\Big(\frac{y+\alpha x}{\sqrt{2}}\Big)G\Big(\frac{y-\alpha x}{\sqrt{2}}\Big)\|X_k\Big](\vec{\xi}) \tag{2.9}$$

if they exist.

REMARK 2.2. (i)Using the simple formula (2.7), we can get the formulas for expressing Wiener integrals (2.8) and (2.9), respectively, with

$$\mathcal{F}_{\alpha,\beta}(F\|X_k)(y,\vec{\xi}) = E_x[F(\alpha(x - x_k + \vec{\xi}_k) + \beta y)] \tag{2.10}$$

and

$$((F*G)_\alpha\|X_k)(y,\vec{\xi}) = E_x[F\Big(\frac{y+\alpha(x - x_k + \vec{\xi}_k)}{\sqrt{2}}\Big)G\Big(\frac{y-\alpha(x - x_k + \vec{\xi}_k)}{\sqrt{2}}\Big)]. \tag{2.11}$$

(ii) When $k = 1$ and $\alpha_1(t) = \frac{1}{\sqrt{T}}$, $X_1(x) = \frac{x(T)}{\sqrt{T}}$ is essentially the same function used by current authors and Skoug [8] and Lee et al.[10]. In this case the conditioning function $X_1(x)$ is depend on the value of $x$ at only one point $T$ in $(0, T]$.

(iii) In particular [8], for the conditioning function $X(x) = x(T)$ the current authors studied the conditional integral transforms and conditional convolutions of the types

$$\mathcal{F}_{\alpha,\beta}(F\|X)(y,\eta) = E_x[F(\alpha x + \beta y)\|x(T) = \eta] \tag{2.12}$$

and

$$((F*G)_\alpha\|X)(y,\eta) = E_x\Big[F\Big(\frac{y+\alpha x}{\sqrt{2}}\Big)G\Big(\frac{y-\alpha x}{\sqrt{2}}\Big)\|x(T)=\eta\Big] \tag{2.13}$$

(iv) In [6], the current authors considered the non vector-valued conditioning function $X(x)=\int_0^T h(u)dx(u)$ where $h(\neq 0)$ is in $L_2[0,T]$.

Next we describe the class of functionals that we work with in this paper. Let $\{\theta_1,\theta_2,\ldots\}$ be a complete orthonormal set of $\mathbb{R}$-valued functions in $L_2[0,T]$. Furthermore assume that each $\theta_j$ is of bounded variation on $[0,T]$. Then for each $y\in K$ and $j\in\{1,2,\ldots\}$, the Riemann-Stieltjes integral $\langle\theta_j,y\rangle\equiv\int_0^T\theta_j(t)\,dy(t)$ exists.

For $0\le\sigma<1$, let $E_\sigma$ be the space of all functionals $F:K\to\mathbb{C}$ of the form

$$F(y)=f(\langle\theta_1,y\rangle,\ldots,\langle\theta_n,y\rangle)=f(\langle\vec{\theta},y\rangle) \tag{2.14}$$

for some positive integer $n$, where $f(\lambda_1,\ldots,\lambda_n)=f(\vec{\lambda})$ is an entire function of $n$ complex variables $\lambda_1,\ldots,\lambda_n$ of exponential type; that is to say

$$|f(\vec{\lambda})|\le A_F\exp\{B_F\sum_{j=1}^n|\lambda_j|^{1+\sigma}\} \tag{2.15}$$

for some positive constants $A_F$ and $B_F$.

In addition we use the notation $F_j(y)=f_j(\langle\vec{\theta},y\rangle)$ where $f_j(\vec{\lambda})=\frac{\partial}{\partial\lambda_j}f(\lambda_1,\ldots,\lambda_n)$ for $j=1,\ldots,n$.

In [8], the current authors and Skoug showed that for all $F$ and $G$ in $E_\sigma$, $\mathcal{F}_{\alpha,\beta}(F\|X)$ and $((F*G)_\alpha\|X)$ exist and belong to $E_\sigma$ for all nonzero complex numbers $\alpha$ and $\beta$ and the condition by $X(x)=x(T)$. Also $\delta F(y|w)$ exists and belongs to $E_\sigma$ for all $y$ and $w$ in $K$.

Now let $\{u_1-\mathcal{P}_ku_1,\ldots,u_m-\mathcal{P}_ku_m\}$ be a maximal independent subset of $\{\theta_1-\mathcal{P}_k\theta_1,\ldots,\theta_n-\mathcal{P}_k\theta_n\}$ with $m\le n$ if it exists, where $\mathcal{P}_k$ is the orthogonal projection given by (2.4). Let $\{\phi_1,\ldots,\phi_m\}$ be the orthonormal set obtained from $\{u_1-\mathcal{P}_ku_1,\ldots,u_m-\mathcal{P}_ku_m\}$ using Gram-Schmidt orthonormalization process. Then we can find $n\times m$ matrix $A=(a_{i,j})$ with

$$\vec{\theta}-\mathcal{P}_k\vec{\theta}=(\sum_{j=1}^m a_{1,j}\phi_j,\ldots,\sum_{j=1}^m a_{n,j}\phi_j) \tag{2.16}$$

where $\vec{\theta} = (\theta_1, \dots, \theta_n)$ and $\vec{\theta} - \mathcal{P}_k\vec{\theta} = (\theta_1 - \mathcal{P}_k\theta_1, \dots, \theta_n - \mathcal{P}_k\theta_n)$.

LEMMA 2.3. *For all $j \in \{1, 2, \dots, n\}$,*

$$\langle \theta_j, x - x_k \rangle = \langle w_k(\theta_j), x \rangle \tag{2.17}$$

*and*

$$\langle \theta_j, \vec{\theta_j} \rangle = \sum_{j=1}^{k} \xi_j(\theta_j, \alpha_j). \tag{2.18}$$

*Proof.* By equations (2.5) and (2.6),

$$\begin{aligned}\langle \theta_j, x - x_k \rangle =& \langle \theta_j, x \rangle - \sum_{i=1}^{k} \gamma_i(x) \langle \theta_j, \beta_i \rangle \\ =& \langle \theta_j, x \rangle - \sum_{i=1}^{k} \langle \alpha_i, x \rangle \langle \theta_j, \beta_i \rangle \\ =& \langle \theta_j - \sum_{i=1}^{k} \langle \theta_j, \beta_i \rangle \alpha_i, x \rangle \\ =& \langle \theta_j - \sum_{i=1}^{k} (\theta_j, \alpha_i) \alpha_i, x \rangle = \langle w_k(\theta_j), x \rangle.\end{aligned}$$

Similarly for all $j \in \{1, 2, \dots, n\}$, we have the equation (2.18). □

THEOREM 2.4. *Let $F \in E_\sigma$ be given by (2.14), and let $X_k$ be given by (2.3). Then the CIT $\mathcal{F}_{\alpha,\beta}(F \| X_k)(y, \vec{\xi})$ exists, belongs to $E_\sigma$ and is given by the formula*

$$\mathcal{F}_{\alpha,\beta}(F \| X_k)(y, \vec{\xi}) = L_k(\vec{\xi}; \langle \vec{\theta}, y \rangle) \tag{2.19}$$

*for all $y \in K$ and a.e.$\vec{\xi} \in \mathbb{R}^k$, where*

$$\begin{aligned}&L_k(\vec{\xi}; \vec{\lambda}) \\ &= (2\pi)^{-m/2} \int_{\mathbb{R}^m} f(\alpha A\vec{u} + \alpha\langle \vec{\theta}, \vec{\xi}_k \rangle + \beta\vec{\lambda}) \exp\Big\{-\frac{1}{2} \sum_{j=1}^{m} u_j^2\Big\}\, d\vec{u}.\end{aligned} \tag{2.20}$$

*Proof.* For each $y \in K$ and a.e. $\vec{\xi} \in \mathbb{R}^k$,

$$\mathcal{F}_{\alpha,\beta}(F \| X_k)(y, \vec{\xi}) = E_x[f(\alpha\langle \vec{\theta}, x - x_k + \vec{\xi}_k)\rangle + \beta\langle \vec{\theta}, y \rangle)].$$

Using (2.14), (2.16) and Lemma 2.3, we have

(2.21)

$$
\begin{aligned}
&\mathcal{F}_{\alpha,\beta}(F\|X_k)(y,\vec{\xi})\\
=&E_x[f(\alpha(\sum_{j=1}^{m} a_{1,j}\langle\phi_j,x\rangle,\dots,\sum_{j=1}^{m} a_{n,j}\langle\phi_j,x\rangle)+\alpha\langle\vec{\theta},\vec{\xi}_k\rangle+\beta\langle\vec{\theta},y\rangle)]\\
=&E_x[f(\alpha A\langle\vec{\phi},x\rangle+\alpha\langle\vec{\theta},\vec{\xi}_k\rangle+\beta\langle\vec{\theta},y\rangle)].
\end{aligned}
$$

By a well-known Wiener integration theorem, we see that the last expression of (2.21) equals $L_k(\vec{\xi};\langle\vec{\theta},y\rangle)$. By [5, Theorem 3.15] $L_k(\vec{\xi};\vec{\lambda})$ is an entire function. Moreover by the inequality (2.15) we have

$$
\begin{aligned}
&|L_k(\vec{\xi};\vec{\lambda})|\\
\le&(2\pi)^{-m/2}A_F\exp\left\{B_F(3|\beta|)^{1+\sigma}\sum_{i=1}^{n}|\lambda_i|^{1+\sigma}\right\}\\
&\cdot\int_{\mathbb{R}^m}\exp\Big\{B_F(3|\alpha|)^{1+\sigma}\sum_{i=1}^{n}\Big(|(A\vec{u})_i|^{1+\sigma}+|\langle\vec{\theta},\vec{\xi}_k\rangle_i|^{1+\sigma}\Big)-\frac{1}{2}\sum_{j=1}^{m}u_j^2\Big\}d\vec{u}\\
=&A_{\mathcal{F}_{\alpha,\beta}F}\exp\Big\{B_{\mathcal{F}_{\alpha,\beta}F}\sum_{i=1}^{n}|\lambda_i|^{1+\sigma}\Big\},
\end{aligned}
$$

where $B_{\mathcal{F}_{\alpha,\beta}F}=B_F(3|\beta|)^{1+\sigma}$, and

$$
\begin{aligned}
A_{\mathcal{F}_{\alpha,\beta}F}=&A_F(2\pi)^{-m/2}\int_{\mathbb{R}^m}\exp\Big\{B_F(3|\alpha|)^{1+\sigma}\\
&\sum_{i=1}^{n}\Big(|(A\vec{u})_i|^{1+\sigma}+|\langle\vec{\theta},\xi_k\rangle_i|^{1+\sigma}\Big)-\frac{1}{2}\sum_{j=1}^{m}u_j^2\Big\}\,d\vec{u}<\infty.
\end{aligned}
$$

Hence $\mathcal{F}_{\alpha,\beta}(F\|X_k)(y,\eta)\in E_\sigma$ as a function of $y$. $\square$

Corollary 2.5. *Let $F\in E_\sigma$ and $X_k$ be given by (2.14) and (2.3), respectively. If $\{\theta_1,\dots,\theta_n,\alpha_1,\dots,\alpha_k\}$ is an orthonormal set of functions in $L_2[0,T]$, then*

$$
\mathcal{F}_{\alpha,\beta}(F\|X_k)(y,\vec{\xi})=\mathcal{F}_{\alpha,\beta}(F)(y) \tag{2.22}
$$

*Proof.* As in the proof of Lemma 2.3, we have

$\langle\theta_j,x_k\rangle=\sum_{i=1}^{k}\gamma_i(x)(\theta_j,\alpha_i)=0$ and $\langle\theta_j,\vec{\xi}_k\rangle=0$.

Now applying (2.5)and (2.6) to the CIT (2.10), yields the equation (2.22) as desired. □

In our next theorem we show that the conditional convolution of functionals from $E_\sigma$ for the general conditioning function $X_k$ is an element of $E_\sigma$.

THEOREM 2.6. *Let $F, G \in E_\sigma$ be given by* (2.14) *with corresponding entire functions $f$ and $g$, respectively. And let $X_k$ be given by* (2.3). *Then the CC $((F * G)_\alpha \| X_k)(y, \vec{\xi})$ exists for all $y \in K$ and a.e. $\vec{\xi} \in \mathbb{R}^k$, belongs to $E_\sigma$, and is given by the formula*

$$((F * G)_\alpha \| X_k)(y, \vec{\xi}) = M_k(\vec{\xi}; \langle \vec{\theta}, y \rangle) \tag{2.23}$$

*where*

$$\begin{aligned} &M_k(\vec{\xi}; \vec{\lambda}) \\ =&(2\pi)^{-m/2} \int_{\mathbb{R}^m} f\Big(\frac{\vec{\lambda} + \alpha A \vec{u} + \alpha \langle \vec{\theta}, \vec{\xi}_k \rangle}{\sqrt{2}}\Big) g\Big(\frac{\vec{\lambda} - \alpha A \vec{u} - \alpha \langle \vec{\theta}, \vec{\xi}_k \rangle}{\sqrt{2}}\Big) \\ &\exp\Big\{-\frac{1}{2} \sum_{j=1}^m u_j^2\Big\}\, d\vec{u}. \end{aligned} \tag{2.24}$$

*Proof.* For each $y \in K$ and a.e. $\vec{\xi} \in \mathbb{R}^k$,

$$\begin{aligned} &((F * G)_\alpha \| X_k)(y, \vec{\xi}) \\ =&E_x\Big[f\Big(\frac{1}{\sqrt{2}}(\langle \vec{\theta}, y \rangle + \alpha \langle \vec{\theta}, y \rangle + \alpha \langle \vec{\theta}, x - x_k \rangle + \alpha \langle \vec{\theta}, \vec{\xi}_k \rangle)\Big) \\ &g\Big(\frac{1}{\sqrt{2}}(\langle \vec{\theta}, y \rangle - \alpha \langle \vec{\theta}, y \rangle - \alpha \langle \vec{\theta}, x - x_k \rangle - \alpha \langle \vec{\theta}, \vec{\xi}_k \rangle)\Big)\Big] \\ =&E_x\Big[f\Big(\frac{1}{\sqrt{2}}(\langle \vec{\theta}, y \rangle + \alpha \langle \vec{\theta}, y \rangle + \alpha A \langle \vec{\phi}, x \rangle + \alpha \langle \vec{\theta}, \vec{\xi}_k \rangle)\Big) \\ &g\Big(\frac{1}{\sqrt{2}}(\langle \vec{\theta}, y \rangle - \alpha \langle \vec{\theta}, y \rangle - \alpha A \langle \vec{\phi}, x \rangle - \alpha \langle \vec{\theta}, \vec{\xi}_k \rangle)\Big)\Big] \end{aligned}$$

By (2.14), (2.16), Lemma 2.3 and a well-known Wiener integration theorem, we see that the last expression above equals $M_k(\vec{\xi}; \langle \vec{\theta}, y \rangle)$. By [5, Theorem 3.15], $M_k(\vec{\xi};\vec{\lambda})$ is an entire function and

$$\begin{aligned}
&|M_k(\vec{\xi};\vec{\lambda})|\\
\le&(2\pi)^{-\frac{m}{2}}A_F A_G\exp\Big\{(B_F+B_G)\Big(\frac{3}{\sqrt{2}}\Big)^{1+\sigma}\sum_{i=1}^n|\lambda_i|^{1+\sigma}\Big\}\\
&\cdot\int_{\mathbb{R}^m}\exp\Big\{(B_F+B_G)\Big(\frac{3|\alpha|}{\sqrt{2}}\Big)^{1+\sigma}\sum_{i=1}^n\Big(|(A\vec{u})_i|^{1+\sigma}+|\langle\vec{\theta},\vec{\xi}_k\rangle_i|^{1+\sigma}\Big)-\frac{1}{2}\sum_{j=1}^m u_j^2\Big\}\,d\vec{u}\\
=&A_{(F*G)_\alpha}\exp\Big\{B_{(F*G)_\alpha}\sum_{i=1}^n|\lambda_i|^{1+\sigma}\Big\},
\end{aligned}$$

where

$$B_{(F*G)_\alpha}=(B_F+B_G)\Big(\frac{3}{\sqrt{2}}\Big)^{1+\sigma}$$

and

$$\begin{aligned}
A_{(F*G)_\alpha}=&A_F A_G(2\pi)^{-\frac{m}{2}}\int_{\mathbb{R}^m}\exp\Big\{(B_F+B_G)\Big(\frac{3|\alpha|}{\sqrt{2}}\Big)^{1+\sigma}\\
&\sum_{i=1}^n\Big(|(A\vec{u})_i|^{1+\sigma}+|\langle\vec{\theta},\vec{\xi}_k\rangle_i|^{1+\sigma}\Big)-\frac{1}{2}\sum_{j=1}^m u_j^2\Big\}\,d\vec{u}<\infty.
\end{aligned}$$

Hence $((F*G)_\alpha\|X_k)(y,\eta)\in E_\sigma$ as a function of $y$. □

In [8, Theorem 2.6], the authors and Skoug showed that for $F\in E_\sigma$, $0<\sigma<1$, the first variation $\delta F(y|w)$ of functionals $F$ in $E_\sigma$ is an element of $E_\sigma$, both as a function of $y$ for fixed $w$ and as a function of $w$ for fixed $y$.

Remark 2.7. Note that in view of Theorems 2.4, and 2.6 above and Theorem 2.6 [8], all of the functionals that arise in Section 3 below are automatically elements of $E_\sigma$.

## 3. Various relationships and formulas involving the CIT, CC and FV

In this section, for the general vector-valued conditioning function $X_k$ given by (2.3), we obtain the various relationships involving the three concepts of CIT, CC and FV for functionals belonging to $E_\sigma$. Once we have shown the existence Theorems 2.4 and 2.6 above and Theorem 2.6

in [8], the proofs of the Theorems 3.1 through 3.5 are similar to those in [8].

Our first formula (3.1) is useful because it allows us to calculate $\mathcal{F}_{\alpha,\beta}(((F*G)_\alpha\|X_k)(\cdot,\vec{\xi}_1)\|X_k)(y,\vec{\xi}_2)$ without ever actually calculating $(F*G)_\alpha$ or $((F*G)_\alpha\|X_k)$.

Theorem 3.1. *For $F$ , $G$ in $E_\sigma$ we have*

$$\begin{aligned}&\mathcal{F}_{\alpha,\beta}(((F*G)_\alpha\|X_k)(\cdot,\vec{\xi}_1)\|X_k)(y,\vec{\xi}_2)\\ =&\mathcal{F}_{\alpha,\beta}(F\|X_k)\Big(\frac{y}{\sqrt{2}},\frac{\vec{\xi}_2+\vec{\xi}_1}{\sqrt{2}}\Big)\mathcal{F}_{\alpha,\beta}(G\|X_k)\Big(\frac{y}{\sqrt{2}},\frac{\vec{\xi}_2-\vec{\xi}_1}{\sqrt{2}}\Big)\end{aligned}\tag{3.1}$$

*for all $y\in K$ and a.e. $\vec{\xi}_1,\vec{\xi}_2\in\mathbb{R}^k$.*

*Proof.* The left hand side of (3.1) exists by Theorems 2.4 and 2.6 while the right hand side of (3.1) exists by Theorem 2.4. The equality in equation (3.1) then follows from (2.7), (2.10) and Wiener process properties. □

Our next formula (3.2), giving the CC of CIT, follows from Theorems 2.4, 2.6 and a well-known Wiener integration formula.

Theorem 3.2. *Let $F$ and $G$ be as in Theorem 2.6. Then for all $y\in K$ and a.e. $\vec{\xi}_i\in\mathbb{R}^k, i=1,2,3,$*

(3.2)

$$\begin{aligned}&((\mathcal{F}_{\alpha,\beta}(F\|X_k)(\cdot,\vec{\xi}_1)*\mathcal{F}_{\alpha,\beta}(G\|X_k)(\cdot,\vec{\xi}_2))_\alpha\|X_k)(y,\vec{\xi}_3)\\ =&(2\pi)^{-3m/2}\int_{\mathbb{R}^{3m}}f(\alpha A\vec{v}+\alpha\langle\vec{\theta},(\vec{\xi}_1)_k\rangle+\frac{\beta}{\sqrt{2}}(\langle\vec{\theta},y\rangle+\alpha A\vec{u}+\alpha\langle\vec{\theta},(\vec{\xi}_3)_k\rangle))\\ &\qquad g(\alpha A\vec{w}+\alpha\langle\vec{\theta},(\vec{\xi}_2)_k\rangle+\frac{\beta}{\sqrt{2}}(\langle\vec{\theta},y\rangle-\alpha A\vec{u}-\alpha\langle\vec{\theta},(\vec{\xi}_3)_k\rangle))\\ &\qquad \exp\{-\frac{1}{2}\sum_{j=1}^m(u_j^2+v_j^2+w_j^2)\}\,d\vec{u}\,d\vec{v}\,d\vec{w}.\end{aligned}$$

In Theorem 3.3 below we obtain a formula for the conditional convolution product with respect to the first argument of the variation of the first variation of functionals from $E_\sigma$.

THEOREM 3.3. *Let $F$ and $G$ be as in Theorem 2.6. Then for a.e. $\vec{\xi} \in \mathbb{R}^k$,*

$$\begin{aligned}(3.3)\qquad &((\delta F(\cdot|w) * \delta G(\cdot|w))_\alpha \| X_k)(y, \vec{\xi}) \\ &= \sum_{j=1}^{n} \sum_{l=1}^{n} \langle \theta_j, w\rangle \langle \theta_l, w\rangle ((F_j * G_l)_\alpha \| X_k)(y, \vec{\xi})\end{aligned}$$

*for all $y$ and $w$ in $K$.*

*Proof.* Applying the additive distribution properties of the CC to the expressions for

$$\delta F(y|w) = \sum_{j=1}^{n} \langle \theta_j, w\rangle F_j(y)$$

and $G$ which is given by (2.14) with corresponding entire function $g$ yields equation (3.3).

□

In our next theorem we obtain a formula for the first variation of the conditional convolution of functionals from $E_\sigma$.

THEOREM 3.4. *Let $F \in E_\sigma$ be given by (2.14). Then for a.e. $\vec{\xi} \in \mathbb{R}^k$,*

$$\begin{aligned}(3.4)\qquad &\delta((F * G)_\alpha \| X_k)(\cdot, \vec{\xi})(y|w) \\ &= \sum_{j=1}^{n} \frac{\langle \theta_j, w\rangle}{\sqrt{2}} \Big[((F_j * G)_\alpha \| X_k)(y, \vec{\xi}) + ((F * G_j)_\alpha \| X_k)(y, \vec{\xi})\Big]\end{aligned}$$

*for all $y$ and $w$ in $K$.*

*Proof.* Using the definition of FV and equation (2.11) it follows that

$$
\begin{aligned}
&\delta((F*G)_\alpha\|X_k)(\cdot,\vec{\xi})(y|w)\\
=&\frac{\partial}{\partial t}E_x[f(\frac{1}{\sqrt{2}}(\langle\vec{\theta},y+tw\rangle+\alpha A\langle\vec{\phi},x\rangle+\alpha\langle\vec{\theta},\vec{\xi}_k\rangle))\\
&\qquad g(\frac{1}{\sqrt{2}}(\langle\vec{\theta},y+tw\rangle-\alpha A\langle\vec{\phi},x\rangle-\alpha\langle\vec{\theta},\vec{\xi}_k\rangle))]|_{t=0}\\
=&\sum_{j=1}^{n}\frac{\langle\theta_j,w\rangle}{\sqrt{2}}E_x[f_j(\frac{1}{\sqrt{2}}(\langle\vec{\theta},y+tw\rangle+\alpha A\langle\vec{\phi},x\rangle+\alpha\langle\vec{\theta},\vec{\xi}_k\rangle))\\
&\qquad g(\frac{1}{\sqrt{2}}(\langle\vec{\theta},y+tw\rangle-\alpha A\langle\vec{\phi},x\rangle-\alpha\langle\vec{\theta},\vec{\xi}_k\rangle))\\
&+f_j(\frac{1}{\sqrt{2}}(\langle\vec{\theta},y+tw\rangle+\alpha A\langle\vec{\phi},x\rangle+\alpha\langle\vec{\theta},\vec{\xi}_k\rangle))\\
&\qquad g(\frac{1}{\sqrt{2}}(\langle\vec{\theta},y+tw\rangle-\alpha A\langle\vec{\phi},x\rangle-\alpha\langle\vec{\theta},\vec{\xi}_k\rangle))]|_{t=0}\\
=&\sum_{j=1}^{n}\frac{\langle\theta_j,w\rangle}{\sqrt{2}}\Big[((F_j*G)_\alpha\|X_k)(y,\vec{\xi})+((F*G_j)_\alpha\|X_k)(y,\vec{\xi})\Big]
\end{aligned}
\tag{3.5}
$$

□

In Theorem 3.5 below we get a formula for the integral transform with respect to the first argument of variation.

THEOREM 3.5. *Let $F\in E_\sigma$ be given by (2.14). Then for a.e. $\vec{\xi}\in\mathbb{R}^k$,*

$$
\mathcal{F}_{\alpha,\beta}(\delta F(\cdot|w)\|X_k)(y,\vec{\xi})=\frac{1}{\beta}\delta\mathcal{F}_{\alpha,\beta}(F\|X_k)(\cdot,\vec{\xi})(y|w)
\tag{3.6}
$$

*for all $y$ and $w$ in $K$.*

In addition we establish the following various formulas (3.7) through (3.11) involving the three concepts where each concept is used exactly once. We omit details because of the calculations are rather long, but similar to those carried out in Section 2 above.

FORMULA 3.6. Let $F$ and $G$ be as in Theorem 2.6. Then for all $y$ and $w$ in $K$ and a.e. $\vec{\xi_i} \in \mathbb{R}^k$, i=1, 2, 3,

$$\begin{aligned}
&\delta[\mathcal{F}_{\alpha,\beta}((F*G)_\alpha\|X_k)(\cdot,\vec{\xi_1})\|(\cdot,\vec{\xi_2})](y|w)\\
&=\delta\mathcal{F}_{\alpha,\beta}(F\|X_k)(\cdot,\frac{\vec{\xi_2}+\vec{\xi_1}}{\sqrt{2}})(\frac{y}{\sqrt{2}}|\frac{w}{\sqrt{2}})\mathcal{F}_{\alpha,\beta}(G\|X_k)(\frac{y}{\sqrt{2}},\frac{\vec{\xi_2}-\vec{\xi_1}}{\sqrt{2}})\\
&\quad+\mathcal{F}_{\alpha,\beta}(F\|X_k)(\frac{y}{\sqrt{2}},\frac{\vec{\xi_2}+\vec{\xi_1}}{\sqrt{2}})\delta\mathcal{F}_{\alpha,\beta}(G\|X_k)(\cdot,\frac{\vec{\xi_2}-\vec{\xi_1}}{\sqrt{2}})(\frac{y}{\sqrt{2}}|\frac{w}{\sqrt{2}})
\end{aligned}\tag{3.7}$$

(3.8)

$$\begin{aligned}
&\delta[(\mathcal{F}_{\alpha,\beta}(F\|X_k)(\cdot,\vec{\xi_1})*\mathcal{F}_{\alpha,\beta}(G\|X_k)(\cdot,\vec{\xi_2}))_\alpha\|X_k)(\cdot,\vec{\xi_3})](y|w)\\
&=\frac{\beta}{\sqrt{2}}\sum_{j=1}^n\langle\theta_j,w\rangle[((\mathcal{F}_{\alpha,\beta}(F_j\|X_k)(\cdot,\vec{\xi_1})*\mathcal{F}_{\alpha,\beta}(G\|X_k)(\cdot,\vec{\xi_2}))_\alpha\|X_k)(y,\vec{\xi_3})\\
&\qquad((\mathcal{F}_{\alpha,\beta}(F\|X_k)(\cdot,\vec{\xi_1})*\mathcal{F}_{\alpha,\beta}(G_j\|X_k)(\cdot,\vec{\xi_2}))_\alpha\|X_k)(y,\vec{\xi_3})]
\end{aligned}$$

$$\begin{aligned}
&\beta\mathcal{F}_{\alpha,\beta}(\delta((F*G)_\alpha\|X_k)(\cdot,\vec{\xi_1})(\cdot|w)\|X_k)(y,\vec{\xi_2})\\
&=\delta\mathcal{F}_{\alpha,\beta}(((F*G)_\alpha\|X_k)(\cdot,\vec{\xi_1})\|X_k)(\cdot,\vec{\xi_2})(y,\vec{\xi_2})
\end{aligned}\tag{3.9}$$

(3.10)

$$\begin{aligned}
&\beta^2\mathcal{F}_{\alpha,\beta}(((\delta F(\cdot|w)*\delta G(\cdot|w))_\alpha\|X_k)(\cdot,\vec{\xi_1})\|X_k)(y,\vec{\xi_2})\\
&=\beta^2\mathcal{F}_{\alpha,\beta}(\delta F(\cdot|w)\|X_k)(\frac{y}{\sqrt{2}},\frac{\vec{\xi_2}+\vec{\xi_1}}{\sqrt{2}})\mathcal{F}_{\alpha,\beta}(\delta G(\cdot|w)\|X_k)(\frac{y}{\sqrt{2}},\frac{\vec{\xi_2}-\vec{\xi_1}}{\sqrt{2}})\\
&=\delta\mathcal{F}_{\alpha,\beta}(F\|X_k)(\cdot,\frac{\vec{\xi_2}+\vec{\xi_1}}{\sqrt{2}})(\frac{y}{\sqrt{2}}|w)\delta\mathcal{F}_{\alpha,\beta}(G\|X_k)(\cdot,\frac{\vec{\xi_2}-\vec{\xi_1}}{\sqrt{2}})(\frac{y}{\sqrt{2}}|w)
\end{aligned}$$

(3.11)

$$\begin{aligned}
&(\mathcal{F}_{\alpha,\beta}(\delta F(\cdot|w)\|X_k)(\cdot,\vec{\xi_1})*\mathcal{F}_{\alpha,\beta}(\delta G(\cdot|w)\|X_k)(\cdot,\vec{\xi_2}))_\alpha\|X_k)(y,\vec{\xi_3})\\
&=\sum_{j=1}^n\sum_{l=1}^n\langle\theta_j,w\rangle\langle\theta_l,w\rangle(((\mathcal{F}_{\alpha,\beta}(F_j\|X_k)(\cdot,\vec{\xi_1}))*(\mathcal{F}_{\alpha,\beta}(G_l\|X_k)(\cdot,\vec{\xi_2})))_\alpha\|X_k)(y,\vec{\xi_3}).
\end{aligned}$$

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