

ON THE NORM ATTAINING OPERATORS

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ABSTRACT. In this paper, we show the norm attaining paranormal operators have a nontrivial invariant subspace. Also, we show the norm attaining quadratically hyponormal weighted shift is subnormal.

1. Introduction

Let \mathcal{H} be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on \mathcal{H} . An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *hyponormal* if $T^*T \geq TT^*$ and *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *paranormal* if $\|T^2x\| \geq \|Tx\|^2$ for all unit vector $x \in \mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called *norm attaining* if there is an $x \in \mathcal{H}$ with $\|x\| = 1$ and $\|Tx\| = \|T\|$. The Bram-Halmos criterion for subnormality states that an operator T is subnormal if and only if

$$\sum_{i,j} (T^i x_j, T^j x_i) \geq 0$$

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for all finite collections $x_0, x_1, \dots, x_k \in \mathcal{H}$ ([3],[6, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$(0.1) \quad \begin{pmatrix} I & T^* & \cdots & T^{*k} \\ T & T^*T & \cdots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \cdots & T^{*k}T^k \end{pmatrix} \geq 0 \quad (\text{all } k \geq 1).$$

Condition (0.1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (0.1) for $k = 1$ is equivalent to the hyponormality of T , while subnormality requires the validity of (0.1) for all k . Let $[A, B] := AB - BA$ denote the commutator of two operators A and B , and define T to be k -hyponormal whenever the $k \times k$ operator matrix

$$(0.2) \quad M_k(T) := ([T^{*j}, T^i]_{i,j=1}^k)$$

is positive. An application of the Choleski algorithm for operator matrices shows that the positivity of (0.2) is equivalent to the positivity of the $(k+1) \times (k+1)$ operator matrix in (0.1); the Bram-Halmos criterion can be then rephrased as saying that T is subnormal if and only if T is k -hyponormal for every $k \geq 1$ ([9]).

Recall ([1],[9],[7]) that $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly k -hyponormal* if

$$LS(T, T^2, \dots, T^k) := \left\{ \sum_{j=1}^k \alpha_j T^j : \alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{C}^k \right\}$$

consists entirely of hyponormal operators, or equivalently, $M_k(T)$ is *weakly positive*, i.e., ([9])

$$(0.3) \quad \left\langle M_k(T) \begin{pmatrix} \lambda_1 x \\ \vdots \\ \lambda_k x \end{pmatrix}, \begin{pmatrix} \lambda_1 x \\ \vdots \\ \lambda_k x \end{pmatrix} \right\rangle \geq 0 \quad \text{for all } x \in \mathcal{H} \text{ and } \lambda_1, \dots, \lambda_k \in \mathbb{C}.$$

If $k = 2$ then T is said to be *quadratically hyponormal* and if $k = 3$ then T is said to be *cubically hyponormal*. Similarly, $T \in \mathcal{L}(\mathcal{H})$ is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that k -hyponormal \Rightarrow weakly k -hyponormal, but the converse is not true in general.

The invariant subspace problem is the simple question: Does every bounded linear operator T on a separable Hilbert space \mathcal{H} over \mathbb{C} have a

non-trivial invariant subspace? Here non-trivial subspace means a closed subspace of \mathcal{H} different from 0 and different from \mathcal{H} . Invariant means that the operator T maps it to itself. The problem is easy to state, however, it is still open. The answer is no in general for (separable) complex Banach spaces. For certain classes of bounded linear operators on complex Hilbert spaces, the problem has an affirmative answer (for example, compact operators, isometry operators, subnormal operators etc). But, the problem is still open for operators on a separable complex Hilbert space.

On the other hand, the celebrated Bishop-Phelps theorem states that the set of norm attaining functionals on a Banach space is norm dense in the dual space ([2]). J. Lindenstrauss ([15]) obtained for a reflexive space X and any Banach space Y that the norm attaining operators in $\mathcal{L}(X, Y)$ are dense and provided also the first counterexample: the set of norm attaining operators from c_0 into any strictly convex space containing c_0 is not dense in the set of all bounded operators. Nevertheless, there are also remarkable situations in which a Bishop-Phelps theorem for operators does hold, such as when the domain space is reflexive ([15]) or, more generally, when it has the Radon-Nikodym property ([4]). After the pioneering paper by Lindenstrauss, several results dealing with denseness of norm attaining operators appeared ([4],[12],[13], etc). Since every Hilbert space is reflexive, norm attaining operators on a Hilbert space \mathcal{H} are dense. In this paper, we will show the norm attaining paranormal operators have a nontrivial invariant subspace.

2. Main results

We start from a basic criterion for norm attaining operators.

LEMMA 1. *If $T \in \mathcal{B}(\mathcal{H})$ is a norm attaining operator if and only if $\|T\|^2 \in \sigma_p(T^*T)$, where $\sigma_p(S)$ denote the point spectrum of $S \in \mathcal{B}(\mathcal{H})$.*

Proof. Observe that $\|Tx\| = \|T\||x\|$ if and only if $\langle (T^*T - \|T\|^2)x, x \rangle = 0$. Since $T^*T - \|T\|^2$ is hermitian, we can see that $\langle (T^*T - \|T\|^2)x, x \rangle = 0$ if and only if $T^*Tx = \|T\|^2x$ or equivalently, $x \in \text{Ker}(T^*T - \|T\|^2I)$. Thus T is a norm attaining operator if and only if $\|T\|^2 \in \sigma_p(T^*T)$. \square

REMARK 2. *Note that if $T \in \mathcal{B}(\mathcal{H})$, then $\|T\|^2 \in \sigma_{ap}(T^*T)$, where $\sigma_{ap}(S)$ denote the approximate point spectrum of $S \in \mathcal{B}(\mathcal{H})$.*

Proof. Since T^*T is positive for any $T \in \mathcal{B}(\mathcal{H})$, T^*T is normaloid (i.e., norm equals to spectral radius). Hence $\|T\|^2 = \|T^*T\|$ is a spectral value of T^*T . From the fact that the boundary of the spectrum is contained in the approximate point spectrum ([11]), the desired result follows. \square

We now have our main theorem:

THEOREM 3. *If $T \in \mathcal{B}(\mathcal{H})$ is a norm attaining paranormal operator then T has a non-trivial invariant subspace, and hence norm attaining hyponormal operators have a non-trivial invariant subspace.*

Proof. Let $M := \{x \in \mathcal{H} : \|Tx\| = \|T\|\|x\|\}$. Observe that $x \in M$ if and only if $x \in \text{Ker}(\|T\|^2I - T^*T)$. Thus M is a closed subspace of \mathcal{H} and $M \neq \{0\}$ by the hypothesis. If $x \in M$ with $\|x\| = 1$, then by the paranormality of T we have

$$\|T^2x\| \geq \|Tx\|^2 = (\|T\|\|x\|)^2 = \|T\|^2.$$

Thus

$$\|T^2x\| \leq \|T\|\|Tx\| = \|T\|^2\|x\| = \|T\|^2 \leq \|T^2x\|.$$

This shows that $\|T^2x\| = \|T\|\|Tx\|$ for $x \in M$ with $\|x\| = 1$ and hence M is invariant under T . If $M \neq \mathcal{H}$, then M is a non-trivial invariant subspace for T . If $M = \mathcal{H}$, then $\frac{T}{\|T\|}$ is an isometry and hence T has a non-trivial invariant subspace. \square

Recall that given a bounded sequence of positive real numbers $\alpha : \alpha_0, \alpha_1, \dots$ (called *weights*), the (*unilateral*) *weighted shift* W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for $\ell^2(\mathbb{Z}_+)$ (where \mathbb{Z}_+ is the set of non-negative integers). It is straightforward to check that W_α can never be *normal*, and that W_α is *hyponormal* if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. We then have:

THEOREM 4. *W_α is norm attaining if and only if $\|W_\alpha\| = \alpha_i$ for some i .*

Proof. Since $W_\alpha^*W_\alpha = \text{diag}\{\alpha_0^2, \alpha_1^2, \dots\}$, we have $\sigma_p(W_\alpha^*W_\alpha) = \{\alpha_0^2, \alpha_1^2, \dots\}$. The desired result now follows from Lemma 1. \square

J. Stampfli [16] showed that for subnormal weighted shifts W_α , a *propagation* phenomenon occurs which forces the flatness of W_α whenever two equal weights are present. Later, A. Joshi proved in [14] that the shift

with weights $\alpha_0 = \alpha_1 = a$, $\alpha_2 = \alpha_3 = \dots = b$, $0 < a < b$, is *not* quadratically hyponormal, and P. Fan [10] established that for $a = 1$, $b = 2$, and $0 < s < \sqrt{5}/5$, $W_\alpha + sW_\alpha^2$ is *not* hyponormal. On the other hand, it was shown in [8, Theorem 2] that a hyponormal weighted shift with *three* equal weights cannot be quadratically hyponormal without being flat: *If W_α is quadratically hyponormal and $\alpha_n = \alpha_{n+1} = \alpha_{n+2}$ for some $n \geq 0$, then $\alpha_1 = \alpha_2 = \alpha_3 = \dots$, i.e., W_α is subnormal.* Furthermore, in [8, Proposition 11] it was shown that, in the presence of quadratic hyponormality, two consecutive pairs of equal weights again force flatness, thereby subnormality. Y. Choi [5] improved this result, that is, if W_α is quadratically hyponormal and $\alpha_n = \alpha_{n+1}$ for some $n \geq 1$, then W_α is flat. Moreover, Y. Choi [5] also showed that if W_α is polynomially hyponormal and $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then W_α is flat.

PROPOSITION 5. (*Propagation*) *Let W_α be a weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$.*

- (i) ([16, Theorem 6]) *Let W_α be subnormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then α is flat, i.e., $\alpha_1 = \alpha_2 = \alpha_3 = \dots$.*
- (ii) ([8, Corollary 6]) *Let W_α be 2-hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then α is flat.*
- (iii) ([5, Theorem 1]) *Let W_α be quadratically hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 1$, then α is flat.*
- (iv) ([5, Theorem 2]) *Let W_α be polynomially hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then α is flat.*

We now show outer propagation phenomena occurs for a norm attaining hyponormal weighted shift W_α .

COROLLARY 6. *Let W_α be a norm attaining hyponormal weighted shift. Then $\alpha_n = \alpha_{n+1} = \dots$ for some $n \geq 0$.*

Proof. By Theorem 4, we have that $\|W_\alpha\| = \alpha_n = \max_i \alpha_i$ for some $n \geq 0$. But since W_α is hyponormal, the corresponding weight sequence is monotonically increasing. Thus, $\alpha_n = \alpha_{n+1} = \dots$. □

Moreover, we have:

COROLLARY 7. (1) *If W_α is a norm attaining quadratically hyponormal weighted shift. Then α is flat and hence W_α is subnormal.*
 (2) *If W_α is a norm attaining 2-hyponormal weighted shift. Then α is flat and hence W_α is subnormal.*

Proof. It follows from Corollary 6 and Proposition 5. \square

Although the norm attaining operators are dense in $\mathcal{B}(\mathcal{H})$, we cannot expect that every hyponormal(or paranormal) operator is a norm attaining operator.

EXAMPLE 8. Let $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ be a strictly increasing bounded sequence. Then W_α is hyponormal (and hence paranormal), but not norm attaining.

Proof. Since the weight sequence is strictly increasing, W_α is clearly hyponormal. However, $\|W_\alpha\|^2$ cannot be in the point spectrum of $W_\alpha^*W_\alpha = \text{diag}\{\alpha_0^2, \alpha_1^2, \dots\}$. Thus, by Lemma 1, W_α is not norm attaining. \square

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