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INTERVAL-VALUED INTUITIONISTIC SMOOTH TOPOLOGICAL SPACES

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Abstract. In this paper, we introduce the concepts of several types of interval-valued intuitionistic fuzzy mappings and several types of interval-valued intuitionistic fuzzy compactness in interval-valued intuitionistic smooth topological spaces and then investigate their properties.

1. Introduction

After Zadeh [15] introduced the concept of fuzzy sets, there have been various generalizations of the concept of fuzzy sets. Chang [4] defined fuzzy topological spaces and Coker [6] defined intuitionistic fuzzy topological spaces. In their definitions of fuzzy topology and intuitionistic fuzzy topology, fuzzyness in the concept of openness of fuzzy subsets and intuitionistic fuzzy subsets was absent. Chattopadhyay, Hazra and Samanta [5,7] introduced the concept of gradation of openness of fuzzy subsets. Zadeh [16] introduced the concept of interval-valued fuzzy sets and Atanassov [1] introduced the concept of intuitionistic fuzzy sets. Atanassov and Gargov [2] introduced the concept of interval-valued intuitionistic fuzzy sets which is a generalization of both interval-valued

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fuzzy sets and intuitionistic fuzzy sets. Mondal and Samanta [8,14] introduced the concept of intuitionistic gradation of openness and defined intuitionistic fuzzy topological spaces. In [11], we defined interval-valued intuitionistic smooth topological spaces. Ramadan, Abbas and Abd El-Latif [13] introduced the concepts of several types of fuzzy continuous mappings and several types of fuzzy compactness in intuitionistic fuzzy topological spaces.

In this paper, we introduce the concepts of several types of interval-valued intuitionistic fuzzy mappings and several types of interval-valued intuitionistic fuzzy compactness in interval-valued intuitionistic smooth topological spaces and then investigate their properties.

2. Preliminaries

Throughout this paper, let X be a nonempty set, $I = [0,1]$, $I_0 = (0,1]$ and $I_1 = [0,1)$. The family of all fuzzy sets of X will be denoted by I^X. By 0_X and 1_X we denote the characteristic functions of ϕ and X, respectively. For any $A \in I^X$, A^c denotes the complement of A, i.e., $A^c = 1_X - A$.

Definition 2.1.[3,5,12]. A *gradation of openness* (for short, GO) on X, which is also called a *smooth topology* on X, is a mapping $\tau : I^X \to I$ satisfying the following conditions:

(O1) $\tau(0_X) = \tau(1_X) = 1$,

(O2) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$ for each $A, B \in I^X$,

(O3) $\tau(\cup_{i\in\Gamma} A_i) \geq \wedge_{i\in\Gamma} \tau(A_i)$ for each subfamily $\{A_i : i \in \Gamma\} \subseteq I^X$.

The pair (X, τ) is called a *smooth topological space* (for short, STS).

Definition 2.2.[8]. An *intuitionistic gradation of openness* (for short, IGO) on X, which is also called an *intuitionistic smooth topology* on X, is an ordered pair (τ, τ^*) of mappings from I^X to I satisfying the following conditions:

(IGO1) $\tau(A) + \tau^*(A) \leq 1$ for each $A \in I^X$,

(IGO2) $\tau(0_X) = \tau(1_X) = 1$ and $\tau^*(0_X) = \tau^*(1_X) = 0$,

(IGO3) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$ and $\tau^*(A \cap B) \leq \tau^*(A) \vee \tau^*(B)$ for each $A, B \in I^X$,

(IGO4) $\tau(\cup_{i\in\Gamma} A_i) \geq \wedge_{i\in\Gamma} \tau(A_i)$ and $\tau^*(\cup_{i\in\Gamma} A_i) \leq \vee_{i\in\Gamma} \tau^*(A_i)$ for each subfamily $\{A_i : i \in \Gamma\} \subseteq I^X$.

The triple (X, τ, τ^*) is called an *intuitionistic smooth topological space* (for short, ISTS). τ and τ^* may be interpreted as gradation of openness and gradation of nonopenness, respectively.

Definition 2.3.[8]. Let (X, τ, τ^*) and (Y, η, η^*) be two ISTSs and let $f : X \to Y$ be a mapping. Then f is called a *gradation preserving mapping* (for short, a GP-mapping) if for each $A \in I^Y$, $\eta(A) \leq \tau(f^{-1}(A))$ and $\eta^*(A) \geq \tau^*(f^{-1}(A))$.

Let $D(I)$ be the set of all closed subintervals of the unit interval I. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$ and $M = [M^L, M^U]$, where M^L and M^U are respectively the lower and the upper end points. Especially, we denote $\mathbf{r} = [r.r]$ for each $r \in I$. The complement of M, denoted by M^c, is defined by $M^c = 1 - M = [1 - M^U, 1 - M^L]$. Note that $M = N$ iff $M^L = N^L$ and $M^U = N^U$ and that $M \leq N$ iff $M^L \leq N^L$ and $M^U \leq N^U$.

Definition 2.4.[17]. A mapping $A = [A^L, A^U] : X \to D(I)$ is called an *interval-valued fuzzy set* (for short, IVFS) on X, where $A(x) = [A^L(x), A^U(x)]$ for each $x \in X$. $A^L(x)$ and $A^U(x)$ are called the *lower* and *upper end points* of $A(x)$, respectively.

Definition 2.5.[9]. Let A and B be IVFSs on X. Then
(i) $A = B$ iff $A^L(x) = B^L(x)$ and $A^U(x) = B^U(x)$ for all $x \in X$.
(ii) $A \subseteq B$ iff $A^L(x) \leq B^L(x)$ and $A^U(x) \leq B^U(x)$ for all $x \in X$.
(iii) The *complement* A^c of A is defined by $A^c(x) = [1 - A^U(x), 1 - A^L(x)]$ for all $x \in X$.
(iv) For a family of IVFSs $\{A_i : i \in \Gamma\}$, the union $\cup_{i\in\Gamma} A_i$ and the intersection $\cap_{i\in\Gamma} A_i$ are respectively defined by

$$\cup_{i\in\Gamma} A_i(x) = [\vee_{i\in\Gamma} A_i^L(x), \vee_{i\in\Gamma} A_i^U(x)],$$
$$\cap_{i\in\Gamma} A_i(x) = [\wedge_{i\in\Gamma} A_i^L(x), \wedge_{i\in\Gamma} A_i^U(x)]$$

for all $x \in X$.

Definition 2.6.[2]. A mapping $A = (\mu_A, \nu_A) : X \to D(I) \times D(I)$ is called an *interval-valued intuitionistic fuzzy set* (for short, IVIFS) on X, where $\mu_A : X \to D(I)$ and $\nu_A : X \to D(I)$ are interval-valued fuzzy sets on X with the condition $\sup_{x\in X} \mu_A^U(x) + \sup_{x\in X} \nu_A^U(x) \leq 1$. The intervals $\mu_A(x) = [\mu_A^L(x), \mu_A^U(x)]$ and $\nu_A(x) = [\nu_A^L(x), \nu_A^U(x)]$ denote the degree of belongingness and the degree of nonbelongingness of the element x to the set A, respectively.

DEFINITION 2.7.[10]. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IVIFSs on X. Then

(i) $A \subseteq B$ iff $\mu_A^L(x) \leq \mu_B^L(x)$, $\mu_A^U(x) \leq \mu_B^U(x)$ and $\nu_A^L(x) \geq \nu_B^L(x)$, $\nu_A^U(x) \geq \nu_B^U(x)$ for all $x \in X$.

(ii) $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

(iii) The *complement* A^c of A is defined by $\mu_{A^c}(x) = \nu_A(x)$ and $\nu_{A^c}(x) = \mu_A(x)$ for all $x \in X$.

(iv) For a family of IVIFSs $\{A_i : i \in \Gamma\}$, the union $\cup_{i\in\Gamma} A_i$ and the intersection $\cap_{i\in\Gamma} A_i$ are respectively defined by

$$\mu_{\cup_{i\in\Gamma} A_i}(x) = \cup_{i\in\Gamma}\mu_{A_i}(x), \nu_{\cup_{i\in\Gamma} A_i}(x) = \cap_{i\in\Gamma}\nu_{A_i}(x),$$
$$\mu_{\cap_{i\in\Gamma} A_i}(x) = \cap_{i\in\Gamma}\mu_{A_i}(x), \nu_{\cap_{i\in\Gamma} A_i}(x) = \cup_{i\in\Gamma}\nu_{A_i}(x)$$

for all $x \in X$.

3. Several types of interval-valued intuitionistic fuzzy mappings

DEFINITION 3.1.[11]. An *interval-valued intuitionistic gradation of openness* (for short, IVIGO) on X, which is also called an *interval-valued intuitionistic smooth topology* on X, is an ordered pair (τ, τ^*) of mappings $\tau = [\tau^L, \tau^U] : I^X \to D(I)$ and $\tau^* = [\tau^{*L}, \tau^{*U}] : I^X \to D(I)$ satisfying the following conditions:

(IVIGO1) $\tau^L(A) \leq \tau^U(A)$, $\tau^{*L}(A) \leq \tau^{*U}(A)$ and $\tau^U(A) + \tau^{*U}(A) \leq 1$ for each $A \in I^X$,

(IVIGO2) $\tau(0_X) = \tau(1_X) = \mathbf{1}$ and $\tau^*(0_X) = \tau^*(1_X) = \mathbf{0}$,

(IVIGO3) $\tau^L(A \cap B) \geq \tau^L(A) \wedge \tau^L(B)$, $\tau^U(A \cap B) \geq \tau^U(A) \wedge \tau^U(B)$ and $\tau^{*L}(A \cap B) \leq \tau^{*L}(A) \vee \tau^{*L}(B)$, $\tau^{*U}(A \cap B) \leq \tau^{*U}(A) \vee \tau^{*U}(B)$ for each $A, B \in I^X$,

(IVIGO4) $\tau^L(\cup_{i\in\Gamma}\ A_i) \geq \wedge_{i\in\Gamma}\ \tau^L(A_i)$, $\tau^U(\cup_{i\in\Gamma}\ A_i) \geq \wedge_{i\in\Gamma}\ \tau^U(A_i)$ and $\tau^{*L}(\cup_{i\in\Gamma}\ A_i) \leq \vee_{i\in\Gamma}\ \tau^{*L}(A_i)$, $\tau^{*U}(\cup_{i\in\Gamma}\ A_i) \leq \vee_{i\in\Gamma}\ \tau^{*U}(A_i)$ for each subfamily $\{A_i : i \in \Gamma\} \subseteq I^X$.

The triple (X, τ, τ^*) is called an *interval-valued intuitionistic smooth topological space* (for short, IVISTS). τ and τ^* may be interpreted as interval-valued gradation of openness and interval-valued gradation of nonopenness, respectively.

Definition 3.2.[11]. An *interval-valued intuitionistic gradation of closedness* (for short, IVIGC) on X, which is also called an *interval-valued intuitionistic smooth cotopology* on X, is an ordered pair $(\mathcal{F}, \mathcal{F}^*)$ of mappings $\mathcal{F} = [\mathcal{F}^L, \mathcal{F}^U] : I^X \to D(I)$ and $\mathcal{F}^* = [\mathcal{F}^{*L}, \mathcal{F}^{*U}] : I^X \to D(I)$ satisfying the following conditions:

(IVIGC1) $\mathcal{F}^L(A) \leq \mathcal{F}^U(A)$, $\mathcal{F}^{*L}(A) \leq \mathcal{F}^{*U}(A)$ and $\mathcal{F}^U(A) + \mathcal{F}^{*U}(A) \leq 1$ for each $A \in I^X$,

(IVIGC2) $\mathcal{F}(0_X) = \mathcal{F}(1_X) = \mathbf{1}$ and $\mathcal{F}^*(0_X) = \mathcal{F}^*(1_X) = \mathbf{0}$,

(IVIGC3) $\mathcal{F}^L(A \cup B) \geq \mathcal{F}^L(A) \wedge \mathcal{F}^L(B)$, $\mathcal{F}^U(A \cup B) \geq \mathcal{F}^U(A) \wedge \mathcal{F}^U(B)$ and $\mathcal{F}^{*L}(A \cup B) \leq \mathcal{F}^{*L}(A) \vee \mathcal{F}^{*L}(B)$, $\mathcal{F}^{*U}(A \cup B) \leq \mathcal{F}^{*U}(A) \vee \mathcal{F}^{*U}(B)$ for each $A, B \in I^X$,

(IVIGC4) $\mathcal{F}^L(\cap_{i\in\Gamma} A_i) \geq \wedge_{i\in\Gamma} \mathcal{F}^L(A_i)$, $\mathcal{F}^U(\cap_{i\in\Gamma} A_i) \geq \wedge_{i\in\Gamma} \mathcal{F}^U(A_i)$ and $\mathcal{F}^{*L}(\cap_{i\in\Gamma} A_i) \leq \vee_{i\in\Gamma} \mathcal{F}^{*L}(A_i)$, $\mathcal{F}^{*U}(\cap_{i\in\Gamma} A_i) \leq \vee_{i\in\Gamma} \mathcal{F}^{*U}(A_i)$ for each subfamily $\{A_i : i \in \Gamma\} \subseteq I^X$.

For an IVIGO (τ, τ^*) and an IVIGC $(\mathcal{F}, \mathcal{F}^*)$ on X, we define

$$\tau_{\mathcal{F}}(A) = \mathcal{F}(A^c),\ \tau^*_{\mathcal{F}^*}(A) = \mathcal{F}^*(A^c),$$
$$\mathcal{F}_\tau(A) = \tau(A^c),\ \mathcal{F}^*_{\tau^*}(A) = \tau^*(A^c)$$

for each $A \in I^X$.

Theorem 3.3.[11]. (i) *(τ, τ^*) is an IVIGO on X if and only if $(\mathcal{F}_\tau, \mathcal{F}^*_{\tau^*})$ is an IVIGC on X,*

(ii) *$(\mathcal{F}, \mathcal{F}^*)$ is an IVIGC on X if and only if $(\tau_{\mathcal{F}}, \tau^*_{\mathcal{F}^*})$ is an IVIGO on X,*

Definition 3.4. Let (X, τ, τ^*) be an IVISTS, $A \in I^X$ and $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$. Then the $([r, s],\ [t, u])$-*interval-valued intuitionistic fuzzy closure* and $([r, s],\ [t, u])$-*interval-valued intuitionistic fuzzy interior* of A are defined by

$cl_{[r,s],[t,u]}(A) = \cap\{K \in I^X : A \subseteq K,\ \mathcal{F}_\tau(K) \geq [r, s],\ \mathcal{F}^*_{\tau^*}(K) \leq [t, u]\}$,
$int_{[r,s],[t,u]}(A) = \cup\{G \in I^X : G \subseteq A,\ \tau(G) \geq [r, s],\ \tau^*(G) \leq [t, u]\}$.

Theorem 3.5. *Let (X, τ, τ^*) be an IVISTS, $A, B \in I^X$ and $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$. Then*

(i) *$A \subseteq cl_{[r,s],[t,u]}(A)$.*

(ii) *$int_{[r,s],[t,u]}(A) \subseteq A$.*

(iii) *$A = cl_{[r,s],[t,u]}(A)$ if $\mathcal{F}_\tau(A) \geq [r, s]$ and $\mathcal{F}^*_{\tau^*}(A) \leq [t, u]$.*

(iv) *$A = int_{[r,s],[t,u]}(A)$ if $\tau(A) \geq [r, s]$ and $\tau^*(A) \leq [t, u]$.*

(v) *$cl_{[r,s],[t,u]}(A) \subseteq cl_{[r',s'],[t',u']}(B)$ if $A \subseteq B$, $[r, s] \leq [r', s']$ and $[t, u] \geq [t', u']$.*

(vi) $int_{[r',s'],[t',u']}(A) \subseteq int_{[r,s],[t,u]}(B)$ *if* $A \subseteq B$, $[r,s] \leq [r',s']$ *and* $[t,u] \geq [t',u']$.
(vii) $cl_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(A)) = cl_{[r,s],[t,u]}(A)$.
(viii) $int_{[r,s],[t,u]}(int_{[r,s],[t,u]}(A)) = int_{[r,s],[t,u]}(A)$.
(ix) $cl_{[r,s],[t,u]}(A \cup B) = cl_{[r,s],[t,u]}(A) \cup cl_{[r,s],[t,u]}(B)$.
(x) $int_{[r,s],[t,u]}(A \cap B) = int_{[r,s],[t,u]}(A) \cap int_{[r,s],[t,u]}(B)$.
(xi) $(cl_{[r,s],[t,u]}(A))^c = int_{[r,s],[t,u]}(A^c)$.
(xii) $(int_{[r,s],[t,u]}(A))^c = cl_{[r,s],[t,u]}(A^c)$.

Proof. The proof is straightforward.

□

Definition 3.6.[11]. Let (X, τ, τ^*) and (Y, η, η^*) be two IVISTSs and let $f : X \to Y$ be a mapping. Then f is called an *interval-valued intuitionistic gradation preserving mapping* (for short, an IVIGP-mapping) if for each $A \in I^Y$, $\eta(A) \leq \tau(f^{-1}(A))$ and $\eta^*(A) \geq \tau^*(f^{-1}(A))$.

Definition 3.7. Let (X, τ, τ^*) and (Y, η, η^*) be two IVISTSs and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$ and let $f : X \to Y$ be a mapping. Then f is called a *weakly* $([r,s],[t,u])$-*interval-valued intuitionistic gradation preserving mapping* (for short, a weakly $([r,s],[t,u])$-IVIGP-mapping) if $\eta(A) \geq [r,s]$ and $\eta^*(A) \leq [t,u]$ implies $\tau(f^{-1}(A)) \geq [r,s]$ and $\tau^*(f^{-1}(A)) \leq [t,u]$ for each $A \in I^Y$.

Note that if $f : (X, \tau, \tau^*) \to (Y, \eta, \eta^*)$ is an IVIGP-mapping, then f is a weakly $([r,s],[t,u])$-IVIGP-mapping, where $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$.

Example 3.8. Every weakly $([r,s],[t,u])$-IVIGP-mapping need not be an IVIGP-mapping.

Let $X = \{a,b\}$ and $Y = \{1,2\}$. Define $G_1 \in I^X$ and $G_2 \in I^Y$ as follows:

$$G_1 = \{(a,0.4),(b,0.4)\},\ G_2 = \{(1,0.4),(2,0.5)\}.$$

Define $\tau, \tau^* : I^X \to D(I)$, $\eta, \eta^* : I^Y \to D(I)$ as follows:

$$\tau(A) = \begin{cases} \mathbf{1} & \text{if } A \in \{0_X, 1_X\}, \\ [0.7, 0.8] & \text{if } A = G_1, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

$$\tau^*(A) = \begin{cases} \mathbf{0} & \text{if } A \in \{0_X, 1_X\}, \\ [0.1, 0.2] & \text{if } A = G_1, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

$$\eta(A) = \begin{cases} \mathbf{1} & \text{if } A \in \{0_Y, 1_Y\}, \\ [0.8, 0.9] & \text{if } A = G_2, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

$$\eta^*(A) = \begin{cases} \mathbf{0} & \text{if } A \in \{0_Y, 1_Y\}, \\ [0.1, 0.2] & \text{if } A = G_2, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

Define the mapping $f : (X, \tau, \tau^*) \to (Y, \eta, \eta^*)$ by $f(a) = 1, f(b) = 2$ and let $[r, s] = [0.5, 0.6]$ and $[t, u] = [0.3, 0.4]$. Then f is a weakly $([r, s], [t, u])$-IVIGP-mapping, but f is not an IVIGP-mapping.

THEOREM 3.9. *Let (X, τ, τ^*) and (Y, η, η^*) be two IVISTSs and $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$ and let $f : X \to Y$ be a mapping. Then f is a weakly $([r, s], [t, u])$-IVIGP-mapping if and only if $f(cl_{[r,s],[t,u]}(A)) \subseteq cl_{[r,s],[t,u]}(f(A))$ for each $A \in I^X$.*

Proof. Suppose that f is a weakly $([r, s], [t, u])$-IVIGP-mapping. For each $A \in I^X$, we have

$$\begin{aligned}
& f^{-1}(cl_{[r,s],[t,u]}(f(A))) \\
&= f^{-1}(\cap\{K \in I^Y : f(A) \subseteq K,\ \mathcal{F}_\eta(K) \geq [r, s],\ \mathcal{F}^*_{\eta^*}(K) \leq [t, u]\}) \\
&= f^{-1}(\cap\{K \in I^Y : f(A) \subseteq K,\ \eta(K^c) \geq [r, s],\ \eta^*(K^c) \leq [t, u]\}) \\
&\supseteq f^{-1}(\cap\{K \in I^Y : f(A) \subseteq K,\ \tau(f^{-1}(K^c)) \geq [r, s],\ \tau^*(f^{-1}(K^c)) \leq [t, u]\}) \\
&= f^{-1}(\cap\{K \in I^Y : f(A) \subseteq K,\ \tau((f^{-1}(K))^c) \geq [r, s], \\
&\qquad \tau^*((f^{-1}(K))^c) \leq [t, u]\}) \\
&\supseteq f^{-1}(\cap\{K \in I^Y : A \subseteq f^{-1}(K),\ \mathcal{F}_\tau(f^{-1}(K)) \geq [r, s], \\
&\qquad \mathcal{F}^*_{\tau^*}(f^{-1}(K)) \leq [t, u]\}) \\
&= \cap\{f^{-1}(K) : K \in I^Y,\ A \subseteq f^{-1}(K),\ \mathcal{F}_\tau(f^{-1}(K)) \geq [r, s], \\
&\qquad \mathcal{F}^*_{\tau^*}(f^{-1}(K)) \leq [t, u]\} \\
&\supseteq \cap\{F \in I^X : A \subseteq F,\ \mathcal{F}_\tau(F) \geq [r, s],\ \mathcal{F}^*_{\tau^*}(F) \leq [t, u]\} \\
&= cl_{[r,s],[t,u]}(A).
\end{aligned}$$

Hence $f(cl_{[r,s],[t,u]}(A)) \subseteq f(f^{-1}(cl_{[r,s],[t,u]}(f(A)))) \subseteq cl_{[r,s],[t,u]}(f(A))$.

Conversely, suppose that $f(cl_{[r,s],[t,u]}(A)) \subseteq cl_{[r,s],[t,u]}(f(A))$ for each $A \in I^X$. Let $A \in I^Y$ with $\eta(A) \geq [r,s]$ and $\eta^*(A) \leq [t,u]$. Then $\mathcal{F}_\eta(A^c) = \eta(A) \geq [r,s]$ and $\mathcal{F}^*_{\eta^*}(A^c) = \eta^*(A) \leq [t,u]$ and so $cl_{[r,s],[t,u]}(A^c) = A^c$. By hypothesis, $f(cl_{[r,s],[t,u]}(f^{-1}(A^c))) \subseteq cl_{[r,s],[t,u]}(f(f^{-1}(A^c))) \subseteq cl_{[r,s],[t,u]}(A^c) = A^c$. Hence $cl_{[r,s],[t,u]}(f^{-1}(A^c)) \subseteq f^{-1}(f(cl_{[r,s],[t,u]}(f^{-1}(A^c)))) \subseteq f^{-1}(A^c)$. Thus $cl_{[r,s],[t,u]}(f^{-1}(A^c)) = f^{-1}(A^c)$. By Definition 3.4, $\mathcal{F}_\tau(f^{-1}(A^c)) \geq [r,s]$ and $\mathcal{F}^*_{\tau^*}(f^{-1}(A^c)) \leq [t,u]$ and so $\tau(f^{-1}(A)) \geq [r,s]$ and $\tau^*(f^{-1}(A)) \leq [t,u]$. Hence f is a weakly $([r,s],[t,u])$-IVIGP-mapping.

□

Corollary 3.10. *Let (X,τ,τ^*) and (Y,η,η^*) be two IVISTSs and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$ and let $f : X \to Y$ be a mapping. If f is a weakly $([r,s],[t,u])$-IVIGP-mapping, then*

(i) *$f(cl_{[r,s],[t,u]}(A)) \subseteq cl_{[r,s],[t,u]}(f(A))$ for each $A \in I^X$,*
(ii) *$cl_{[r,s],[t,u]}(f^{-1}(A)) \subseteq f^{-1}(cl_{[r,s],[t,u]}(A))$ for each $A \in I^Y$,*
(iii) *$f^{-1}(int_{[r,s],[t,u]}(A)) \subseteq int_{[r,s],[t,u]}(f^{-1}(A))$ for each $A \in I^Y$.*

Proof. (i) It follows from Theorem 3.9.

(ii) Let $A \in I^Y$. Then $f^{-1}(A) \in I^X$. By (i), we have

$$\begin{aligned} cl_{[r,s],[t,u]}(f^{-1}(A)) &\subseteq f^{-1}(f(cl_{[r,s],[t,u]}(f^{-1}(A)))) \\ &\subseteq f^{-1}(cl_{[r,s],[t,u]}(f(f^{-1}(A)))) \\ &\subseteq f^{-1}(cl_{[r,s],[t,u]}(A)). \end{aligned}$$

(iii) Let $A \in I^Y$. By (ii), we have

$$\begin{aligned} f^{-1}(int_{[r,s],[t,u]}(A)) &= (f^{-1}(cl_{[r,s],[t,u]}(A^c)))^c \\ &\subseteq (cl_{[r,s],[t,u]}(f^{-1}(A^c)))^c \\ &= int_{[r,s],[t,u]}(f^{-1}(A)). \end{aligned}$$

□

Definition 3.11. Let (X,τ,τ^*) and (Y,η,η^*) be two IVISTSs and let $f : X \to Y$ be a mapping. Then f is called an *interval-valued intuitionistic fuzzy open mapping* (for short, an IVIFO-mapping) if for each $A \in I^X$, $\eta(f(A)) \geq \tau(A)$ and $\eta^*(f(A)) \leq \tau^*(A)$.

Definition 3.12. Let (X,τ,τ^*) and (Y,η,η^*) be two IVISTSs and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$ and let $f : X \to Y$ be a mapping. Then f is called a weakly $([r,s],[t,u])$-*interval-valued intuitionistic fuzzy open mapping* (for short, a weakly $([r,s],[t,u])$-IVIFO-mapping) if $\tau(A) \geq [r,s]$ and $\tau^*(A) \leq [t,u]$ implies $\eta(f(A)) \geq [r,s]$ and $\eta^*(f(A)) \leq [t,u]$ for each $A \in I^X$.

Note that if $f : (X, \tau, \tau^*) \to (Y, \eta, \eta^*)$ is an IVIFO-mapping, then f is a weakly $([r,s],[t,u])$-IVIFO-mapping, where $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$.

Example 3.13. Every weakly $([r,s],[t,u])$-IVIFO-mapping need not be an IVIFO-mapping.

Let $X = \{a,b\}$ and $Y = \{1,2\}$. Define $G_1 \in I^X$ and $G_2 \in I^Y$ as follows:

$$G_1 = \{(a, 0.4), (b, 0.5)\},\ G_2 = \{(1, 0.4), (2, 0.5)\}.$$

Define $\tau, \tau^* : I^X \to D(I)$, $\eta, \eta^* : I^Y \to D(I)$ as follows:

$$\tau(A) = \begin{cases} \mathbf{1} & \text{if } A \in \{0_X, 1_X\}, \\ [0.8, 0.9] & \text{if } A = G_1, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

$$\tau^*(A) = \begin{cases} \mathbf{0} & \text{if } A \in \{0_X, 1_X\}, \\ [0.1, 0.2] & \text{if } A = G_1, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

$$\eta(A) = \begin{cases} \mathbf{1} & \text{if } A \in \{0_Y, 1_Y\}, \\ [0.7, 0.8] & \text{if } A = G_2, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

$$\eta^*(A) = \begin{cases} \mathbf{0} & \text{if } A \in \{0_Y, 1_Y\}, \\ [0.1, 0.2] & \text{if } A = G_2, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

Define the mapping $f : (X, \tau, \tau^*) \to (Y, \eta, \eta^*)$ by $f(a) = 1, f(b) = 2$ and let $[r,s] = [0.5, 0.6]$ and $[t,u] = [0.3, 0.4]$. Then f is a weakly $([r,s],[t,u])$-IVIFO-mapping, but f is not an IVIFO-mapping.

Theorem 3.14. *Let (X, τ, τ^*) and (Y, η, η^*) be two IVISTSs and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$ and let $f : X \to Y$ be a mapping. Then f is a weakly $([r,s],[t,u])$-IVIFO-mapping if and only if $f(int_{[r,s],[t,u]}(A)) \subseteq int_{[r,s],[t,u]}(f(A))$ for each $A \in I^X$.*

Proof. Suppose that f is a weakly $([r,s],[t,u])$-IVIFO-mapping. For each $A \in I^X$, we have

$$\begin{aligned}
&f(int_{[r,s],[t,u]}(A))\\
&= f(\cup\{G \in I^X : G \subseteq A,\ \tau(G) \geq [r,s],\ \tau^*(G) \leq [t,u]\})\\
&\subseteq \cup\{f(G) : G \in I^X,\ G \subseteq A,\ \tau(G) \geq [r,s],\ \tau^*(G) \leq [t,u]\}\\
&\subseteq \cup\{f(G) : G \in I^X,\ f(G) \subseteq f(A),\ \eta(f(G)) \geq [r,s],\ \eta^*(f(G)) \leq [t,u]\}\\
&\subseteq \cup\{U \in I^X : U \subseteq f(A),\ \eta(U) \geq [r,s],\ \eta^*(U) \leq [t,u]\}\\
&= int_{[r,s],[t,u]}(f(A)).
\end{aligned}$$

Thus $f(int_{[r,s],[t,u]}(A)) \subseteq int_{[r,s],[t,u]}(f(A))$.

Conversely, suppose that $f(int_{[r,s],[t,u]}(A)) \subseteq int_{[r,s],[t,u]}(f(A))$ for each $A \in I^X$. Let $A \in I^X$ with $\tau(A) \geq [r,s]$ and $\tau^*(A) \leq [t,u]$. Then $int_{[r,s],[t,u]}(A) = A$. By hypothesis, $f(A) = f(int_{[r,s],[t,u]}(A)) \subseteq int_{[r,s],[t,u]}(f(A))$. Hence $int_{[r,s],[t,u]}(f(A)) = f(A)$. By Definition 3.4, $\eta(f(A)) \geq [r,s]$ and $\eta^*(f(A)) \leq [t,u]$. Hence f is a weakly $([r,s],[t,u])$-IVIFO-mapping.

□

COROLLARY 3.15. *Let (X,τ,τ^*) and (Y,η,η^*) be two IVISTSs and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$ and let $f : X \to Y$ be a mapping. If f is a weakly $([r,s],[t,u])$-IVIFO-mapping, then*

(i) *$f(int_{[r,s],[t,u]}(A)) \subseteq int_{[r,s],[t,u]}(f(A))$ for each $A \in I^X$,*

(ii) *$int_{[r,s],[t,u]}(f^{-1}(A)) \subseteq f^{-1}(int_{[r,s],[t,u]}(A))$ for each $A \in I^Y$,*

(iii) *$f^{-1}(cl_{[r,s],[t,u]}(A)) \subseteq cl_{[r,s],[t,u]}(f^{-1}(A))$ for each $A \in I^Y$.*

Proof. (i) It follows from Theorem 3.14.

(ii) Let $A \in I^Y$. Then $f^{-1}(A) \in I^X$. By (i), we have

$$\begin{aligned}
f(int_{[r,s],[t,u]}(f^{-1}(A))) &\subseteq int_{[r,s],[t,u]}(f(f^{-1}(A)))\\
&\subseteq int_{[r,s],[t,u]}(A).
\end{aligned}$$

Hence we have

$$\begin{aligned}
int_{[r,s],[t,u]}(f^{-1}(A)) &\subseteq f^{-1}(f(int_{[r,s],[t,u]}(f^{-1}(A))))\\
&\subseteq f^{-1}(int_{[r,s],[t,u]}(A)).
\end{aligned}$$

(iii) Let $A \in I^Y$. By (ii), we have

$$\begin{aligned}(f^{-1}(cl_{[r,s],[t,u]}(A)))^c &= f^{-1}(int_{[r,s],[t,u]}(A^c)) \\ &\supseteq int_{[r,s],[t,u]}(f^{-1}(A^c)) \\ &= (cl_{[r,s],[t,u]}(f^{-1}(A)))^c.\end{aligned}$$

Hence $f^{-1}(cl_{[r,s],[t,u]}(A)) \subseteq cl_{[r,s],[t,u]}(f^{-1}(A))$.

□

Definition 3.16. Let (X, τ, τ^*) and (Y, η, η^*) be two IVISTSs and let $f : X \to Y$ be a mapping. Then f is called an *interval-valued intuitionistic fuzzy closed mapping* (for short, an IVIFC-mapping) if for each $A \in I^X$, $\mathcal{F}_\eta(f(A)) \geq \mathcal{F}_\tau(A)$ and $\mathcal{F}^*_{\eta^*}(f(A)) \leq \mathcal{F}^*_{\tau^*}(A)$.

Definition 3.17. Let (X, τ, τ^*) and (Y, η, η^*) be two IVISTSs and $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$ and let $f : X \to Y$ be a mapping. Then f is called a weakly $([r, s], [t, u])$*-interval-valued intuitionistic fuzzy closed mapping* (for short, a weakly $([r, s], [t, u])$-IVIFC-mapping) if $\mathcal{F}_\tau(A) \geq [r, s]$ and $\mathcal{F}^*_{\tau^*}(A) \leq [t, u]$ implies $\mathcal{F}_\eta(f(A)) \geq [r, s]$ and $\mathcal{F}^*_{\eta^*}(f(A)) \leq [t, u]$ for each $A \in I^X$.

Note that if $f : (X, \tau, \tau^*) \to (Y, \eta, \eta^*)$ is an IVIFC-mapping, then f is a weakly $([r, s], [t, u])$-IVIFC-mapping, where $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$.

Example 3.18. Every weakly $([r, s], [t, u])$-IVIFC-mapping need not be an IVIFC-mapping.

Let $X = \{a, b\}$ and $Y = \{1, 2\}$. Define $G_1 \in I^X$ and $G_2 \in I^Y$ as follows:

$$G_1 = \{(a, 0.5), (b, 0.5)\},\ G_2 = \{(1, 0.5), (2, 0.5)\}.$$

Define $\tau, \tau^* : I^X \to D(I)$, $\eta, \eta^* : I^Y \to D(I)$ as follows:

$$\tau(A) = \begin{cases} \mathbf{1} & \text{if } A \in \{0_X, 1_X\}, \\ [0.8, 0.9] & \text{if } A = G_1, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

$$\tau^*(A) = \begin{cases} \mathbf{0} & \text{if } A \in \{0_X, 1_X\}, \\ [0.1, 0.2] & \text{if } A = G_1, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

$$\eta(A) = \begin{cases} \mathbf{1} & \text{if } A \in \{0_Y, 1_Y\}, \\ [0.7, 0.8] & \text{if } A = G_2, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

$$\eta^*(A) = \begin{cases} \mathbf{0} & \text{if } A \in \{0_Y, 1_Y\}, \\ [0.1, 0.2] & \text{if } A = G_2, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

Define the mapping $f : (X, \tau, \tau^*) \to (Y, \eta, \eta^*)$ by $f(a) = 1, f(b) = 2$ and let $[r, s] = [0.5, 0.6]$ and $[t, u] = [0.3, 0.4]$. Then f is a weakly $([r, s], [t, u])$-IVIFC-mapping, but f is not an IVIFC-mapping.

THEOREM 3.19. *Let (X, τ, τ^*) and (Y, η, η^*) be two IVISTSs and $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$ and let $f : X \to Y$ be a mapping. Then f is a weakly $([r, s], [t, u])$-IVIFC-mapping if and only if $cl_{[r,s],[t,u]}(f(A)) \subseteq f(cl_{[r,s],[t,u]}(A))$ for each $A \in I^X$.*

Proof. Suppose that f is a weakly $([r, s], [t, u])$-IVIFC-mapping. For each $A \in I^X$, we have

$$\begin{aligned}
&\mathcal{F}_\tau(cl_{[r,s],[t,u]}(A)) \\
&= \mathcal{F}_\tau(\cap\{K \in I^X : A \subseteq K,\ \mathcal{F}_\tau(K) \geq [r, s],\ \mathcal{F}^*_{\tau^*}(K) \leq [t, u]\}) \\
&= [\mathcal{F}^L_\tau(\cap\{K \in I^X : A \subseteq K,\ \mathcal{F}_\tau(K) \geq [r, s],\ \mathcal{F}^*_{\tau^*}(K) \leq [t, u]\}), \\
&\quad \mathcal{F}^U_\tau(\cap\{K \in I^X : A \subseteq K,\ \mathcal{F}_\tau(K) \geq [r, s],\ \mathcal{F}^*_{\tau^*}(K) \leq [t, u]\})] \\
&\geq [\wedge\{\mathcal{F}^L_\tau(K) : A \subseteq K,\ \mathcal{F}_\tau(K) \geq [r, s],\ \mathcal{F}^*_{\tau^*}(K) \leq [t, u]\}, \\
&\quad \wedge \{\mathcal{F}^U_\tau(K) : A \subseteq K,\ \mathcal{F}_\tau(K) \geq [r, s],\ \mathcal{F}^*_{\tau^*}(K) \leq [t, u]\}] \\
&\geq [r, s],
\end{aligned}$$

$$\begin{aligned}
&\mathcal{F}^*_{\tau^*}(cl_{[r,s],[t,u]}(A)) \\
&= \mathcal{F}^*_{\tau^*}(\cap\{K \in I^X : A \subseteq K,\ \mathcal{F}_\tau(K) \geq [r, s],\ \mathcal{F}^*_{\tau^*}(K) \leq [t, u]\}) \\
&= [\mathcal{F}^{*\,L}_{\tau^*}(\cap\{K \in I^X : A \subseteq K,\ \mathcal{F}_\tau(K) \geq [r, s],\ \mathcal{F}^*_{\tau^*}(K) \leq [t, u]\}), \\
&\quad \mathcal{F}^{*\,U}_{\tau^*}(\cap\{K \in I^X : A \subseteq K,\ \mathcal{F}_\tau(K) \geq [r, s],\ \mathcal{F}^*_{\tau^*}(K) \leq [t, u]\})] \\
&\leq [\vee\{\mathcal{F}^{*\,L}_{\tau^*}(K) : A \subseteq K,\ \mathcal{F}_\tau(K) \geq [r, s],\ \mathcal{F}^*_{\tau^*}(K) \leq [t, u]\}, \\
&\quad \vee \{\mathcal{F}^{*\,U}_{\tau^*}(K) : A \subseteq K,\ \mathcal{F}_\tau(K) \geq [r, s],\ \mathcal{F}^*_{\tau^*}(K) \leq [t, u]\}] \\
&\leq [t, u].
\end{aligned}$$

Thus $\mathcal{F}_\tau(cl_{[r,s],[t,u]}(A)) \geq [r,s]$ and $\mathcal{F}^*_{\tau^*}(cl_{[r,s],[t,u]}(A)) \leq [t,u]$. Since f is a weakly $([r,s],[t,u])$-IVIFC-mapping, $\mathcal{F}_\eta(f(cl_{[r,s],[t,u]}(A))) \geq [r,s]$ and $\mathcal{F}^*_{\eta^*}(f\ (cl_{[r,s],[t,u]}(A))) \leq [t,u]$. By Theorem 3.5, we have

$$f(cl_{[r,s],[t,u]}(A)) = cl_{[r,s],[t,u]}(f(cl_{[r,s],[t,u]}(A))) \supseteq cl_{[r,s],[t,u]}(f(A)).$$

Conversely, suppose that $cl_{[r,s],[t,u]}(f(A)) \subseteq f(cl_{[r,s],[t,u]}(A))$ for each $A \in I^X$. Let $A \in I^X$ with $\mathcal{F}_\tau(A) \geq [r,s]$ and $\mathcal{F}^*_{\tau^*}(A) \leq [t,u]$. Then by Theorem 3.5, $cl_{[r,s],[t,u]}(A) = A$. By hypothesis,

$$cl_{[r,s],[t,u]}(f(A)) \subseteq f(cl_{[r,s],[t,u]}(A)) = f(A) \subseteq cl_{[r,s],[t,u]}(f(A)).$$

Thus $f(A) = cl_{[r,s],[t,u]}(f(A))$. Hence

$$\begin{aligned}
&\mathcal{F}_\eta(f(A)) \\
&= \mathcal{F}_\eta(cl_{[r,s],[t,u]}(f(A))) \\
&= \mathcal{F}_\eta(\cap\{K \in I^Y : f(A) \subseteq K,\ \mathcal{F}_\eta(K) \geq [r,s],\ \mathcal{F}^*_{\eta^*}(K) \leq [t,u]\}) \\
&= [\mathcal{F}^L_\eta(\cap\{K \in I^Y : f(A) \subseteq K,\ \mathcal{F}_\eta(K) \geq [r,s],\ \mathcal{F}^*_{\eta^*}(K) \leq [t,u]\}), \\
&\quad \mathcal{F}^U_\eta(\cap\{K \in I^Y : f(A) \subseteq K,\ \mathcal{F}_\eta(K) \geq [r,s],\ \mathcal{F}^*_{\eta^*}(K) \leq [t,u]\})] \\
&\geq [\wedge\{\mathcal{F}^L_\eta(K) : f(A) \subseteq K,\ \mathcal{F}_\eta(K) \geq [r,s],\ \mathcal{F}^*_{\eta^*}(K) \leq [t,u]\}, \\
&\quad \wedge\{\mathcal{F}^U_\eta(K) : f(A) \subseteq K,\ \mathcal{F}_\eta(K) \geq [r,s],\ \mathcal{F}^*_{\eta^*}(K) \leq [t,u]\}] \\
&\geq [r,s],
\end{aligned}$$

$$\begin{aligned}
&\mathcal{F}^*_{\eta^*}(f(A)) \\
&= \mathcal{F}^*_{\eta^*}(cl_{[r,s],[t,u]}(f(A))) \\
&= \mathcal{F}^*_{\eta^*}(\cap\{K \in I^Y : f(A) \subseteq K,\ \mathcal{F}_\eta(K) \geq [r,s],\ \mathcal{F}^*_{\eta^*}(K) \leq [t,u]\}) \\
&= [{\mathcal{F}^*_{\eta^*}}^L(\cap\{K \in I^Y : f(A) \subseteq K,\ \mathcal{F}_\eta(K) \geq [r,s],\ \mathcal{F}^*_{\eta^*}(K) \leq [t,u]\}), \\
&\quad {\mathcal{F}^*_{\eta^*}}^U(\cap\{K \in I^Y : f(A) \subseteq K,\ \mathcal{F}_\eta(K) \geq [r,s],\ \mathcal{F}^*_{\eta^*}(K) \leq [t,u]\})] \\
&\leq [\vee\{{\mathcal{F}^*_{\eta^*}}^L(K) : f(A) \subseteq K,\ \mathcal{F}_\eta(K) \geq [r,s],\ \mathcal{F}^*_{\eta^*}(K) \leq [t,u]\}, \\
&\quad \vee\{{\mathcal{F}^*_{\eta^*}}^U(K) : f(A) \subseteq K,\ \mathcal{F}_\eta(K) \geq [r,s],\ \mathcal{F}^*_{\eta^*}(K) \leq [t,u]\}] \\
&\leq [t,u].
\end{aligned}$$

Hence f is a weakly $([r,s],[t,u])$-IVIFC-mapping.

$\square$

4. Several types of compactness in interval-valued intuitionistic smooth topological spaces

DEFINITION 4.1. Let $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$. Then

(i) An IVISTS (X, τ, τ^*) is called $([r, s], [t, u])$-*interval-valued intuitionistic fuzzy compact* if for every family $\{G_i : i \in \Gamma\}$ in $\{G \in I^X : \tau(G) > [r, s], \tau^*(G) < [t, u]\}$ such that $\cup_{i\in\Gamma} G_i = 1_X$, there exists a finite subset Γ_0 of Γ such that $\cup_{i\in\Gamma_0} G_i = 1_X$.

(ii) An IVISTS (X, τ, τ^*) is called $([r, s], [t, u])$-*interval-valued intuitionistic fuzzy nearly compact* if for every family $\{G_i : i \in \Gamma\}$ in $\{G \in I^X : \tau(G) > [r, s], \tau^*(G) < [t, u]\}$ such that $\cup_{i\in\Gamma} G_i = 1_X$, there exists a finite subset Γ_0 of Γ such that $\cup_{i\in\Gamma_0} int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(G_i)) = 1_X$.

(iii) An IVISTS (X, τ, τ^*) is called $([r, s], [t, u])$-*interval-valued intuitionistic fuzzy almost compact* if for every family $\{G_i : i \in \Gamma\}$ in $\{G \in I^X : \tau(G) > [r, s], \tau^*(G) < [t, u]\}$ such that $\cup_{i\in\Gamma} G_i = 1_X$, there exists a finite subset Γ_0 of Γ such that $\cup_{i\in\Gamma_0} cl_{[r,s],[t,u]}(G_i) = 1_X$.

THEOREM 4.2. *Let $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$. If (X, τ, τ^*) is $([r, s], [t, u])$-interval-valued intuitionistic fuzzy compact, then (X, τ, τ^*) is $([r, s], [t, u])$-interval-valued intuitionistic fuzzy nearly compact.*

Proof. Let (X, τ, τ^*) be $([r, s], [t, u])$-interval-valued intuitionistic fuzzy compact. Then for every family $\{G_i : i \in \Gamma\}$ in $\{G \in I^X : \tau(G) > [r, s], \tau^*(G) < [t, u]\}$ such that $\cup_{i\in\Gamma} G_i = 1_X$, there exists a finite subset Γ_0 of Γ such that $\cup_{i\in\Gamma_0} G_i = 1_X$. Since $\tau(G_i) > [r, s]$ and $\tau^*(G_i) < [t, u]$ for each $i \in \Gamma$, by Theorem 3.5 $G_i = int_{[r,s],[t,u]}(G_i)$ for each $i \in \Gamma$. Thus $G_i = int_{[r,s],[t,u]}(G_i) \subseteq int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(G_i))$ for each $i \in \Gamma$. Hence $1_X = \cup_{i\in\Gamma_0} G_i \subseteq \cup_{i\in\Gamma_0} int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(G_i))$. So $\cup_{i\in\Gamma_0} int_{[r,s],[t,u]}$ $(cl_{[r,s],[t,u]}(G_i)) = 1_X$. Hence (X, τ, τ^*) is $([r, s], [t, u])$-interval-valued intuitionistic fuzzy nearly compact.

□

THEOREM 4.3. *Let $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$. If (X, τ, τ^*) is $([r, s], [t, u])$-interval-valued intuitionistic fuzzy nearly compact, then (X, τ, τ^*) is $([r, s], [t, u])$-interval-valued intuitionistic fuzzy almost compact.*

Proof. The proof is similar to Theorem 4.2.

□

THEOREM 4.4. *Let* (X, τ, τ^*) *and* (Y, η, η^*) *be two IVISTSs and* $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ *with* $s + u \leq 1$ *and let* $f : X \to Y$ *be a surjective IVIGP-mapping. If* (X, τ, τ^*) *is* $([r, s], [t, u])$*-interval-valued intuitionistic fuzzy compact, then so is* (Y, η, η^*).

Proof. Let $\{G_i : i \in \Gamma\}$ be a family in $\{G \in I^Y : \eta(G) > [r, s], \eta^*(G) < [t, u]\}$ such that $\cup_{i\in\Gamma} G_i = 1_Y$. Then $1_X = f^{-1}(1_Y) = f^{-1}(\cup_{i\in\Gamma} G_i) = \cup_{i\in\Gamma} f^{-1}(G_i)$. Since f is an IVIGP-mapping, $\tau(f^{-1}(G_i)) \geq \eta(G_i) > [r, s]$ and $\tau^*(f^{-1}(G_i)) \leq \eta^*(G_i) < [t, u]$ for each $i \in \Gamma$. Since (X, τ, τ^*) is $([r, s], [t, u])$-interval-valued intuitionistic fuzzy compact, there exists a finite subset Γ_0 of Γ such that $\cup_{i\in\Gamma_0} f^{-1}(G_i) = 1_X$. Since f is surjective, $1_Y = f(1_X) = f(\cup_{i\in\Gamma_0} f^{-1}(G_i)) = \cup_{i\in\Gamma_0} f(f^{-1}(G_i)) = \cup_{i\in\Gamma_0} G_i$. Thus $\cup_{i\in\Gamma_0} G_i = 1_Y$. Hence (Y, η, η^*) is $([r, s], [t, u])$-interval-valued intuitionistic fuzzy compact.

□

THEOREM 4.5. *Let* (X, τ, τ^*) *and* (Y, η, η^*) *be two IVISTSs and* $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ *with* $s + u \leq 1$ *and let* $f : X \to Y$ *be a surjective IVIGP-mapping. If* (X, τ, τ^*) *is* $([r, s], [t, u])$*-interval-valued intuitionistic fuzzy almost compact, then so is* (Y, η, η^*).

Proof. Let $\{G_i : i \in \Gamma\}$ be a family in $\{G \in I^Y : \eta(G) > [r, s], \eta^*(G) < [t, u]\}$ such that $\cup_{i\in\Gamma} G_i = 1_Y$. Then $1_X = f^{-1}(1_Y) = f^{-1}(\cup_{i\in\Gamma} G_i) = \cup_{i\in\Gamma} f^{-1}(G_i)$. Since f is an IVIGP-mapping, $\tau(f^{-1}(G_i)) \geq \eta(G_i) > [r, s]$ and $\tau^*(f^{-1}(G_i)) \leq \eta^*(G_i) < [t, u]$ for each $i \in \Gamma$. Since (X, τ, τ^*) is $([r, s], [t, u])$-interval-valued intuitionistic fuzzy almost compact, there exists a finite subset Γ_0 of Γ such that $\cup_{i\in\Gamma_0} cl_{[r,s],[t,u]}(f^{-1}(G_i)) = 1_X$. Since f is an IVIGP-mapping, by Theorem 4.6[11] $f(cl_{[r,s],[t,u]}(f^{-1}(G_i)) \subseteq cl_{[r,s],[t,u]}(f(f^{-1}(G_i)))$ for each $i \in \Gamma$. Since f is surjective, we have

$$\begin{aligned} 1_Y = f(1_X) &= f(\cup_{i\in\Gamma_0} cl_{[r,s],[t,u]}(f^{-1}(G_i))) \\ &= \cup_{i\in\Gamma_0} f(cl_{[r,s],[t,u]}(f^{-1}(G_i))) \\ &\subseteq \cup_{i\in\Gamma_0} cl_{[r,s],[t,u]}(f(f^{-1}(G_i))) \\ &= \cup_{i\in\Gamma_0} cl_{[r,s],[t,u]}(G_i). \end{aligned}$$

Thus $\cup_{i\in\Gamma_0} cl_{[r,s],[t,u]}(G_i) = 1_Y$. Hence (Y, η, η^*) is $([r, s], [t, u])$-interval-valued intuitionistic fuzzy almost compact.

□

Theorem 4.6. *Let (X,τ,τ^*) and (Y,η,η^*) be two IVISTSs and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s+u \leq 1$ and let $f : X \to Y$ be a surjective IVIGP and weakly $([r,s],[t,u])$-IVIFO-mapping. If (X,τ,τ^*) is $([r,s],[t,u])$-interval-valued intuitionistic fuzzy nearly compact, then so is (Y,η,η^*).*

Proof. Let $\{G_i : i \in \Gamma\}$ be a family in $\{G \in I^Y : \eta(G) > [r,s], \eta^*(G) < [t,u]\}$ such that $\cup_{i\in\Gamma} G_i = 1_Y$. Then $1_X = f^{-1}(1_Y) = f^{-1}(\cup_{i\in\Gamma} G_i) = \cup_{i\in\Gamma} f^{-1}(G_i)$. Since f is an IVIGP-mapping, $\tau(f^{-1}(G_i)) \geq \eta(G_i) > [r,s]$ and $\tau^*(f^{-1}(G_i)) \leq \eta^*(G_i) < [t,u]$ for each $i \in \Gamma$. Since (X,τ,τ^*) is $([r,s],[t,u])$-interval-valued intuitionistic fuzzy nearly compact, there exists a finite subset Γ_0 of Γ such that $\cup_{i\in\Gamma_0} int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(G_i))) = 1_X$. Since f is a surjective IVIGP and weakly $([r,s],[t,u])$-IVIFO-mapping, by Theorem 4.6[11] and Corollary 3.15 we have

$$\begin{aligned}
1_Y = f(1_X) &= f(\cup_{i\in\Gamma_0} int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(G_i)))) \\
&= \cup_{i\in\Gamma_0} f(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(G_i)))) \\
&\subseteq \cup_{i\in\Gamma_0} int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f(f^{-1}(G_i)))) \\
&= \cup_{i\in\Gamma_0} int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(G_i)).
\end{aligned}$$

Thus $\cup_{i\in\Gamma_0} int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(G_i)) = 1_Y$. Hence (Y,η,η^*) is $([r,s],[t,u])$-interval-valued intuitionistic fuzzy nearly compact.

□

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