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AVERAGE OF CLASS NUMBERS OF SOME FAMILY OF ARTIN-SCHREIER EXTENSIONS OF RATIONAL FUNCTION FIELDS

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Abstract. In this paper we obtain average of class numbers of some family of Artin-Schreier extensions of rational function field $\mathbb{F}_q(t)$, where q is a power of 3.

1. Introduction

The average of class numbers of a family of global fields has been studied by many authors. This problem was initiated by Gauss who made two famous conjectures on average values of class numbers of of orders in quadratic fields. These conjectures were proved by Lipschitz in imaginary quadratic fields case and by Siegel [8] in real quadratic fields case. Takhtadzjan and Vinogradov [9] gave an average formula for class numbers of quadratic fields with prime discriminants. Let $k = \mathbb{F}_q(t)$ be a rational function field over the finite field $\mathbb{F}_q$, where q is a power of a prime number p, and $\mathbb{A} = \mathbb{F}_q[t]$ be the polynomial ring. Assume that q is odd. In [3], Hoffstein and Rosen gave an average of class numbers of orders of quadratic extensions of k and also an average value of class numbers of maximal orders of quadratic extensions of k. When $q \equiv 1 \bmod 3$, Rosen [7] gave an average of class numbers of orders of Kummer extensions of k and Jung [5] obtained an average of class numbers of maximal orders of Kummer extensions $K = k(\sqrt[3]{P})$ of k,

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where P runs over monic irreducible polynomials. In [2], when q is even, Chen obtained an average of class numbers of orders of quadratic extensions of k. In [1], Bae, Jung and Kang extended Chen's result to any Artin-Schreier extensions of k. Let $K_u = k(\alpha_u)$ be the Artin-Schreier extension of k generated by a root α_u of $x^p - x = u$, where $u = \frac{B}{A} \in k$ is normalized (see §2.1). Then $G(K_u) = A$ which is a monic polynomial in $\mathbb{A}$ is uniquely determined by the field K_u. In [1], when $p = 2$, Bae, Jung and Kang gave an average of class numbers of maximal orders of quadratic Artin-Schreier extensions K_u of k with monic irreducible $G(K_u)$. In this paper, when $p = 3$, we study the average of class numbers of maximal orders of Artin-Schreier extensions K_u of k with monic irreducible $G(K_u)$. In §2, we recall some basic facts on the Artin-Schreier extensions of k and L-functions associated to maximal orders of Artin-Schreier extensions with class number formulas. In §3, when $p = 3$, we give averages of class numbers of maximal orders of real/inert imaginary/ramified imaginary Artin-Schreier extensions K_u of k with monic irreducible $G(K_u)$.

2. Preliminaries

Let $k = \mathbb{F}_q(t)$ and $\mathbb{A} = \mathbb{F}_q[t]$, where q is a power of a prime p. Let $\infty_k = (1/t)$ be the infinite prime of k. We denote by $\mathbb{A}^+$ the set of monic polynomials in $\mathbb{A}$ and by $\mathcal{P}$ the set of monic irreducible polynomials in $\mathbb{A}$. Write $\mathbb{A}_n = \{N \in \mathbb{A} : \deg(N) = n\}$, $\mathbb{A}_n^+ = \mathbb{A}^+ \cap \mathbb{A}_n$ and $\mathcal{P}_n = \mathcal{P} \cap \mathbb{A}_n$. For any $0 \neq N \in \mathbb{A}$, let $|N| = \#(\mathbb{A}/N\mathbb{A}) = q^{\deg(N)}$, $\Phi(N) = \#(\mathbb{A}/N\mathbb{A})^\times$, where $\#X$ denotes the cardinality of a set X, and $\operatorname{sgn}(N)$ denote the leading coefficient of N. Let $\zeta_{\mathbb{A}}(s) = \frac{1}{1-q^{1-s}}$ be the zeta function of $\mathbb{A}$.

2.1. Artin-Schreier extensions. Let $\wp(x) = x^p - x$ be the Artin-Schreier operator. For $u = \frac{B}{A} \in k$ with $A \in \mathbb{A}^+, B \in \mathbb{A}$ and $\gcd(A, B) = 1$, we say that u is *normalized* if it satisfies the following conditions: (1) if $A = \prod_{i=1}^{r} P_i^{e_i}$, then $p \nmid e_i$ for each $1 \leq i \leq r$, (2) if $\deg(B) > \deg(A)$, then $p \nmid (\deg(B) - \deg(A))$, (3) if $\deg(B) = \deg(A)$, then $\operatorname{sgn}(B) \notin \wp(\mathbb{F}_q)$. Let $K_u = k(\alpha_u)$ be the Artin-Schreier extension of k generated by a root α_u of $\wp(x) = u$. Let $\mathcal{O}_u$ be the integral closure of $\mathbb{A}$ in K_u. If we write $u = f(t) + \frac{B_1}{A}$ with $f(t) \in \mathbb{A}$ and $\deg(B_1) < \deg(A)$, then $f(t)$ and A are uniquely determined by the field K_u. Also, if K is a $\mathbb{Z}/p\mathbb{Z}$-extension of k, then there exists such a normalized $u \in k$ such that $K = K_u$.

Let $G(K) = A$ be the denominator of u as above. The discriminant d_u of $\mathcal{O}_u$ over $\mathbb{A}$ is $(A \cdot \mathrm{rad}(A))^{p-1}$, where $\mathrm{rad}(A)$ denotes the product of the distinct monic irreducible divisors of A (see [1, Corollary 2.7]). The local discriminant d_{∞_k} at ∞_k is $\infty_k^{(p-1)(\deg(f)+1)}$ if $\deg(f) > 0$ and 1 otherwise. The discriminant d_{K_u} of K_u is defined to be $d_u \cdot d_{\infty_k}$. We say that the Artin-Schreier extension K/k is real, inert imaginary or ramified imaginary according as ∞_k splits completely, is inert or ramifies in K. Then, the extension K_u/k is real, inert imaginary or ramified imaginary according as $\deg(B) < \deg(A)$, $\deg(A) = \deg(B)$ or $\deg(A) < \deg(B)$. (See [4] for details.)

2.2. L-functions and class number formulas. Fix an isomorphism $\psi : \mathbb{F}_p \to \mu_p$ sending 1 to a primitive p-th root ζ_p of unity, where μ_p is the group of p-th roots of unity in $\mathbb{C}$. For $u \in k$ and $P \in \mathcal{P}$ which is prime to the denominator of u, define $[u, P) \in \mathbb{F}_p$ by $(P, K_u/k)(\alpha_u) = \alpha_u + [u, P)$, where $(P, K_u/k)$ is the Artin automorphism at P. Extend this to $N \in \mathbb{A}^+$, which is prime to the denominator of u, by multiplicativity. For any $N \in \mathbb{A}^+$, define $\{\frac{u}{N}\} = \psi([u, N))$ if N is prime to the denominator of u and $\{\frac{u}{N}\} = 0$ otherwise. Let χ_u be the character defined by $\chi_u(N) = \{\frac{u}{N}\}$. For $0 \leq i \leq p-1$, the L-function $L(s, \chi_u^i)$ associated to χ_u^i is defined by

$$L(s, \chi_u^i) = \sum_{N \in \mathbb{A}^+} \frac{\chi_u^i(N)}{|N|^s} = \sum_{n=0}^{\infty} \sigma_n^{(i)}(u) q^{-ns} \quad \text{with} \quad \sigma_n^{(i)}(u) = \sum_{N \in \mathbb{A}_n^+} \chi_u^i(N).$$

It is well known that $L(s, \chi_u^i)$ is a polynomial in q^{-s} of degree $\deg(\mathrm{rad}(A)) + \deg(B) - 1$ or $\deg(A) + \deg(\mathrm{rad}(A)) - 1$ according as ∞_k ramifies in K_u or otherwise for $1 \leq i \leq p-1$.

Let $u = \frac{B}{P} \in k$ be a normalized one with $B \in \mathbb{A}$ and $P \in \mathcal{P}_m$ and K_u be the associated Artin-Schreier extension of k. Let h_u and R_u be the ideal class number and regulator of $\mathcal{O}_u$, respectively. Since $d_u = P^{2(p-1)}$ and d_{∞_k} is $\infty_k^{(p-1)(c+1)}$ if $c = \deg(B) - \deg(P) > 0$ and 1 otherwise, by [1, Proposition 5.1], we have

$$\prod_{i=1}^{p-1} L(1, \chi_u^i) = \begin{cases} (q-1)^{p-1} q^{(1-p)m} h_u R_u & \text{if } \infty_k \text{ splits,} \\ \frac{q^p - 1}{q-1} p^{-1} q^{(1-p)m} h_u & \text{if } \infty_k \text{ is inert,} \\ q^{\frac{(p-1)((p-1)(c+1)-2)}{2} - (p-1)m} h_u & \text{if } \infty_k \text{ is ramified.} \end{cases} \tag{2.1}$$

3. Average value of ideal class numbers of Artin-Schreier extensions

In this section, when $p = 3$, we study the averages of class numbers of maximal orders of real/inert imaginary/ramified imaginary Artin-Schreier extensions K_u of k with monic irreducible $G(K_u)$, respectively.

3.1. Real Artin-Schreier extension. For $P \in \mathcal{P}$, let $\mathfrak{F}_P = \{B \in \mathbb{A} : B \neq 0, \deg(B) < \deg(P)\}$ and $\mathcal{F}_P$ be the set of real Artin-Schreier extensions K of k with $G(K) = P$. It is easy to show that for any $B_1, B_2 \in \mathfrak{F}_P$, $K_{B_1/P} = K_{B_2/P}$ if and only if $B_1 = B_2$. Hence, the map $B \mapsto K_{B/P}$ is a bijection from $\mathfrak{F}_P$ onto $\mathcal{F}_P$.

THEOREM 3.1. *Assume that $p = 3$. Then we have*

$$\frac{\sum_{P\in\mathcal{P}_m}\sum_{B\in\mathfrak{F}_P} h_{B/P}R_{B/P}}{(q^m-1)\#\mathcal{P}_m} = \frac{\zeta_{\mathbb{A}}(2)^3\zeta_{\mathbb{A}}(3)^2}{\zeta_{\mathbb{A}}(6)}\, q^{2(m-1)} + O\left(16^m q^{\frac{3m}{2}}\right).$$

Proof. Let

$$F_m(s) = \frac{\sum_{P\in\mathcal{P}_m}\sum_{B\in\mathfrak{F}_P}\prod_{i=1}^{2} L(s,\chi^i_{B/P})}{(q^m-1)\#\mathcal{P}_m}.$$

Since $L(s,\chi^i_{B/P})$ is a polynomial in q^{-s} of degree $2m-1$, we have

$$\prod_{i=1}^{2} L(s,\chi^i_{B/P}) = \sum_{n=0}^{4m-2}\sum_{\substack{n_1+n_2=n\\ N_1\in\mathbb{A}^+_{n_1},N_2\in\mathbb{A}^+_{n_2}}} \chi_{B/P}(N_1N_2^2)q^{-ns}.$$

Then,

$$F_m(s) = \frac{\sum_{n=0}^{4m-2}\sum_{P\in\mathcal{P}_m}\sum_{B\in\mathfrak{F}_P} a_n(B/P)q^{-ns}}{(q^m-1)\#\mathcal{P}_m}$$

with

$$a_n(B/P) = \sum_{\substack{n_1+n_2=n\\ N_1\in\mathbb{A}^+_{n_1},N_2\in\mathbb{A}^+_{n_2}}} \chi_{B/P}(N_1N_2^2).$$

Following the arguments of [7, §2], we have $|a_n(B/P)| \le \binom{4m-2}{n} q^{\frac{n}{2}}$ and

$$\left|\sum_{n=m}^{4m-2} a_n(B/P)q^{-ns}\right| \le \frac{2^{4m-2}q^{m(\frac{1}{2}-\sigma)}}{1-q^{\frac{1}{2}-\sigma}}$$

for $s \in \mathbb{C}$ with $\sigma = \mathrm{Re}(s) > \frac{1}{2}$. Thus, we have

$$\left| \frac{\sum_{n=m}^{4m-2} \sum_{P \in \mathcal{P}_m} \sum_{B \in \mathfrak{F}_P} a_n(B/P) q^{-ns}}{(q^m - 1)\#\mathcal{P}_m} \right| \leq \frac{2^{4m-2} q^{m(\frac{1}{2}-\sigma)}}{1 - q^{\frac{1}{2}-\sigma}}. \tag{3.1}$$

In particular, when $s = 1$, the summation is less than or equal to $16^m q^{-\frac{m}{2}}$.

Now, we consider

$$\begin{aligned} f_m(s) &= \frac{\sum_{n=0}^{m-1} \sum_{P \in \mathcal{P}_m} \sum_{B \in \mathfrak{F}_P} a_n(B/P) q^{-ns}}{(q^m - 1)\#\mathcal{P}_m} \\ &= \frac{\sum_{n=0}^{m-1} \sum_{\substack{n_1+n_2=n \\ N_1 \in \mathbb{A}_{n_1}^+, N_2 \in \mathbb{A}_{n_2}^+}} \sum_{P \in \mathcal{P}_m} \sum_{B \in \mathfrak{F}_P} \chi_{B/P}(N_1 N_2^2) q^{-ns}}{(q^m - 1)\#\mathcal{P}_m}. \end{aligned}$$

Note that $P \nmid N_1 N_2$ for any $N_1 \in \mathbb{A}_{n_1}^+, N_2 \in \mathbb{A}_{n_2}^+$ with $n_1 + n_2 = n$ since $n < m$. By definition, if $N_1 N_2^2$ is cube, we have $\chi_{B/P}(N_1 N_2^2) = 1$. If $N_1 N_2^2$ is not cube, by [6, Corollary 2.2], we have

$$\sum_{B \in \mathfrak{F}_P} \chi_{B/P}(N_1 N_2^2) = -1.$$

Hence, we have $f_m(s) = f_m^{(1)}(s) + f_m^{(2)}(s)$ with

$$f_m^{(1)}(s) = \sum_{n=0}^{m-1} \sum_{\substack{n_1+n_2=n \\ N_1 \in \mathbb{A}_{n_1}^+, N_2 \in \mathbb{A}_{n_2}^+ \\ N_1 N_2^2 : \text{cube}}} q^{-ns}$$

and

$$f_m^{(2)}(s) = -\frac{1}{q^m - 1} \sum_{n=0}^{m-1} \sum_{\substack{n_1+n_2=n \\ N_1 \in \mathbb{A}_{n_1}^+, N_2 \in \mathbb{A}_{n_2}^+ \\ N_1 N_2^2 : \text{not cube}}} q^{-ns}.$$

For $s \in \mathbb{C}$ with $\sigma = \mathrm{Re}(s) > \frac{1}{3}$, as $m \to \infty$, we have

$$\begin{aligned} \left| f_m^{(2)}(s) \right| &\leq \frac{1}{q^m - 1} \sum_{n=0}^{m-1} \sum_{\substack{n_1+n_2=n \\ N_1 \in \mathbb{A}_{n_1}^+, N_2 \in \mathbb{A}_{n_2}^+}} q^{-n\sigma} \\ &= \frac{1}{1 - q^{-m}} \left(\frac{q^{-m} - q^{-m\sigma}}{(1 - q^{1-\sigma})^2} - \frac{m q^{-m\sigma}}{1 - q^{1-\sigma}} \right) \to 0. \end{aligned} \tag{3.2}$$

Let $L(s)=\sum_{(N_1,N_2)}|N_1|^{-s}|N_2|^{-s}$, where (N_1,N_2) runs over monic polynomials $N_1,N_2\in\mathbb{A}^+$ such that $P\nmid N_1N_2$ and $N_1N_2^2$ is a cube. Then, as in [7, §2], we have $L(s)=\frac{\zeta_{\mathbb{A}}(2s)\zeta_{\mathbb{A}}(3s)^2}{\zeta_{\mathbb{A}}(6s)}$ and

$$\left|f_m^{(1)}(s)-L(s)\right|\le Cm^2q^{\frac{m}{3}(1-3\sigma)} \tag{3.3}$$

for some constant C and $\sigma=\mathrm{Re}(s)>\frac{1}{3}$. Hence, by (3.2) and (3.3), for $s\in\mathbb{C}$ with $\sigma=\mathrm{Re}(s)>\frac{1}{3}$, we have

$$f_m(s)=\frac{\zeta_{\mathbb{A}}(2s)\zeta_{\mathbb{A}}(3s)^2}{\zeta_{\mathbb{A}}(6s)}+O\left(m^2q^{\frac{m}{3}(1-3\sigma)}\right). \tag{3.4}$$

Finally, since $Cm^2q^{-\frac{2m}{3}}=o(16^mq^{-\frac{m}{2}})$, by (3.1) and (3.4), we have

$$F_m(1)=\frac{\zeta_{\mathbb{A}}(2)\zeta_{\mathbb{A}}(3)^2}{\zeta_{\mathbb{A}}(6)}+O\left(16^mq^{-\frac{m}{2}}\right). \tag{3.5}$$

Hence, by (2.1) and (3.5), we have

$$\frac{\sum_{P\in\mathcal{P}_m}\sum_{B\in\mathfrak{F}_P}h_{B/P}R_{B/P}}{(q^m-1)\#\mathcal{P}_m}=\frac{\zeta_{\mathbb{A}}(2)\zeta_{\mathbb{A}}(3)^2}{(q-1)^2\zeta_{\mathbb{A}}(6)}q^{2m}+O(16^mq^{\frac{3m}{2}}).$$

Since $\frac{1}{q-1}=\zeta_{\mathbb{A}}(2)q^{-1}$, we get the result. □

3.2. Inert imaginary Artin-Schreier extension. Let $\{0,\xi_1,\dots,\xi_{p-1}\}$ be a set of representatives of $\mathbb{F}_q/\wp(\mathbb{F}_q)$. For $P\in\mathcal{P}$, let $\mathfrak{G}_P=\{\xi_aP+B: B\in\mathfrak{F}_P, 1\le a\le p-1\}$ and $\mathcal{G}_P$ be the set of inert imaginary Artin-Schreier extensions K of k with $G(K)=P$. For any $B_1,B_2\in\mathfrak{F}_P$ and $1\le a,b\le p-1$, $K_{\xi_a+B_1/P}=K_{\xi_b+B_2/P}$ if and only if $a=b$ and $B_1=B_2$. Thus, the map $\xi_aP+B\mapsto K_{\xi_a+B/P}$ is a bijection from $\mathfrak{G}_P$ onto $\mathcal{G}_P$.

Theorem 3.2. *Assume that $p=3$. Then we have*

$$\frac{\sum_{P\in\mathcal{P}_m}\sum_{B\in\mathfrak{F}_P}\sum_{a=1}^{2}h_{\xi_a+B/P}}{2(q^m-1)\#\mathcal{P}_m}=\frac{3\zeta_{\mathbb{A}}(3)^2\zeta_{\mathbb{A}}(4)}{\zeta_{\mathbb{A}}(6)}q^{2(m-1)}+O(16^mq^{\frac{3m}{2}}).$$

Proof. For a positive integer m and $a\in\{1,2\}$, let

$$G_{m,a}(s)=\frac{\sum_{P\in\mathcal{P}_m}\sum_{B\in\mathfrak{F}_P}\prod_{i=1}^{2}L(s,\chi^i_{\xi_a+B/P})}{2(q^m-1)\#\mathcal{P}_m},$$

and $G_m(s) = G_{m,1}(s) + G_{m,2}(s)$. Since $L(s, \chi^i_{\xi_a+B/P})$ is a polynomial in q^{-s} of degree $2m-1$, we have

$$\prod_{i=1}^{2} L(s, \chi^i_{\xi_a+B/P}) = \sum_{n=0}^{4m-2} \sum_{\substack{n_1+n_2=n \\ N_1\in\mathbb{A}^+_{n_1}, N_2\in\mathbb{A}^+_{n_2}}} \chi_{\xi_a+B/P}(N_1N_2^2)q^{-ns}.$$

Then,

$$G_{m,a}(s) = \frac{\sum_{n=0}^{4m-2}\sum_{P\in\mathcal{P}_m}\sum_{B\in\mathfrak{F}_P} a_n(\xi_a+B/P)q^{-ns}}{2(q^m-1)\#\mathcal{P}_m}$$

with

$$a_n(\xi_a+B/P) = \sum_{\substack{n_1+n_2=n \\ N_1\in\mathbb{A}^+_{n_1}, N_2\in\mathbb{A}^+_{n_2}}} \chi_{\xi_a+B/P}(N_1N_2^2).$$

Following the arguments of [7, §2], we have $|a_n(\xi_a+B/P)| \le \binom{4m-2}{n} q^{\frac{n}{2}}$ and

$$\left|\sum_{n=m}^{4m-2} a_n(\xi_a+B/P)q^{-ns}\right| \le \frac{2^{4m-2}q^{m(\frac{1}{2}-\sigma)}}{1-q^{\frac{1}{2}-\sigma}}$$

for $\sigma = \mathrm{Re}(s) > \frac{1}{2}$. Thus, we have

$$\left|\frac{\sum_{n=m}^{4m-2}\sum_{P\in\mathcal{P}_m}\sum_{B\in\mathfrak{F}_P} a_n(\xi_a+B/P)q^{-ns}}{2(q^m-1)\#\mathcal{P}_m}\right| \le \frac{2^{4m-2}q^{m(\frac{1}{2}-\sigma)}}{1-q^{\frac{1}{2}-\sigma}}. \tag{3.6}$$

In particular, when $s = 1$, the summation is less than or equal to $16^m q^{-\frac{m}{2}}$.

Now, we consider

$$\begin{aligned} g_{m,a}(s) &= \frac{\sum_{n=0}^{m-1}\sum_{P\in\mathcal{P}_m}\sum_{B\in\mathfrak{F}_P} a_n(\xi_a+B/P)q^{-ns}}{2(q^m-1)\#\mathcal{P}_m} \\ &= \frac{\sum_{n=0}^{m-1}\sum_{\substack{n_1+n_2=n \\ N_1\in\mathbb{A}^+_{n_1}, N_2\in\mathbb{A}^+_{n_2}}}\sum_{P\in\mathcal{P}_m}\sum_{B\in\mathfrak{F}_P} \chi_{\xi_a+B/P}(N_1N_2^2)q^{-ns}}{2(q^m-1)\#\mathcal{P}_m}. \end{aligned}$$

Note that $P \nmid N_1N_2$ for any $N_1 \in \mathbb{A}^+_{n_1}, N_2 \in \mathbb{A}^+_{n_2}$ with $n_1+n_2 = n$ since $n < m$. By definition, if $N_1N_2^2$ is cube, we have $\chi_{\xi_a+B/P}(N_1N_2^2) = 1$. If $N_1N_2^2$ is not cube, by [6, Corollary 2.2], we have

$$\sum_{B\in\mathfrak{F}_P} \chi_{\xi_a+B/P}(N_1N_2^2) = \{\tfrac{\xi_a}{N_1N_2^2}\} \sum_{B\in\mathfrak{F}_P} \chi_{B/P}(N_1N_2^2) = -\{\tfrac{\xi_a}{N_1N_2^2}\}.$$

Hence, we have $g_{m,a}(s) = \frac{1}{2} f_m^{(1)}(s) + g_{m,a}^{(2)}(s)$ with

$$g_{m,a}^{(2)}(s) = -\frac{1}{2(q^m-1)} \sum_{n=0}^{m-1} \sum_{\substack{n_1+n_2=n \\ N_1 \in \mathbb{A}_{n_1}^+, N_2 \in \mathbb{A}_{n_2}^+ \\ N_1 N_2^2 : \text{not cube}}} \{\tfrac{\xi_a}{N_1 N_2^2}\} q^{-ns}.$$

For $s \in \mathbb{C}$ with $\sigma = \mathrm{Re}(s) > \frac{1}{3}$, as $m \to \infty$, we have

$$\left|g_{m,a}^{(2)}(s)\right| \le \frac{1}{q^m-1} \sum_{n=0}^{m-1} \sum_{\substack{n_1+n_2=n \\ N_1 \in \mathbb{A}_{n_1}^+, N_2 \in \mathbb{A}_{n_2}^+}} q^{-n\sigma} \to 0 \tag{3.7}$$

as in (3.2). Let $g_m(s) = g_{m,1}(s) + g_{m,2}(s) = f_m^{(1)}(s) + g_{m,1}^{(2)}(s) + g_{m,2}^{(2)}(s)$. Then, by (3.4) and (3.7), for $s \in \mathbb{C}$ with $\sigma = \mathrm{Re}(s) > \frac{1}{3}$, we have

$$g_m(s) = \frac{\zeta_{\mathbb{A}}(2s)\zeta_{\mathbb{A}}(3s)^2}{\zeta_{\mathbb{A}}(6s)} + O\left(m^2 q^{\frac{m}{3}(1-3\sigma)}\right). \tag{3.8}$$

Finally, since $Cm^2 q^{-\frac{2m}{3}} = o(16^m q^{-\frac{m}{2}})$, by (3.6) and (3.8), we have

$$G_m(1) = \frac{\zeta_{\mathbb{A}}(2)\zeta_{\mathbb{A}}(3)^2}{\zeta_{\mathbb{A}}(6)} + O\left(16^m q^{-\frac{m}{2}}\right). \tag{3.9}$$

Hence, by (2.1) and (3.9), we have

$$\frac{\sum_{P \in \mathcal{P}_m} \sum_{B \in \mathfrak{F}_P} \sum_{a=1}^{2} h_{\xi_a + B/P}}{2(q^m-1)\#\mathcal{P}_m} = \frac{3(q-1)}{q^3-1} \frac{\zeta_{\mathbb{A}}(2)\zeta_{\mathbb{A}}(3)^2}{\zeta_{\mathbb{A}}(6)} q^{2m} + O(16^m q^{\frac{3m}{2}}).$$

Since $q - 1 = \frac{q}{\zeta_{\mathbb{A}}(2)}$ and $\frac{1}{q^3-1} = \frac{\zeta_{\mathbb{A}}(4)}{q^3}$, we get the result. □

3.3. Ramified imaginary Artin-Schreier extension. For $P \in \mathcal{P}$ and positive integer c with $p \nmid c$, let $\mathfrak{H}_{P,c} = \{B \in \mathbb{A} : P \nmid B, \deg(B) = \deg(P) + c\}$ and $\mathcal{H}_{P,c}$ be the set of ramified imaginary Artin-Schreier extensions K of k with $G(K) = P$ and whose discriminant d_K is $P^{2(p-1)} \cdot \infty_k^{(p-1)(c+1)}$. For any $B, B' \in \mathfrak{H}_{P,c}$, we have that $K_{B/P} = K_{B'/P}$ if and only if $B' = B + P(D^p - D)$ for some $D \in \mathbb{A}$. We say that $B, B' \in \mathfrak{H}_{P,c}$ are equivalent, denoted by $B \sim B'$, if $B' = B + P(D^p - D)$ for some $D \in \mathbb{A}$. Let $[B]$ be the equivalence class of $B \in \mathfrak{H}_{p,c}$ with respect to $\sim$, and $\tilde{\mathfrak{H}}_{P,c} = \{[B] : B \in \mathfrak{H}_{P,c}\}$. Then, the map $[B] \mapsto K_{B/P}$ is a bijection from $\tilde{\mathfrak{H}}_{P,c}$ onto $\mathcal{H}_{P,c}$. For $B \in \mathfrak{H}_{P,c}$, we have a surjective map

$$\{D \in \mathbb{A} : \deg(D) \le [c/p]\} \to [B], \ D \mapsto B + P(D^p - D). \tag{3.10}$$

For $D, E \in \mathbb{A}$ with $\deg(D), \deg(E) \le [c/p]$, we have that $B + P(D^p - D) = B + P(E^p - E)$ if and only if $D - E \in \mathbb{F}_p$. Hence, the map in (3.10) is p to 1, so we have $\#[B] = \frac{q^{[c/p]}}{p}$. Since $\#\mathfrak{H}_{P,c} = \#\mathbb{A}_{\deg(P)+c} - \#\mathbb{A}_c = q^c(q-1)(q^{\deg(P)} - 1)$, we have

$$\#\tilde{\mathfrak{H}}_{P,c} = pq^{c-[c/p]}(q-1)(q^{\deg(P)} - 1).$$

THEOREM 3.3. *Assume that $p = 3$. Then we have*

$$\frac{\sum_{P\in\mathcal{P}_m}\sum_{[B]\in\tilde{\mathfrak{H}}_{P,c}} h_{B/P}}{\tilde{I}_q(m,c)} = \frac{\zeta_{\mathbb{A}}(2)\zeta_{\mathbb{A}}(3)^2}{\zeta_{\mathbb{A}}(6)} q^{2m+2c} + O\left(4^{2m+c} q^{\frac{3m}{2}+2c}\right),$$

where $\tilde{I}_q(m,c) = 3q^{c-[c/3]}(q-1)(q^m - 1)\#\mathcal{P}_m$.

Proof. For a positive integers m and c with $3 \nmid c$, let

$$H_{m,c}(s) = \frac{\sum_{P\in\mathcal{P}_m}\sum_{[B]\in\tilde{\mathfrak{H}}_{P,c}} \prod_{i=1}^{2} L(s, \chi^i_{B/P})}{\tilde{I}_q(m,c)}.$$

Since $L(s, \chi^i_{B/P})$ is a polynomial in q^{-s} of degree $2m + c - 1$, we have

$$\prod_{i=1}^{2} L(s, \chi^i_{B/P}) = \sum_{n=0}^{4m+2c-2} \sum_{\substack{n_1+n_2=n \\ N_1\in\mathbb{A}^+_{n_1}, N_2\in\mathbb{A}^+_{n_2}}} \chi_{B/P}(N_1N_2^2) q^{-ns}.$$

Then,

$$H_{m,c}(s) = \frac{3\sum_{n=0}^{4m+2c-2}\sum_{P\in\mathcal{P}_m}\sum_{B\in\mathfrak{H}_{P,c}} a_n(B/P) q^{-ns}}{q^{[c/3]}\tilde{I}_q(m,c)}$$

with

$$a_n(B/P) = \sum_{\substack{n_1+n_2=n \\ N_1\in\mathbb{A}^+_{n_1}, N_2\in\mathbb{A}^+_{n_2}}} \chi_{B/P}(N_1N_2^2).$$

Following the arguments of [7, §2], we have $|a_n(B/P)| \le \binom{4m+2c-2}{n} q^{\frac{n}{2}}$ and

$$\left|\sum_{n=m}^{4m+2c-2} a_n(B/P) q^{-ns}\right| \le 2^{4m+2c-2} q^{m(\frac{1}{2}-\sigma)}(1 - q^{\frac{1}{2}-\sigma})^{-1}$$

for $\sigma = \mathrm{Re}(s) > \frac{1}{3}$. Thus, we have

(3.11)

$$\left| \frac{3\sum_{n=m}^{4m+2c-2} \sum_{P\in\mathcal{P}_m} \sum_{B\in\mathfrak{H}_{P,c}} a_n(B/P)q^{-ns}}{q^{[c/3]}\tilde{I}_q(m,c)} \right| \le \frac{2^{4m+2c-2}q^{m(\frac{1}{2}-\sigma)}}{1-q^{\frac{1}{2}-\sigma}}.$$

In particular, when $s = 1$, the summation is less than or equal to $4^{2m+c}q^{-\frac{m}{2}}$.

Now, we consider

$$\begin{aligned} h_{m,c}(s) &= \frac{3\sum_{n=0}^{m-1} \sum_{P\in\mathcal{P}_m} \sum_{B\in\mathfrak{H}_{P,c}} a_n(B/P)q^{-ns}}{q^{[c/3]}\tilde{I}_q(m,c)} \\ &= \frac{3\sum_{n=0}^{m-1} \sum_{\substack{n_1+n_2=n \\ N_1\in\mathbb{A}^+_{n_1}, N_2\in\mathbb{A}^+_{n_2}}} \sum_{P\in\mathcal{P}_m} \sum_{B\in\mathfrak{H}_{P,c}} \chi_{B/P}(N_1N_2^2)q^{-ns}}{q^{[c/3]}\tilde{I}_q(m,c)}. \end{aligned}$$

Note that $P \nmid N_1N_2$ for any $N_1 \in \mathbb{A}^+_{n_1}, N_2 \in \mathbb{A}^+_{n_2}$ with $n_1 + n_2 = n$ since $n < m$. By definition, if $N_1N_2^2$ is cube, we have $\chi_{B/P}(N_1N_2^2) = 1$. If $N_1N_2^2$ is not cube, by [6, Corollary 2.4], we have

$$\sum_{B\in\mathfrak{H}_{P,c}} \chi_{B/P}(N_1N_2^2) = -\sum_{C\in\mathbb{A}_c} \left\{\frac{C}{N_1N_2^2}\right\}.$$

Hence, we have $h_{m,c}(s) = f_m^{(1)}(s) + h_{m,c}^{(2)}(s)$ with

$$h^{(2)}_{m,c}(s) = -\frac{3}{q^{[c/3]}\tilde{I}_q(m,c)} \sum_{n=0}^{m-1} \sum_{\substack{n_1+n_2=n \\ N_1\in\mathbb{A}^+_{n_1}, N_2\in\mathbb{A}^+_{n_2} \\ N_1N_2^2:\text{not cube}}} \sum_{P\in\mathcal{P}_m} \sum_{C\in\mathbb{A}_c} \{\tfrac{C}{N_1N_2^2}\}q^{-ns}.$$

For $\sigma = \mathrm{Re}(s) > \frac{1}{2}$, by using the fact that $\sum_{C\in\mathbb{A}_c} |\{\frac{C}{N_1N_2^2}\}| \le \#\mathbb{A}_c = (q-1)q^c$, we have

$$\begin{aligned} |h^{(2)}_{m,c}(s)| &\le \frac{3}{q^{[c/3]}\tilde{I}_q(m,c)} \sum_{n=0}^{m-1} \sum_{\substack{n_1+n_2=n \\ N_1\in\mathbb{A}^+_{n_1}, N_2\in\mathbb{A}^+_{n_2}}} \sum_{P\in\mathcal{P}_m} \sum_{C\in\mathbb{A}_c} \left|\left\{\frac{C}{N_1N_2^2}\right\}\right| q^{-n\sigma} \\ &\le \frac{1}{q^m-1} \sum_{n=0}^{m-1} q^{-n\sigma} \sum_{\substack{n_1+n_2=n \\ N_1\in\mathbb{A}^+_{n_1}, N_2\in\mathbb{A}^+_{n_2}}} 1 = \frac{1}{q^m-1} \sum_{n=0}^{m-1} (n+1)q^{(1-\sigma)n}. \end{aligned}$$

If $\sigma = 1$, we have

$$\frac{1}{q^m - 1}\sum_{n=0}^{m-1}(n+1)q^{(1-\sigma)n} = \frac{m(m+1)}{2(q^m-1)} \to 0$$

and if $\sigma \neq 1$, we have

$$\frac{1}{q^m - 1}\sum_{n=0}^{m-1}(n+1)q^{(1-\sigma)n} = \frac{1}{1-q^{-m}}\left(\frac{q^{-m}-q^{-\sigma m}}{(1-q^{1-\sigma})^2} - \frac{mq^{-\sigma m}}{1-q^{1-\sigma}}\right) \to 0$$

as $m \to \infty$. Hence, for $\sigma = \mathrm{Re}(s) > \frac{1}{2}$, we have

$$h^{(2)}_{m,c}(s) \to 0 \tag{3.12}$$

as $m \to \infty$. Hence, by (3.4) and (3.12), for $s \in \mathbb{C}$ with $\sigma = \mathrm{Re}(s) > \frac{1}{3}$, we have

$$h_{m,c}(s) = \frac{\zeta_{\mathbb{A}}(2s)\zeta_{\mathbb{A}}(3s)^2}{\zeta_{\mathbb{A}}(6s)} + O\left(m^2 q^{\frac{m}{3}(1-3\sigma)}\right). \tag{3.13}$$

Finally, since $Cm^2q^{-\frac{2m}{3}} = o(4^{2m+c}q^{-\frac{m}{2}})$, we have

$$H_{m,c}(1) = \frac{\zeta_{\mathbb{A}}(2)\zeta_{\mathbb{A}}(3)^2}{\zeta_{\mathbb{A}}(6)} + O(4^{2m+c}q^{-\frac{m}{2}}). \tag{3.14}$$

Hence, by (2.1) and (3.14), we have

$$\frac{\sum_{P\in\mathcal{P}_m}\sum_{[B]\in\widetilde{\mathfrak{H}}_{P,c}} h_{B/P}}{\tilde{I}_q(m,c)} = \frac{\zeta_{\mathbb{A}}(2)\zeta_{\mathbb{A}}(3)^2}{\zeta_{\mathbb{A}}(6)}q^{2m+2c} + O\left(4^{2m+c}q^{\frac{3m}{2}+2c}\right).$$

□

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