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ABELIAN PROPERTY CONCERNING FACTORIZATION MODULO RADICALS

Dong Hyeon Chae, Jeong Min Choi, Dong Hyun Kim, Jae Eui Kim, Jae Min Kim, Tae Hyeong Kim, Ji Young Lee, Yang Lee, You Sun Lee, Jin Hwan Noh and Sung Ju Ryu*

Abstract. In this note we describe some classes of rings in relation to Abelian property of factorizations by nilradicals and Jacobson radical. The ring theoretical structures are investigated for various sorts of such factor rings which occur in the process.

1. Introduction

Throughout this note every ring is an associative ring with identity unless otherwise stated. Let R be a ring. The polynomial (resp., power series) ring with an indeterminate x over R is denoted by $R[x]$ (resp., $R[[x]]$) and for any polynomial (resp., power series) $f(x)$ in $R[x]$ (resp., $R[[x]]$), let $C_{f(x)}$ denote the set of all coefficients of $f(x)$. Use the notation that $\bar{R} = R/I$ and $\bar{r} = r + I$, where I is an ideal of R. $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n). Denote the n by n full

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* Corresponding author.
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(resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $U_n(R)$). Use E_{ij} for the matrix with (i,j)-entry 1 and zeros elsewhere. Following the literature, $D_n(R) = \{(a_{ij}) \in U_n(R) \mid a_{11} = \cdots = a_{nn}\}$ and $N_n(R) = \{(b_{ij}) \in D_n(R) \mid b_{11} = \cdots = b_{nn} = 0\}$.

Let $J(R)$, $N_*(R)$, $N^*(R)$, and $N(R)$ to denote the Jacobson radical, the lower nilradical (i.e., intersection of all prime ideals), the upper nilradical (i.e., sum of all nil ideals), and the set of all nilpotent elements in R (possibly without identity), respectively. It is well-known that $N^*(R) \subseteq J(R)$ and $N_*(R) \subseteq N^*(R) \subseteq N(R)$. A ring R is usually called *semiprimitive* (resp., *semiprime*) if $J(R) = 0$ (resp., $N_*(R) = 0$).

A ring is usually called *reduced* if it has no nonzero nilpotents. A ring is usually called *Abelian* if every idempotent is central. Reduced rings are easily shown to be Abelian. It is obvious that the class of Abelian rings is closed under subrings.

Let R be a ring and $n \geq 2$. We use $V_n(R)$ to denote the ring of all matrices (a_{ij}) in $D_n(R)$ such that $a_{st} = a_{(s+1)(t+1)}$ for $s = 1, \ldots, n-2$ and $t = 2, \ldots, n-1$, following the literature, i.e.,

$$V_n(R) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} \mid a_1, a_2, \ldots, a_n \in R \right\}.$$

It is well-known that $V_n(R)$ is isomorphic to the factor ring $R[x]/x^nR[x]$, via the corresponding

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} \mapsto a_1 + a_2\bar{x} + \cdots + a_n\bar{x}^{n-1},$$

where $\bar{x} = x + x^nR[x]$. We use this fact freely. The following is a simple extension of [15, Lemma 8] and [10, Lemma 2].

PROPOSITION 1.1. *For a ring R and $n \geq 2$, the following conditions are equivalent:*

(1) *R is Abelian;*
(2) *$R[x]$ is Abelian;*
(3) *$D_n(R)$ is Abelian;*

(4) *$V_n(R)$ is Abelian;*
(5) *$R[x]/x^nR[x]$ is Abelian.*

Proof. The equivalence of the conditions (1), (2), and (3) is shown by [15, Lemma 8] and [10, Lemma 2]. (3) implying (4), and (4) implying (1) are obvious because $V_n(R)$ is a subring of $D_n(R)$, and R is a subring of $V_n(R)$. The equivalence of the conditions (4) and (5) follows the isomorphism of $V_n(R)$ and $R[x]/x^nR[x]$. □

Considering Proposition 1.1, one may ask whether Abelian property passes to factor rings. But the answer is negative by the following.

Example 1.2. Let F be a field and $A = F\langle X\rangle$ be the free algebra generated by a set X of noncommuting indeterminates over F, where the cardinality of X is ≥ 2. Then A is a domain and so it is Abelian. Let a be taken arbitrarily in X. Consider next an ideal I of R generated by $a^2 - a$, and set $R = A/I$. Let $x \in X$ coincide with its image $x + I$ in R for simplicity. Then $a^2 = a$ (i.e., a is an idempotent in R), but $ab \neq ba$ for all $b \in X\backslash\{a\}$. Thus R is a non-Abelian ring.

In the following arguments, we see two sorts of rings which are closed under factor rings modulo nilradicals. For a reduced ring R, Armendariz [4, Lemma 1] proved that

$$ab = 0 \text{ for all } a \in C_{f(x)}, b \in C_{g(x)} \text{ whenever } f(x)g(x) = 0$$

where $f(x), g(x) \in R[x]$. Based on this result, Rege et al. [20] called a ring (possibly without identity) *Armendariz* if it satisfies this property. So reduced rings are clearly Armendariz. Armendariz rings are Abelian by [11, Corollary 8] or the proof of [2, Theorem 6]. We use this fact without referring.

Let R be an Armendariz ring. Then $R/N_*(R)$ is an Armendariz ring by [8, Theorem 1.4(2)]. Moreover $N_*(R) = N^*(R)$ by [14, Lemma 2.3(5)], and so $R/N^*(R)$ is also Armendariz. Thus both $R/N_*(R)$ and $R/N^*(R)$ are Abelian. This result can be obtained also by the following argument.

Let I be an ideal of a ring R. Following the literature, we say that idempotents modulo I can be *lifted* (or I is *idempotent-lifting*) provided that for every $f \in R$ such that $f^2 - f \in I$ there exists $e^2 = e \in R$ such that $e - f \in I$. A nil ideal is an important example by [18, Proposition 3.6.1].

Proposition 1.3. (1) *Let R be an Abelian ring and N be an ideal of R. If idempotents modulo N can be lifted, then R/N is an Abelian ring.*

(2) *Let R be an Abelian ring. Then R/N is an Abelian ring for any nil ideal N of R.*

(3) *Let R be an Armendariz ring. Then R/N is Abelian for any nil ideal N of R; especially, $R/N_*(R)$ and $R/N^*(R)$ are both Abelian.*

Proof. (1) Assume that idempotents modulo N can be lifted. Consider next $\bar{R} = R/N$, and let $\bar{f}$ be an idempotent in $\bar{R}$. Then there exists $e^2 = e \in R$ such that $\bar{e} = \bar{f}$ because idempotents modulo N can be lifted. But since e is central in R, we get that

$$\bar{f}\bar{r} = \bar{e}\bar{r} = \overline{er} = \overline{re} = \bar{r}\bar{e} = \bar{r}\bar{f}$$

for all $r \in R$. Thus $\bar{R}$ is an Abelian ring.

(2) Since idempotents modulo nil ideals can be lifted by [18, Proposition 3.6.1], R/N is an Abelian ring by (1).

(3) is an immediate consequence of (2) because Armendariz rings are Abelian, noting that the lower nilradical and upper nilradical are both nil ideals. □

Considering Proposition 1.3(3), it is natural to ask whether any factor ring of an Armendariz ring is Abelian. However the answer is negative by Example 1.2. In fact, A is a domain (hence Armendariz), but the factor ring $R = A/I$ is non-Abelian.

We recall next three kinds of well-known definitions. A ring R is called *semilocal* if $R/J(R)$ is semisimple Artinian, and a semilocal ring R is called *semiperfect* if idempotents modulo $J(R)$ can be lifted. A ring R is called *local* if $R/J(R)$ is a division ring. Local rings are clearly semiperfect, and another important case of semiperfect rings is when the Jacobson radical is nil by [18, Proposition 3.6.1]. Local rings are Abelian obviously.

Remark 1.4. The factor rings of semiperfect rings modulo Jacobson radicals need not be Abelian as can be seen by $Mat_n(D)$ over a division ring D when $n \geq 2$. Let R be a semiperfect ring. If R is Abelian then $R/J(R)$ is Abelian by Proposition 1.3(1) because $J(R)$ is idempotent-lifting. Note that $R/J(R)$ is an Abelian ring if and only if $R/J(R)$ is a finite direct product of division rings.

Note that the Jacobson radicals of right Artinian rings are nilpotent by [17, Theorem 2.4.12]. So the factor ring $R/J(R)$ of an Abelian ring R is Abelian by Proposition 1.3(2) when R is right Artinian. There exist many right Artinian Abelian rings by help of by [10, Lemma 2]. In this note, we will study Abelian property of various kinds of factor rings, concentrating on factorizing by nilradicals and Jacobson radicals, motivated by the preceding results.

2. Abelian factor rings modulo nil and Jacobson radicals

In this section we study Abelian property of factor rings factorized by lower nilradicals, upper nilradicals, and Jacobson radicals. Let R be a ring. It is well-known that $N_*(Mat_n(R)) = Mat_n(N_*(R))$. So the factor ring $Mat_n(R)/N_*(Mat_n(R))$ cannot be Abelian when $n \geq 2$ because $Mat_n(R)/N_*(Mat_n(R))$ is isomorphic to $Mat_n(R/N_*(R))$. So the following definition makes sense.

Definition 2.1. A ring R is called *Abelian over lower nilradical* (simply, *Alnr*) if $R/N_*(R)$ is an Abelian ring.

Armendariz rings are Alnr by Proposition 1.3(3). Commutative rings are clearly Alnr. Let R be a ring such that $N_*(R) = N(R)$. Then $R/N_*(R)$ is a reduced (hence Abelian) ring, so R is Alnr. Thus one may ask whether $N_*(R) = N(R)$ if R is an Alnr ring. However the answer is negative as can be seen by the following.

Example 2.2. We refer to the construction of [12, Example 1.2]. Let S be a reduced ring and $M_n = D_{2^n}(S)$ for all $n \geq 1$. Define a map $\sigma : M_n \to M_{n+1}$ by $B \mapsto \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$. Then M_n can be considered as a subring of M_{n+1} via σ (i.e., $B = \sigma(B)$ for $B \in M_n$). Set $R = \cup_{n=1}^{\infty} M_n$. Then R is semiprime by [13, Theorem 2.2(2)]. But

$$N^*(R) = \cup_{n=1}^{\infty} N_{2^n}(S) = N(R),$$

noting $R/N^*(R) \cong S$. Thus $N_*(R) = 0 \subsetneq N(R)$.

Let $E^2 = E \in R$. Then there exists $k \geq 1$ such that

$$E = \begin{pmatrix} f & 0 & 0 & \cdots & 0 \\ 0 & f & 0 & \cdots & 0 \\ 0 & 0 & f & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f \end{pmatrix} \in D_{2^k}(S)$$

with $f^2 = f \in S$, by [10, Lemma 2]. So E is central in R because f is central in S, entailing that R is Abelian. This implies that R is Alnr because $R \cong R/0 = R/N_*(R)$.

Let R be a ring. It is well-known that $N^*(Mat_n(R)) = Mat_n(N^*(R))$. So the factor ring $Mat_n(R)/N^*(Mat_n(R))$ cannot be Abelian when $n \geq 2$ because $Mat_n(R)/N^*(Mat_n(R))$ is isomorphic to $Mat_n(R/N^*(R))$. So the following definition makes sense.

Definition 2.3. A ring R will be called *Abelian over upper nilradical* (simply, *Aunr*) if $R/N^*(R)$ is an Abelian ring.

Armendariz rings are Aunr by Proposition 1.3(3). Let R be a ring such that $N^*(R) = N(R)$. Then $R/N^*(R)$ is a reduced (hence Abelian) ring, so R is Aunr. Thus it is natural to ask whether $N^*(R) = N(R)$ if R is an Aunr ring. However the answer is negative as can be seen by the following.

Example 2.4. We apply the construction in [3, Example 4.8]. Let F be a field and $A = F\langle a, b\rangle$ be the free algebra generated by noncommuting indeterminates a, b over F. Consider next an ideal I of R generated by a^2, and set $R = A/I$. Then R is Armendariz by the argument in [3, Example 4.8], so R is Aunr by Proposition 1.3(3). Let a, b coincide with their images of a, b in R for simplicity. $a^2 = 0$, but $(ab)^n \neq 0$ for all $n \geq 1$. This implies $a \notin N^*(R)$ and $N^*(R) \subsetneq N(R)$. So $R/N^*(R)$ is not reduced as can be seen by $a + N^*(R) \neq 0$ and $(a + N^*(R))^2 = 0$.

Aunr rings need not be Alnr as the following shows.

Example 2.5. We also refer to the construction of [12, Example 1.2]. Let S be a reduced ring and $L_n = U_{2^n}(S)$ for all $n \geq 1$. Define a map $\sigma : L_n \to L_{n+1}$ by $B \mapsto \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$. Then L_n can be considered as a subring of L_{n+1} via σ (i.e., $B = \sigma(B)$ for $B \in L_n$). Set $R = \cup_{n=1}^{\infty} L_n$. Then R is semiprime by [13, Theorem 2.2(1)]. But

$$N^*(R) = \{B \in R \mid \text{ all the diagonal entries of } B \text{ are zero}\} = N(R).$$

So $R/N^*(R)$ is isomorphic to a subring of $\prod_{n=1}^{\infty} S_n$, where $S_n = S$ for all $n \geq 1$. Thus $R/N^*(R)$ is a reduced ring, and so R is Aunr.

However $R \cong R/N_*(R) = R/0$ is non-Abelian as can be seen by the noncentral matrices E_{ii} for all $i \geq 1$, noting E_{ii} is an idempotent. In fact, $E_{ii}E_{i(i+1)} = E_{i(i+1)} \neq 0 = E_{i(i+1)}E_{ii}$. Thus R is not Alnr.

We study next the structure of rings whose factor rings modulo Jacobson radicals are Abelian rings. Let R be a ring. It is well-known that $J(Mat_n(R)) = Mat_n(J(R))$. So the factor ring $Mat_n(R)/J(Mat_n(R))$ cannot be Abelian when $n \geq 2$ because $Mat_n(R)/J(Mat_n(R))$ is isomorphic to $Mat_n(R/J(R))$. So the following definition makes sense.

Definition 2.6. A ring R will be called *Abelian over Jacobson radical* (simply, *Ajr*) if $R/J(R)$ is an Abelian ring.

Ajr rings need not be Alnr (Aunr) as the following shows. For a ring R, $R[[x]]$ denote the power series ring with an indeterminate x over R.

Example 2.7. (1) There exists an Ajr ring but not Alnr. Let S be a semiprimitive domain (e.g., $\mathbb{Z}$), and construct R by the method in Example 2.5. Then R is semiprime by the argument. Note that

$$J(R) = \{B \in R \mid \text{ all the diagonal entries of } B \text{ are zero}\} = N(R).$$

So $R/J(R)$ is isomorphic to a subring of $\prod_{n=1}^{\infty} S_n$, where $S_n = S$ for all $n \geq 1$. Thus $R/J(R)$ is a semiprimitive domain, and so R is Ajr. But R is not Alnr by the argument in Example 2.5.

(2) There exists an Ajr ring but neither Alnr nor Aunr. Let D be a division ring and $n \geq 2$. Consider a subring

$$R = \{\sum_{i=0}^{\infty} a_i x^i \in Mat_n(D)[[x]] \mid a_0 \in U_n(D) \text{ and } a_j \in Mat_n(D) \text{ for all } j \geq 1\}$$

of $Mat_n(D)[[x]]$. Then

$$J(R) = N_n(D) + xMat_n(D)[[x]],$$

entailing $R/J(R) \cong \prod_{k=1}^{n} S_k$, where $S_k = D$ for all k. So R is Ajr.

Set $E = Mat_n(D)$. We claim $N^*(R) = 0$. To see that, let $0 \neq f(x) = \sum_{i=m}^{\infty} a_i x^i \in N(R)$ with $m \geq 0$ and $a_m \neq 0$. Then $a_m \in N(Mat_n(D))$. Compute $J = Rf(x)R$. Then J contains power series

$$\{b_{m+2}x^{m+2} + b_{m+3}x^{m+3} + \cdots \mid b_{m+2} \in Ea_mE\},$$

where $b_{m+2}x^{m+2} + b_{m+3}x^{m+3} + \cdots$ is obtained from $(Ex)f(x)(Ex)$. But $Ea_mE = E$, so J contains non-nilpotent power series (e.g., $x^{m+2} + \cdots$). This implies $f(x) \notin N^*(R)$. Thus $N_*(R) = 0 = N^*(R)$ because the nonzero nilpotent $f(x)$ is taken arbitrarily. Consider next the idempotent E_{11} in R. Then $E_{11}(E_{12}x) = E_{12}x \neq 0 = (E_{12}x)E_{11}$, so $R(\cong R/0 = R/N^*(R) = R/N_*(R))$ is non-Abelian. Therefore R is neither Alnr nor Aunr.

Alnr (Aunr) rings need not be Ajr by the following.

Example 2.8. We apply the ring in [9, Example 3]. Let R_0 be the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$, where p is an odd prime. We next set R be the quaternions over R_0. Then R is clearly a domain (hence Abelian), and so R is both Alnr and Aunr because $N_*(R) = N^*(R) = N(R) = 0$. But $J(R) = pR$, and $R/J(R)$ is isomorphic to $Mat_2(\mathbb{Z}_p)$ by the argument in [7, Exercise 2A]. Since $Mat_2(\mathbb{Z}_p)$ is not Abelian, R is not Ajr.

Armendariz rings need not be Ajr by the ring R in Example 2.8, noting that domains are clearly Armendariz.

A ring R is called (*von Neumann*) *regular* if for every $a \in R$ there exists $b \in R$ such that $aba = a$, in [6]. Every regular ring R is clearly semiprimitive because ab is a nonzero idempotent for all $0 \neq a \in R$. So we have the following equivalence for regular rings.

Proposition 2.9. *For a regular ring R the following conditions are equivalent:*
(1) *R is Alnr;*
(2) *R is Aunr;*
(3) *R is Ajr;*
(4) *R is Abelian;*
(5) *R is reduced;*
(6) *R is Armendariz.*

Proof. The proof follows [6, Theorem 3.2] and the fact that $N_*(R) = N^*(R) = J(R) = 0$ for a regular ring R, reduced rings are Armendariz, and Armendariz rings are Abelian. □

Following the literature, a ring R is called *π-regular* if for each $a \in R$ there exist a positive integer $n = n(a)$, depending on a, and $b \in R$ such that $a^n = a^n b a^n$. Regular rings are obviously π-regular, letting $n(a) = 1$ for all a. Let A be a division ring, then both $D_n(A)$ and $U_n(A)$ are π-regular by [5, Corollary 6]. They are clearly not regular when $n \geq 2$ because

$$J(D_n(A)) = J(U_n(A)) = N_*(D_n(A)) = N_*(U_n(A)) = N(D_n(A)) \\ = N(U_n(A)) = N_n(A) \neq 0.$$

Considering Proposition 2.9, it is natural to ask whether an Ajr is reduced if it is a π-regular ring. But the answer is negative by the following.

Example 2.10. Let S be a division ring, and construct R by the method in Example 2.5. Then R is Ajr by the argument in Example 2.7. But R is π-regular by [5, Corollary 6] because $R = \cup_{i=1}^{\infty} L_n$, and R is clearly not reduced.

It is easily checked that the Jacobson radicals of π-regular rings are nil. In fact, assume on the contrary that there exists $a \in J(R)$ with $a \notin N(R)$. Then $a^n b a^n = a^n$ for some $n \geq 1$ and $b \in R$. Since $a \notin N(R)$, $a^n b$ is a nonzero idempotent that is contained in $J(R)$. This induces a contradiction. So we get the following.

Proposition 2.11. *Let R be a π-regular ring. Then R is Ajr if and only if R is Aunr.*

Proof. Recall that $J(R) = N^*(R)$ for a π-regular ring R. So Ajr coincides with Aunr. □

Based on Proposition 2.11, one may conjecture that a π-regular ring is Ajr if and only if it is Alnr. But the ring $R = \cup_{i=1}^{\infty} U_{2^n}(S)$ in Example 2.5 erases the possibility. R is π-regular by the argument in Example 2.10 when S is a division ring. R is Aunr (if and only if Ajr by Proposition 2.11), but R is not Alnr.

But if $R/J(R)$ is a regular ring then we get the following equivalence.

Proposition 2.12. *Let R be a ring such that $R/J(R)$ is a regular ring. Then the following conditions are equivalent:*

(1) *R is Ajr;*

(2) *$R/J(R)$ is reduced.*

Proof. It suffices to show (1) implying (2). If R is Ajr then $R/J(R)$ is an Abelian ring. So $R/J(R)$ is reduced by [6, Theorem 3.2]. $\square$

Recall that Aunr rings need not be Alnr. But, in fact, we do not know any example of an Alnr ring that is not Aunr.

`Question.` Are Alur rings Aunr?

3. Polynomial rings concerning Alnr, Aunr, and Ajr

In this section we study the structure of polynomial rings concerning Alnr, Aunr, and Ajr rings. We observe first the equivalence of R being Alnr and $R[x]$ being Alnr.

THEOREM 3.1. *A ring R is Alnr if and only if so is $R[x]$.*

Proof. For any ring R, we have $N_*(R[x]) = N_*(R)[x]$ by [1, Theorem 3]. Let R be an Alnr ring. Then $R/N_*(R)$ is an Abelian ring. So $\frac{R}{N_*(R)}[x]$ is also an Abelian ring by [15, Lemma 8(1)]. But $\frac{R}{N_*(R)}[x]$ is isomorphic to $\frac{R[x]}{N_*(R)[x]}$, and recall $N_*(R[x]) = N_*(R)[x]$. Thus we have that

$$\frac{R[x]}{N_*(R[x])} = \frac{R[x]}{N_*(R)[x]} \text{ is Abelian,}$$

proving that $R[x]$ is Alnr.

Conversely suppose that $R[x]$ is Alnr. Then $R[x]/N_*(R[x])$ is an Abelian ring. So, both $R[x]/N_*(R)[x]$ and $(R/N_*(R))[x]$ are Abelian by the argument above. This implies that $R/N_*(R)$ is Abelian because the class of Abelian rings is closed under subrings. Thus R is Alnr. $\square$

COROLLARY 3.2. *If R is an Abelian ring then $R[x]$ is an Alnr ring.*

Proof. Let R be an Abelian ring. Then R is Alnr by Proposition 1.3(2), so we obtain the corollary by Theorem 3.1. $\square$

Recall that a ring is called *right Goldie* if it has no infinite direct sum of right ideals and has the ascending chain condition on right annihilators.

PROPOSITION 3.3. *For a right Goldie ring R the following conditions are equivalent:*

(1) *$R[x]$ is an Alnr ring;*
(2) *$R[x]$ is an Aunr ring;*
(3) *$R[x]$ is an Ajr ring.*
(4) *R is an Alnr ring.*

Proof. Note first that $J(R[x]) = N[x]$ for some nil ideal N of R by [1, Theorem 1]. Since R is right Goldie, N is nilpotent by [19], entailing that $N[x]$ is also nilpotent. So $N[x] \subseteq N_*(R[x])$, and this yields

$$J(R[x]) = N[x] = N_*(R[x]) = N^*(R[x]).$$

Therefore the proof is complete by help of Theorem 3.1. □

If given rings are Armendariz then we get more results as the following shows.

PROPOSITION 3.4. *If R is an Armendariz ring then we have the following results:*
(1) *R is an Alnr ring;*
(2) *R is an Aunr ring;*
(3) *$R[x]$ is an Alnr ring;*
(4) *$R[x]$ is an Aunr ring;*
(5) *$R[x]$ is an Ajr ring.*

Proof. Let R be an Armendariz ring. Then R is both Alnr and Aunr by Proposition 1.3(3). So $R[x]$ is Alnr by Theorem 3.1. Moreover we have

$$J(R[x]) = N_*(R[x]) = N^*(R[x]) = N^*(R)[x] = N_*(R)[x]$$

by [16, Theorem 1.3] because R is Armendariz. This yields

$$R[x]/J(R[x]) = R[x]/N^*(R[x]) = R[x]/N_*(R[x]),$$

completing the proof because $R[x]/N_*(R[x])$ is an Abelian ring. □

The fact "if R is an Armendariz ring then $R[x]$ is an Ajr ring" in Proposition 3.4 can be shown also by the following.

PROPOSITION 3.5. *If R is an Abelian ring then $R[x]$ is an Ajr ring.*

Proof. Let R be an Abelian ring. Note that $J(R[x]) = N[x]$ for some nil ideal N of R by [1, Theorem 1]. So we have

$$R[x]/J(R[x]) = R[x]/N[x] \cong (R/N)[x].$$

But R/N is Abelian by Proposition 1.3(2), and so $(R/N)[x]$ is Abelian by [15, Lemma 8(1)]. This implies that $R[x]$ is Ajr. □

The converse of Proposition 3.5 need not hold as the following shows. Let K be a field and $R = U_n(K)$ $(n \geq 2)$. Consider $R[x]$ and note $R[x] \cong U_n(K[x])$. Since

$$J(U_n(K[x])) = \{(a_{ij}) \in U_n(K[x]) \mid a_{ii} = 0 \text{ for all } i\},$$

we have that $R[x]/J(R[x])$ is isomorphic to the n-copies of $K[x]$, through $U_n(K[x])/J(U_n(K[x]))$. $R[x]/J(R[x])$ is a reduced ring. So $R[x]$ is Ajr, however R is non-Abelian.

In Proposition 3.4, one may ask whether R being an Ajr ring. But the answer is negative by the ring R in Example 2.8. In fact, R is a domain (hence Armendariz), but it is not Ajr in spite of $R[x]$ being a domain (hence Ajr). However we have an affirmative situation in relation to power series rings, comparing this with Theorem 3.1.

Proposition 3.6. *A ring R is Ajr if and only if so is $R[[x]]$.*

Proof. Note that $J(R[[x]]) = J(R)+xR[[x]]$ for any ring R, so $R/J(R) \cong R[[x]]/J(R[[x]])$. This fact completes the proof. □

References

[1] S.A. Amitsur, *Radicals of polynomial rings*, Canad. J. Math. **8** (1956), 355–361.

[2] D. Anderson, V. Camillo, *Armendariz rings and Gaussian rings*, Comm. Algebra **26** (1998), 2265–2272.

[3] R. Antoine, *Nilpotent elements and Armendariz rings*, J. Algebra **319** (2008), 3128–3140.

[4] E.P. Armendariz, *A note on extensions of Baer and P.P.-rings*, J. Austral. Math. Soc. **18** (1974), 470–473.

[5] G.F. Birkenmeier, J.Y. Kim, J.K. Park, *A connection between weak regularity and the simplicity of prime factor rings*, Proc. Amer. Math. Soc. **122** (1994), 53–58.

[6] K.R. Goodearl, *Von Neumann Regular Rings*, Pitman, London, 1979.

[7] K.R. Goodearl and R.B. Warfield, Jr., *An Introduction to Noncommutative Noetherian Rings*, Cambridge University Press, Cambridge-New York-Port Chester-Melbourne-Sydney, 1989.

[8] J. Han, H.K. Kim, Y. Lee, *Armendariz property over prime radicals*, J. Korean Math. Soc. **50** (2013), 973–989.

[9] Y. Hirano, D.V. Huynh and J.K. Park, *On rings whose prime radical contains all nilpotent elements of index two*, Arch. Math. **66** (1996), 360–365.

[10] C. Huh, H.K. Kim, Y. Lee, *p.p. rings and generalized p.p. rings*, J. Pure Appl. Algebra **167** (2002), 37–52.

[11] C. Huh, Y. Lee, A. Smoktunowicz, *Armendariz rings and semicommutative rings*, Comm. Algebra **30** (2002), 751–761.

[12] S.U. Hwang, Y.C. Jeon and Y. Lee, *Structure and topological conditions of NI rings*, J. Algebra **302** (2006), 186–199.
[13] Y.C. Jeon, H.K. Kim, Y. Lee and J.S. Yoon, *On weak Armendariz rings*, Bull. Korean Math. Soc. 46 (2009), 135–146.
[14] N.K. Kim, K.H. Lee, Y. Lee, *Power series rings satisfying a zero divisor property*, Comm. Algebra **34** (2006), 2205–2218.
[15] N.K. Kim, Y. Lee, *Armendariz rings and reduced rings*, J. Algebra **223** (2000), 477–488.
[16] T.K. Kwak, Y. Lee, A.Ç. Özcan, *On Jacobson and nil radicals related to polynomial rings*, J. Korean Math. Soc. **53** (2016), 415–431.
[17] T.Y. Lam, *Lectures on Modules and Rings*, Springer-Verlag, New York, 1991.
[18] J. Lambek, *Lectures on Rings and Modules*, Blaisdell Publishing Company, Waltham-Massachusetts-Toronto-London, 1966.
[19] C. Lanski, *Nil subrings of Goldie rings are nilpotent*, Canad. J . Math. **21** (1969), 904–907.
[20] M.B. Rege and S. Chhawchharia, *Armendariz rings*, Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), 14–17.

Dong Hyeon Chae, Jeong Min Choi, Dong Hyun Kim,
Jae Eui Kim, Jae Min Kim, Tae Hyeong Kim and Ji Young Lee
Department of Mathematics
Pusan Science High School
Pusan 46235, Korea
E-mail: chae3488@naver.com (Dong Hyeon Chae)
E-mail: jmchoi0516@naver.com (Jeong Min Choi)
E-mail: ehdgud99@gmail.com (Dong Hyun Kim)
E-mail: sophia9287@naver.com (Jae Eui Kim)
E-mail: kjm0245@naver.com (Jae Min Kim)
E-mail: kth1030@hanmail.net (Tae Hyeong Kim)
E-mail: 6851ljy@gmail.com (Ji Young Lee)

Yang Lee
Department of Mathematics
Pusan National University
Pusan 46241, Korea
E-mail: ylee@pusan.ac.kr

You Sun Lee and Jin Hwan Noh
Department of Mathematics
Pusan Science High School
Pusan 46235, Korea
E-mail: amss5006@naver.com (You Sun Lee)
E-mail: xsest5@naver.com (Jin Hwan Noh)

Sung Ju Ryu
Department of Mathematics
Pusan National University
Pusan 46241, Korea
E-mail: sjryu@pusan.ac.kr