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LIPSCHITZ CONTINUOUS AND COMPACT COMPOSITION OPERATOR ACTING BETWEEN SOME WEIGHTED GENERAL HYPERBOLIC-TYPE CLASSES

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Abstract. In this paper, we study Lipschitz continuous, the boundedness and compactness of the composition operator C_ϕ acting between the general hyperbolic Bloch type-classes $\mathcal{B}^*_{p,\log,\alpha}$ and general hyperbolic Besov-type classes $F^*_{p,\log}(p,q,s)$. Moreover, these classes are shown to be complete metric spaces with respect to the corresponding metrics.

1. Introduction

Let ϕ be an analytic self-map of the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ denote the classes of analytic functions in the unit disc $\mathbb{D}$. Let $B(\mathbb{D})$ be a subset of $H(\mathbb{D})$ denote the classes of all the hyperbolic function classes in $\mathbb{D}$, such that $|f(z)| < 1$. A function $f \in B(\mathbb{D})$ belongs to α-Bloch space $\mathcal{B}^\alpha, 0 < \alpha < \infty$ if

$$\|f\|_{\mathcal{B}_\alpha} = \sup_{z\in\mathbb{D}} (1-|z|)^\alpha |f'(z)| < \infty.$$

The little α-Bloch space $\mathcal{B}_{\alpha,0}$ consisting of all $f \in \mathcal{B}_\alpha$ such that

$$\lim_{|z|\to 1^-} (1-|z|^2)|f'(z)| = 0.$$

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If (X, d) is a metric space, we denote the open and closed balls with center x and radius $r > 0$ by

$B(x, r) := \{y \in X : d(y, x) < r\}$ and $\bar{B}(x, r) := \{y \in X : d(x, y) \leq r\}$, respectively.

Hyperbolic function classes are usually defined by using either the hyperbolic derivative $f^*(z) = \frac{|f'(z)|}{1-|f(z)|^2}$ of $f \in B(\mathbb{D})$, or the hyperbolic distance $\rho(f(z), 0) := \frac{1}{2}\log\left(\frac{1+|f(z)|}{1-|f(z)|}\right)$ between $f(z)$ and zero.

2. Preliminaries and basic concepts

The hyperbolic $\mathcal{B}^*_\alpha$ (see [3]) is defined as the set of $f \in B(\mathbb{D})$ for which

$$\mathcal{B}^*_\alpha = \{f : f \text{ analytic in } \mathbb{D} \text{ and } \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha f^*(z) < \infty\}.$$

The little hyperbolic Bloch space $\mathcal{B}^*_{\alpha,0}$ is a subspace of $\mathcal{B}^*_\alpha$ consisting of all $f \in \mathcal{B}^*_\alpha$ such that

$$\lim_{|z|\to 1^-}(1-|z|^2)^\alpha f^*(z) = 0.$$

Quite recently, the author in [3] gave the following definitions for (p, α)-Bloch spaces $\mathcal{B}_{p,\alpha}$ and $\mathcal{B}_{p,\alpha,0}$ for $f \in H(\mathbb{D})$

$$\|f\|_{\mathcal{B}_{p,\alpha}} = \frac{p}{2}\sup_{z\in\mathbb{D}}|f(z)|^{\frac{p}{2}-1}|f'(z)|(1-|z|^2)^\alpha < \infty,$$

and

$$\lim_{|z|\to 1}|f(z)|^{\frac{p}{2}-1}|f'(z)|(1-|z|^2)^\alpha = 0,$$

where $2 \leq p < \infty$ and $0 < \alpha < 1$.

Also in [3], the first author introduced the following generalized hyperbolic derivative:

$$f^*_p(z) = \frac{p}{2}\frac{|f(z)|^{\frac{p}{2}-1}|f'(z)|}{1-|f(z)|^p}, \quad f(z) \in H(\mathbb{D}),$$

when $p = 2$ we obtain the usual hyperbolic derivative as defined above. A function $f \in B(\mathbb{D})$ is said to belong to the generalized (p, α) hyperbolic Bloch-type class $\mathcal{B}^*_{p,\alpha}$ if

$$\|f\|_{\mathcal{B}^*_{p,\alpha}} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha f^*_p(z) < \infty,$$

the little generalized (p,α) hyperbolic Bloch-type class $\mathcal{B}^*_{p,\alpha,0}$ consists of all $f \in \mathcal{B}^*_{p,\alpha}$ such that

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha f^*_p(z) = 0.$$

Remark 2.1. It should be remarked that, the Schwarz-Pick lemma implies $\mathcal{B}^*_{p,\alpha} \equiv B(\mathbb{D})$ for all $1 \le \alpha < \infty$ with $\|f\|_{\mathcal{B}^*_{p,\alpha}} \le 1$, hence the class $\mathcal{B}^*_{p,\alpha}$ is of interest only when $0 < \alpha < 1$.

Denote by

$$g(z,a) = \log\left|\frac{1-\bar{a}z}{z-a}\right| = \log\frac{1}{|\varphi_a(z)|}$$

the Green's function of $\mathbb{D}$ with logarithmic singularity at $a \in \mathbb{D}$.

Now, we give the following definitions of the generalized hyperbolic Bloch-type classes $\mathcal{B}^*_{p,\log,\alpha}$ and the generalized hyperbolic Besov-type classes $F^*_{p,\log}(p,q,s)$:

Definition 2.1. Let $2 \le p, \alpha < \infty$, the generalized hyperbolic Bloch-type classes $\mathcal{B}^*_{p,\log,\alpha}$ consisting of all $f \in B(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}^*_{p,\log,\alpha}} = \sup_{z\in\mathbb{D}} f^*_p(z)(1-|z|^2)^\alpha\left(\log\frac{2}{1-|z|^2}\right) < \infty,$$

the little generalized $(p,\log,\alpha)$ hyperbolic Bloch-type classes $\mathcal{B}^*_{p,\log,\alpha,0}$ consists of all $f \in \mathcal{B}^*_{p,\log,\alpha}$ such that

$$\lim_{|z|\to 1} f^*_p(z)(1-|z|^2)^\alpha\left(\log\frac{2}{1-|z|^2}\right) = 0.$$

Definition 2.2. Let $2 \le p < \infty, 0 < s < \infty$ and $-2 < q < \infty$, the hyperbolic class $F^*_{p,\log}(p,q,s)$ consists of all functions $f \in B(\mathbb{D})$ for which

$$\|f\|^p_{F^*_{p,\log}(p,q,s)} = \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} (f^*_p(z))^p(1-|z|^2)^q g^s(z,a)\left(\log\frac{2}{1-|z|^2}\right)^p dA(z) < \infty.$$

Moreover, we say that $f \in F^*_{p,\log}(p,q,s)$ belongs to the class $F^*_{p,\log,0}(p,q,s)$ if

$$\lim_{|a|\to 1}\int_{\mathbb{D}} (f^*_p(z))^p(1-|z|^2)^q g^s(z,a)\left(\log\frac{2}{1-|z|^2}\right)^p dA(z) = 0.$$

Note that the hyperbolic classes are not linear spaces, since they consist of functions that are self-maps of $\mathbb{D}$. Thus, the result in this paper is a generalization of the recent results of Pérez-González, Rättyä and Taskinen [9]. The study of composition operator C_ϕ acting on spaces of analytic functions has engaged many analysts for many years (see e.g. [1, 2, 4–8, 12]).

Recall that a linear operator $T : X \to Y$ is said to be bounded if there exists a constant $C > 0$ such that $\|T(f)\|_Y \leq C\|f\|_X$ for all maps $f \in X$. By elementary functional analysis, it is well-known that a linear operator between normed spaces is bounded if and only if it is continuous, and the boundedness is trivially also equivalent to the Lipschitz-continuity. Moreover, $T : X \to Y$ is said to be compact if it takes bounded sets in X to sets in Y which have compact closure. For Banach spaces X and Y contained in $B(\mathbb{D})$ or $H(\mathbb{D})$, $T : X \to Y$ is compact if and only if for each bounded sequence $(x_n) \in X$, the sequence $(Tx_n) \in Y$ contains a subsequence converging to a function $f \in Y$.

Two quantities A and B are said to be equivalent if there exist two finite positive constants C_1 and C_2 such that $C_1B \leq A \leq C_2B$, written as $A \approx B$. Throughout this paper, the letter C denotes different positive constants which are not necessarily the same from line to line.
Now, we introduce the following definitions:

Definition 2.3. A composition operator $C_\phi : \mathcal{B}^*_{p,\log,\alpha} \to F^*_{p,\log}(p, q, s)$ is said to be bounded, if there is a positive constant C such that $\|C_\phi f\|_{F^*_{p,\log}(p,q,s)} \leq C\|f\|_{\mathcal{B}^*_{p,\log,\alpha}}$ $\forall\ f \in \mathcal{B}^*_{p,\log,\alpha}$.

Definition 2.4. A composition operator $C_\phi : \mathcal{B}^*_{p,\log,\alpha} \to F^*_{p,\log}(p, q, s)$ is said to be compact, if it maps any ball in $\mathcal{B}^*_{p,\log,\alpha}$ onto a pre-compact set in $F^*_{p\log}(p, q, s)$.

The following lemma follows by standard arguments similar to the result in (see [11]). Hence we omit the proof.

Lemma 2.1. *Assume ϕ is a holomorphic mapping from $\mathbb{D}$ into itself and let $2 \leq p < \infty$, $0 < \alpha < 1$, $0 < s < \infty$, and $-2 < q < \infty$. Then the composition operator $C_\phi : \mathcal{B}^*_{p,\log,\alpha} \to F^*_{p,\log}(p, q, s)$ is compact if and only if for any bounded sequence $(f_n)_{n\in N} \in \mathcal{B}^*_{p,\log,\alpha}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$ we have*

$$\lim_{n\to\infty} \|C_\phi f_n\|_{F^*_{p,\log}(p,q,s)} = 0.$$

THEOREM 2.1. *Let* $0 < p,\ s < \infty$, $-2 < q < \infty$, $0 < r < 1$, $\alpha = \frac{q+2}{p}$ *and* $q + s > -1$. *If*

$$(f^*(a))^p \leq \frac{1}{\pi r^2} \int_{D(0,r)} \left(\frac{|f'(\varphi_a(w))|}{1 - |f(\varphi_a(w))|^2} \right)^p dA(w).$$

Then the following are equivalent:
(A) $f \in \mathcal{B}^*_{p,\alpha,\,\log}$,
(B) $f \in F^*_{p,\,\log}(p,q,s)$,

$$(C) \sup_{a\in\mathbb{D}} \left(\log \frac{2}{1-|a|^2} \right)^p \int_{\mathbb{D}} (f^*_p(z))^p (1-|z|^2)^{\alpha p-2} (1-|\varphi_a(z)|^2)^s dA(z) < \infty,$$

$$(D) \sup_{a\in\mathbb{D}} \left(\log \frac{2}{1-|a|^2} \right)^p \int_{\mathbb{D}} (f^*_p(z))^p (1-|z|^2)^{\alpha p-2} g^s(z,a) dA(z) < \infty.$$

Proof. The proof is similar to the main results in [10].
Now, we can find a natural metric on the generalized hyperbolic $(p, \log, \alpha)$-Bloch class $\mathcal{B}^*_{p,\log,\alpha}$ and the class $F^*_{p,\log}(p,q,s)$.
Let $2 \leq p < \infty, 0 < s < \infty, -2 < q < \infty$, and $0 < \alpha < 1$.
First, we can find a natural metric in $\mathcal{B}^*_{p,\log,\alpha}$ by defining

$$d(f,g;\mathcal{B}^*_{p,\log,\alpha}) := d_{\mathcal{B}^*_{p,\log,\alpha}}(f,g) + \|f-g\|_{\mathcal{B}_{p,\log,\alpha}} + |f(0)-g(0)|^{\frac{p}{2}},$$

$$d_{\mathcal{B}^*_{p,\log,\alpha}}(f,g) := \sup_{a\in\mathbb{D}} \left| \frac{f'(z)|f(z)|^{\frac{p}{2}-1}}{1-|f(z)|^p} - \frac{g'(z)|g(z)|^{\frac{p}{2}-1}}{1-|g(z)|^p} \right| (1-|z|^2)^\alpha \left(\log \frac{2}{1-|z|^2} \right).$$

For $f, g \in F^*_{p,\log}(p,q,s)$, define their distance by

$$d(f,g;F^*_{p,\log}(p,q,s)) := d_{F^*_{p,\log}(p,q,s)}(f,g) + \|f-g\|_{F_{p,\log}(p,q,s)} + |f(0)-g(0)|,$$

where

$$d_{F^*_{p,\log}(p,q,s)}(f,g) := \left(\sup_{z\in\mathbb{D}} \left(\log \frac{2}{1-|a|^2} \right)^p \int_{\mathbb{D}} |f^*_p(z) - g^*_p(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) \right)^{\frac{1}{p}}.$$

□

Now we prove the following results.

PROPOSITION 2.1. *The class $\mathcal{B}^*_{p,\log,\alpha}$ equipped with the metric $d(.,.;\mathcal{B}^*_{p,\log,\alpha})$ is a complete metric space. Moreover, $\mathcal{B}^*_{p,\log,\alpha,0}$ is a closed (and therefore complete) subspace of $\mathcal{B}^*_{p,\log,\alpha}$.*

Proof. For $f, g, h \in \mathcal{B}^*_{p,\log,\alpha}$. Then

- $d(f, g; \mathcal{B}^*_{p,\log,\alpha}) \geq 0$,
- $d(f, f; \mathcal{B}^*_{p,\log,\alpha}) = 0$,
- $d(f, g; \mathcal{B}^*_{p,\log,\alpha}) = 0$ implies $f = g$.
- $d(f, g; \mathcal{B}^*_{p,\log,\alpha}) = d(g, f; \mathcal{B}^*_{p,\log,\alpha})$,
- $d(f, h; \mathcal{B}^*_{p,\log,\alpha}) \leq d(f, g; \mathcal{B}^*_{p,\log,\alpha}) + d(g, h; \mathcal{B}^*_{p,\log,\alpha})$.

Hence, d is metric on $\mathcal{B}^*_{p,\log,\alpha}$.

For the completeness proof, let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in the metric space $(\mathcal{B}^*_{p,\log,\alpha}, d)$, that is, for any $\varepsilon > 0$ there is an $N = N(\varepsilon) \in \mathbb{N}$ such that $d(f_n, f_m) < \varepsilon$, for all $n, m > N$. Since $f_n \in B(\mathbb{D})$ such that f_n converges to f uniformly on compact subsets of $\mathbb{D}$. Let $m > N$ and

$$f^*_{m,p}(z) = \frac{p}{2}\frac{|f_m(z)|^{\frac{p}{2}-1}|f'_m(z)|}{1-|f_m(z)|^p}.$$

Then the uniform convergence yields

$$\begin{aligned}
&\left|f^*_p(z) - f^*_{m,p}(z)\right|(1-|z|^2)^{\alpha}\left(\log\frac{2}{1-|z|^2}\right)\\
&= \lim_{n\to\infty}\left|f^*_{n,p}(z) - f^*_{m,p}(z)\right|(1-|z|^2)^{\alpha}\left(\log\frac{2}{1-|z|^2}\right)\\
&\leq \lim_{n\to\infty} d(f_n, f_m; \mathcal{B}^*_{p,\log,\alpha}) \leq \varepsilon. \qquad (1)
\end{aligned}$$

This yields

$$\|f\|_{\mathcal{B}^*_{p,\log,\alpha}} \leq \varepsilon + \|f_m\|_{\mathcal{B}^*_{p,\log,\alpha}}.$$

Thus $f \in \mathcal{B}^*_{p,\log,\alpha}$ as desired. Moreover, (1) and the completeness of the $(p, \log, \alpha)$-Bloch-space imply that $(f_n)_{n=1}^{\infty}$ converges to f with respect to the metric d. The second part of the assertion follows by (1). □

PROPOSITION 2.2. *The class $F^*_{p,\log}(p, q, s)$ equipped with the metric $d(.,.;F^*_{p,\log}(p, q, s))$ is a complete metric space. Moreover, $F^*_{p,\log,0}(p, q, s)$ is a closed (and therefore complete) subspace of $F^*_{p,\log}(p, q, s)$.*

Proof. For $f, g, h \in F^*_{p,\log}(p, q, s)$. Then

- $d(f, g; F^*_{p,\log}(p, q, s)) \geq 0$,
- $d(f, f; F^*_{p,\log}(p, q, s)) = 0$,

- $d(f,g;F^*_{p,\log}(p,q,s))=0$ implies $f=g$.
- $d(f,g;F^*_{p,\log}(p,q,s))=d(g,f;F^*_{p,\log}(p,q,s))$,
- $d(f,h;F^*_{p,\log}(p,q,s))\le d(f,g;F^*_{p,\log}(p,q,s))+d(g,h;F^*_{p,\log}(p,q,s))$.

Hence, d is metric on $F^*_{p,\log}(p,q,s)$.

For the completeness proof, let $(f_n)_{n=0}^{\infty}$ be a Cauchy sequence in the metric space $F^*_{p,\log}(p,q,s)$, that is, for any $\varepsilon>0$ there is an $N=N(\varepsilon)\in\mathbb{N}$ such that $d(f_n,f_m)<\varepsilon$, for all $n,m>N$. Since $f_n\in B(\mathbb{D})$ such that f_n converges to f uniformly on compact subsets of $\mathbb{D}$. Let $m>N$ and $0<r<1$. Let

$$f^*_{m,p}(z)=\frac{p}{2}\frac{|f_m(z)|^{\frac{p}{2}-1}|f'_m(z)|}{1-|f_m(z)|^p}.$$

Then Fatou's lemma yields

$$\begin{aligned}&\int_{D(0,r)}\big(f^*_p(z)-f^*_{m,p}(z)\big)(1-|z|^2)^q g^s(z,a)dA(z)\\&=\int_{D(0,r)}\lim_{n\to\infty}\left|f^*_{n,p}(z)-f^*_{m,p}(z)\right|^p(1-|z|^2)^q g^s(z,a)dA(z)\\&\le\lim_{n\to\infty}\int_{\mathbb{D}}\left|f^*_{n,p}(z)-f^*_{m,p}(z)\right|^p(1-|z|^2)^q g^s(z,a)dA(z)\le\varepsilon^p.\end{aligned}$$

By letting $r\to 1^-$, it follows from the above inequality and $(a+b)^p\le 2^p(a^p+b^p)$ that

$$\begin{aligned}&\int_{\mathbb{D}}(f^*(z))^p(1-|z|^2)^q g^s(z,a)dA(z)\\&\le 2^p\varepsilon^p+2^p\int_{\mathbb{D}}(f^*_{m,p}(z))^p(1-|z|^2)^q g^s(z,a)dA(z).\end{aligned}\tag{2}$$

This yields

$$\|f\|^p_{F^*_{p,\log}(p,q,s)}\le 2^p\varepsilon^p+2^p\|f_m\|^p_{F^*_{p,\log}(p,q,s)},$$

and thus $f\in F^*_{p,\log}(p,q,s)$. We also find that $f_n\to f$ with respect to the metric of $F^*_{p,\log}(p,q,s)$. The second part of the assertion follows by (2). □

3. Lipschitz continuous and boundedness of C_ϕ

For $0<\alpha<1,\ 2\le p<\infty$. Let $f,g\in\mathcal{B}^*_{p,\log,\alpha}$. Then, we will suppose that

$$(f_p^*(z)+g_p^*(z))\ge\frac{C}{(1-|z|^2)^\alpha\big(\log\frac{2}{1-|z|^2}\big)}>0,\tag{3}$$

for some constant C and for each $z\in\mathbb{D}$.

Let $0<\alpha<1,\quad 0<s<\infty$, and $-2<q<\infty$. We define the following notation:

$$\psi_\phi(\alpha,p,q,s;a)=\ell^p(a)\int_{\mathbb{D}}\frac{|\phi'(z)|^p(1-|z|^2)^q}{(1-|\phi(z)|^p)^{p\alpha}\big(\log\frac{2}{1-|\phi(z)|^2}\big)^p}\,g^s(z,a)dA(z),$$

where $\ell^p(a)=\big(\log\frac{2}{1-|a|^2}\big)^p$.
Now, we give the following result.

THEOREM 3.1. *Assume ϕ is a holomorphic mapping from $\mathbb{D}$ into itself and let $0<\alpha<1,\ 2\le p<\infty, 0\le s<\infty, -2<q<\infty$. Suppose that (3) is satisfied. Then the following statements are equivalent:*
(i) $C_\phi:\mathcal{B}^*_{p,\log,\alpha}\to F^*_{p,\log}(p,q,s)$ is bounded;
(ii) $C_\phi:\mathcal{B}^*_{p,\log,\alpha}\to F^*_{p,\log}(p,q,s)$ is Lipschitz continuous;
(iii) $\sup_{a\in\mathbb{D}}\psi_\phi(\alpha,p,q,s;a)<\infty$.

Proof. To prove (i)⇔ (iii), first assume that (iii) holds and that $f\in\mathcal{B}^*_{p,\log,\alpha}$, then, we obtain

$$\begin{aligned}&\sup_{a\in\mathbb{D}}\big(\log\frac{2}{1-|a|^2}\big)^p\int_{\mathbb{D}}((f_p\circ\phi)^*(z))^p(1-|z|^2)^q g^s(z,a)dA(z)\\ &=\sup_{a\in\mathbb{D}}\big(\log\frac{2}{1-|a|^2}\big)^p\int_{\mathbb{D}}(f_p^*(\phi(z)))^p|\phi'(z)|^p(1-|z|^2)^q g^s(z,a)dA(z)\\ &\le\|f\|^p_{\mathcal{B}^*_{p,\log,\alpha}}\sup_{a\in\mathbb{D}}\psi_\phi(\alpha,p,q,s;a)<\infty.\end{aligned}$$

Hence, it follows that (i) holds.
Conversely, assuming that (i) holds, then there exists a constant C such that

$$\|C_\phi f\|_{F^*_{p,\log}(p,q,s)}\le C\|f\|_{\mathcal{B}^*_{p,\log,\alpha}}.$$

For giving $f \in \mathcal{B}^*_{p,\log,\alpha}$, the function $f_t(z) = f(tz)$, where $0 < t < 1$, belongs to $\mathcal{B}^*_{p,\log,\alpha}$ with the property $\|f_t\|_{\mathcal{B}^*_{p,\log,\alpha}} \leq \|f\|_{\mathcal{B}^*_{p,\log,\alpha}}$. Let $f,\ g$ be the functions from (3), we have

$$f_p^*(z) + g_p^*(z) \geq \frac{C}{(1-|z|^2)^\alpha\big(\log\frac{2}{1-|a|^2}\big)} > 0$$

for all $z \in \mathbb{D}$, then

$$\frac{|\phi'(z)|}{(1-|\phi(z)|^2)^\alpha\big(\log\frac{2}{1-|a|^2}\big)} \leq (f_p \circ \phi)^*(z) + (g_p \circ \phi)^*(z),$$

thus,

$$\begin{aligned}
&\ell^p(a)\int_{\mathbb{D}} \frac{|t\phi'(z)|^p}{(1-|t\phi(z)^p|)^{p\alpha}\big(\log\frac{2}{1-|\phi(z)|^2}\big)^p}(1-|z|^2)^q g^s(z,a)dA(z)\\
&\leq \ell^p(a)\int_{\mathbb{D}} \left(\big((f_p\circ\phi)^*(z)\big)^p + \big((g_p\circ\phi)^*(z)\big)^p\right)(1-|z|^2)^q g^s(z,a)dA(z)\\
&\leq C(\|C_\phi f\|^p_{F^*_{p,\log}(p,q,s)} + \|C_\phi g\|^p_{F^*_{p,\log}(p,q,s)})\\
&\leq C\|C_\phi\|^p(\|f\|^p_{\mathcal{B}^*_{p,\log,\alpha}} + \|g\|^p_{\mathcal{B}^*_{p,\log,\alpha}}),
\end{aligned}$$

so (iii) is satisfied.

To prove (ii)$\Leftrightarrow$ (iii), assume first that $C_\phi : \mathcal{B}^*_{p,\log,\alpha} \to F^*_{p,\log}(p,q,s)$ is Lipschitz continuous, that is, there exists a positive constant C such that

$$d(f\circ\phi, g\circ\phi; F^*_{p,\log}(p,q,s)) \leq Cd(f,g;\mathcal{B}^*_{p,\log,\alpha}), \quad \text{for all } f,g \in \mathcal{B}^*_{p,\log,\alpha}.$$

Taking $g = 0$, we get

$$\|f\circ\phi\|_{F^*_{p,\log}(p,q,s)} \leq C\big(\|f\|_{\mathcal{B}^*_{p,\log,\alpha}} + \|f\|_{\mathcal{B}_{p,\log,\alpha}} + |f(0)|^{\frac{p}{2}}\big), \quad \text{for all } f \in \mathcal{B}^*_{p,\log,\alpha}. \tag{4}$$

The assertion (iii) for $\alpha = 1$, follows by choosing $f(z) = z$ in (4).
If $0 < \alpha < 1$ and $\big(\log\frac{2}{1-|z|^2}\big) \approx \big(\log\frac{2}{1-|a|^2}\big)$ then

$$\begin{aligned}
|f(z)|^{\frac{p}{2}} &\leq \frac{2}{p}\left|\int_0^z |f(s)|^{\frac{p}{2}-1}f'(s)ds + |f(0)^{\frac{p}{2}}\right|\\
&\leq \frac{2}{p}\left[\|f\|_{\mathcal{B}_{p,\log,\alpha}}\frac{1}{\big(\log\frac{2}{1-|a|^2}\big)}\int_0^{|z|}\frac{ds}{(1-s^2)^\alpha} + |f(0)|^{\frac{p}{2}}\right]\\
&\leq C\frac{\|f\|_{\mathcal{B}_{p,\log,\alpha}}}{1-\alpha} + \frac{2}{p}|f(0)|^{\frac{p}{2}}
\end{aligned}$$

this yields

$$\big|f(\phi(0)) - g(\phi(0))\big|^{\frac{p}{2}} \;\leq\; C\frac{\|f-g\|_{\mathcal{B}_{p,\log,\alpha}}}{(1-\alpha)} + \frac{2}{p}|f(0)-g(0)|^{\frac{p}{2}}$$

Moreover, from (3), for $f, g \in \mathcal{B}^*_{p,\log,\alpha}$, we deduce that

$$\big(|f^*_p(z)| + |g^*_p(z)|\big)(1-|z|^2)^\alpha\Big(\log\frac{2}{1-|z|^2}\Big) \geq C > 0, \qquad \text{for all } z \in \mathbb{D}.$$

Therefore,

$$\begin{aligned} &\|f\|_{\mathcal{B}^*_{p,\log,\alpha}} + \|g\|_{\mathcal{B}^*_{p,\log,\alpha}} + \|f\|_{\mathcal{B}_{p,\log,\alpha}} + \|g\|_{\mathcal{B}_{p,\log,\alpha}} + |f(0)|^{\frac{p}{2}} + |g(0)|^{\frac{p}{2}} \\ \geq\;& C\int_{\mathbb{D}} \frac{|\phi'(z)|^p(1-|z|^2)^q}{(1-|\phi(z)^p|)^{p\alpha}\big(\log\frac{2}{1-|z|^2}\big)^p} g^s(z,a)dA(z), \end{aligned}$$

for which the assertion (iii) follows .
Assume now that (iii) is satisfied, we have

$$\begin{aligned} &d(f\circ\phi, g\circ\phi; F^*_{p,\log}(p,q,s)) = d_{F^*_{p,\log}(p,q,s)}(f\circ\phi, g\circ\phi) \\ &+\|f\circ\phi - g\circ\phi\|_{F_{P,\log}(p,q,s)} + \big|f(\phi(0)) - g(\phi(0))^{\frac{p}{2}}\big| \\ &\leq d_{\mathcal{B}^*_{p,\log,\alpha}}(f,g)\bigg(\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|\phi'(z)|^p(1-|z|^2)^q}{(1-(\phi(z))^p)^{p,\alpha}\big(\log\frac{2}{1-|z|^2}\big)^p}\; g^s(z,a)dA(z)\bigg)^{\frac{1}{p}} \\ &+\|f-g\|_{\mathcal{B}_{p,\log,\alpha}}\bigg(\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|\phi'(z)|^p(1-|z|^2)^q}{(1-(\phi(z))^p)^{p,\alpha}\big(\log\frac{2}{1-|z|^2}\big)^p}\; g^s(z,a)dA(z)\bigg)^{\frac{1}{p}} \\ &+\frac{\|f-g\|_{\mathcal{B}_{p,\log,\alpha}}}{1-\alpha} + |f(0)-g(0)|^{\frac{p}{2}} \leq C\, d(f,g;\mathcal{B}^*_{p,\log,\alpha}). \end{aligned}$$

Thus $C_\phi : \mathcal{B}^*_{p,\log,\alpha} \to F_{p,\log}(p,q,s)$ is Lipschitz continuous and the proof is established. □

Remark 3.1. We know that a composition operator $C_\phi : \mathcal{B}^*_{p,\log,\alpha} \to F^*_{p,\log}(p,q,s)$ is said to be bounded if there is a positive constant C such that $\|C_\phi f\|_{F^*_{p,\log}(p,q,s)} \leq C\|f\|_{\mathcal{B}^*_{p,\log,\alpha}}$, for all $f \in \mathcal{B}^*_{p,\log,\alpha}$. Theorem 3.1 shows that $C_\phi : \mathcal{B}^*_{p,\log,\alpha} \to F^*_{p,\log}(p,q,s)$ is bounded if and only if it is Lipschitz continuous, that is, if there exists a positive constant C such that $d(f\circ\phi, g\circ\phi; F^*_{p,\log}(p,q,s)) \leq Cd(f,g;\mathcal{B}^*_{p,\log,\alpha})$, for all $f, g \in \mathcal{B}^*_{p,\log,\alpha}$.

By elementary functional analysis, a linear operator between normed spaces is bounded if and only if it is continuous, since the boundedness is trivially also equivalent to the Lipschitz-continuity. Our result for composition operator in hyperbolic spaces is the correct and natural generalization of the linear operator theory.

4. Compactness of $C_\phi : \mathcal{B}^*_{p,\log,\alpha} \to F^*_{p,\log}(p,q,s)$

Recall that a composition operator $C_\phi : \mathcal{B}^*_{p,\log,\alpha} \to F^*_{p,\log}(p,q,s)$ is said to be compact, if it maps any ball in $\mathcal{B}^*_{p,\log,\alpha}$ onto a pre-compact set in $F^*_{p,\log}(p,q,s)$.
Now, we give the following important results.

PROPOSITION 4.1. *Assume ϕ is a holomorphic mapping from $\mathbb{D}$ into itself. Let $2 \le p < \infty$, $-2 < q < \infty$, $0 < \alpha < 1$ and $0 \le s < \infty$. If $C_\phi : \mathcal{B}^*_{p,\log,\alpha} \to F_{p,\log}(p,q,s)$ is compact, it maps closed balls onto compact sets.*

Proof. If $B \subset \mathcal{B}^*_{p,\log,\alpha}$ is a closed ball and $g \in F^*_{p,\log}(p,q,s)$ belongs to the closure of $C_\phi(B)$, we can find a sequence $(f_n)_{n=1}^\infty \subset B$ such that $f_n \circ \phi$ converges to $g \in F^*_{p,\log}(p,q,s)$ as $n \to \infty$. But $(f_n)_{n=1}^\infty$ is a normal family, hence it has a subsequence $(f_{n_j})_{j=1}^\infty$ converging uniformly on the compact subsets of $\mathbb{D}$ to an analytic function f. As in earlier arguments of Proposition 2.1, we get a positive estimate which shows that f must belong to the closed ball B. On the other hand, also the sequence $(f_{n_j} \circ \phi)_{j=1}^\infty$ converges uniformly on compact subsets to an analytic function, which is $g \in F^*_{p,\log}(p,q,s)$. We get $g = f \circ \phi$, i.e. g belongs to $C_\phi(B)$. Thus, this set is closed and also compact. □

Compactness of composition operator acting between $\mathcal{B}^*_{p,\log,\alpha}$ and $F^*_{p,\log}(p,q,s)$ classes can be characterized in the following result.

THEOREM 4.1. *Assume ϕ is a holomorphic mapping from $\mathbb{D}$ into itself. Let $2 \le p < \infty$, $-2 < q < \infty$, $0 < \alpha < 1$ and $0 \le s < \infty$. Then the following statements are equivalent:*
(i) $C_\phi : \mathcal{B}^*_{p,\log,\alpha} \to F^*_{p,\log}(p,q,s)$ is compact.

(ii) $\lim_{r\to 1^-} \sup_{a\in\mathbb{D}} \psi_\phi(\alpha,p,q,s;a) = 0.$

Proof. We first assume that (ii) holds. Let $B := \bar{B}(g,\delta) \subset \mathcal{B}^*_{p,\log,\alpha}$, $g \in \mathcal{B}^*_{p,\log,\alpha}$ and $\delta > 0$, be a closed ball, and let $(f_n)_{n=1}^\infty \subset B$ be any sequence. We show that its image has a convergent subsequence in $F^*_{p,\log}(p,q,s)$, which proves the compactness of C_ϕ by definition.

Again, $(f_n)_{n=1}^{\infty} \subset B(\mathbb{D})$ is normal, hence, there is a subsequence $(f_{n_j})_{j=1}^{\infty}$ which converges uniformly on the compact subsets of $\mathbb{D}$ to an analytic function f. By Cauchy formula for the derivative of an analytic function, also the sequence $(f'_{n_j})_{j=1}^{\infty}$ converges uniformly on the compact subsets of $\mathbb{D}$ to f'. It follows that also the sequences $(f_{n_j} \circ \phi)_{j=1}^{\infty}$ and $(f'_{n_j} \circ \phi)_{j=1}^{\infty}$ converge uniformly on the compact subsets of $\mathbb{D}$ to $f \circ \phi$ and $f' \circ \phi$, respectively. Moreover, $f \in B \subset \mathcal{B}^*_{p,\log,\alpha}$ since for any fixed $R, 0 < R < 1$, the uniform convergence yield

$$
\begin{aligned}
&\sup_{|z|\le R}\left|\frac{f'(z)|f(z)|^{\frac{p}{2}-1}}{1-|f(z)|^p} - \frac{g'(z)|g(z)|^{\frac{p}{2}-1}}{1-|g(z)|^p}\right|(1-|z|^2)^{\alpha}\left(\log\frac{2}{1-|z|^2}\right)\\
&+\sup_{|z|\le R}|f'(z)-g'(z)||f(z)-g(z)|^{\frac{p}{2}-1}(1-|z|^2)^{\alpha}\left(\log\frac{2}{1-|z|^2}\right)\\
&+|f(0)-g(0)|^{\frac{p}{2}-1}\\
&=\lim_{j\to\infty}\sup_{|z|\le R}\left|\frac{f'_{n_j}(z)|f_{n_j}(z)|^{\frac{p}{2}-1}}{1-|f_{n_j}(z)|^p} - \frac{g'(z)|g(z)|^{\frac{p}{2}-1}}{1-|g(z)|^p}\right|(1-|z|^2)^{\alpha}\left(\log\frac{2}{1-|z|^2}\right)\\
&+\lim_{j\to\infty}\Big(\sup_{|z|\le R}|f'_{n_j}(z)-g'(z)||f_{n_j}(z)-g_(z)|^{\frac{p}{2}-1}(1-|z|^2)^{\alpha}\left(\log\frac{2}{1-|z|^2}\right)\\
&+|f_{n_j}(0)-g(0)|^{\frac{p}{2}-1}\Big)< \delta.
\end{aligned}
$$

Hence, $d(f,g;\mathcal{B}^*_{p,\log,\alpha}) \le \delta$.

Let $\varepsilon > 0$. Since (ii) is satisfied, we may fix $r, 0 < r < 1$, such that

$$
\sup_{a\in\mathbb{D}}\int_{|\phi(z)|>r}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^p)^{p\alpha}\left(\log\frac{2}{1-|\phi(z)|^2}\right)^p}(1-|z|^2)^q g^s(z,a)dA(z) \le \varepsilon.
$$

By the uniform convergence, we may fix $N_1 \in \mathbb{N}$ such that

$$
|f_{n_j}\circ\phi(0) - f\circ\phi(0)| \le \varepsilon, \qquad \text{for all } j \ge N_1. \tag{5}
$$

The condition (ii) is known to imply the compactness of $C_\phi : \mathcal{B}_{p,\log,\alpha} \to F_{p,\log}(p,q,s)$, hence possibly to passing once more to a subsequence and adjusting the notations, we may assume that

$$
\|f_{n_j}\circ\phi - f\circ\phi\|_{F_{p,\log}(p,q,s)} \le \varepsilon, \quad \text{for all } j \ge N_2;\ \ N_2 \in \mathbb{N}. \tag{6}
$$

Since $(f_{n_j})_{j=1}^{\infty} \subset B$ and $f \in B$, it follows that

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \ell^p(a) \int_{|\phi(z)|>r} \big[(f_{p,n_j}\circ\phi)^*(z) - (g_p\circ\phi)^*(z)\big]^p (1-|z|^2)^q \, g^s(z,a)\, dA(z) \\
\leq\ & \frac{p}{2} \sup_{a\in\mathbb{D}} \ell^p(a) \int_{|\phi(z)|>r} Ł(f_{n_j}, g, \phi)(1-|z|^2)^q \, g^s(z,a)\, dA(z) \\
\leq\ & d_{\mathcal{B}^*_{p,\log,\alpha}}(f_{n_j}, g) \sup_{a\in\mathbb{D}} \ell^p(a) \int_{|\phi(z)|>r} \frac{|\phi'(z)|^p (1-|z|^2)^q}{(1-|\phi(z)|^p)^{\alpha p}\big(\log\frac{2}{1-|z|^2}\big)^p} \, g^s(z,a)\, dA(z),
\end{aligned}$$

where

$$Ł(f_{n_j}, g, \phi) = \left| \frac{((f_{n_j}\circ\phi)'(z))|((f_{n_j}\circ\phi)(z)))|^{\frac{p}{2}-1}}{1-|(f_{n_j}\circ\phi)(z)|^p} - \frac{(g\circ\phi)'(z)|((g_{n_j}\circ\phi)(z)))|^{\frac{p}{2}-1}}{1-|(g\circ\phi)(z)|^p} \right|^p$$

hence,

$$\sup_{a\in\mathbb{D}} \ell^p(a) \int_{|\phi(z)|>r} \big[(f_{p,n_j}\circ\phi)^*(z) - (g_p\circ\phi)^*(z)\big]^p (1-|z|^2)^q \, g^s(z,a)\, dA(z) \leq C\varepsilon. \tag{7}$$

On the other hand, by the uniform convergence on the compact disc $\mathbb{D}$, we can find an $N_3 \in \mathbb{N}$ such that for all $j \geq N_3$,

$$Ł_1(f_{n_j}, g, \phi) = \left| \frac{(f'_{n_j}(\phi(z))|((f_{n_j}\circ\phi)(z)))|^{\frac{p}{2}-1}}{1-|(f_{n_j}\circ\phi)(z)|^p} - \frac{g'_{n_j}(\phi(z))|((g_{n_j}\circ\phi)(z))|^{\frac{p}{2}-1}}{1-|(g\circ\phi)(z)|^p} \right| \leq \varepsilon.$$

For all z with $|\phi(z)| \leq r$. Hence, for such j,

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \ell^p(a) \int_{|\phi(z)|\leq r} \big[(f_{p,n_j}\circ\phi)^*(z) - (g_p\circ\phi)^*(z)\big]^p (1-|z|^2)^q \, g^s(z,a)\, dA(z) \\
&\leq \sup_{a\in\mathbb{D}} \ell^p(a) \int_{|\phi(z)|\leq r} Ł_1(f_{n_j}, g, \phi)|\phi'(z)|^p (1-|z|^2)^q \, g^s(z,a)\, dA(z) \\
&\leq \varepsilon \left(\sup_{a\in\mathbb{D}} \ell^p(a) \int_{|\phi(z)|\leq r} \frac{|\phi'(z)|^p (1-|z|^2)^q}{1-(|\phi(z)|^p)^{\alpha p}} g^s(z,a) dA(z) \right)^{\frac{1}{p}} \leq C\varepsilon,
\end{aligned} \tag{8}$$

where C is bounded which is obtained from (iii) of Theorem 3.1. Combining (5), (6), (7) and (8) we deduce that $f_{n_j} \to f$ in $F^*_{p,\log}(p,q,s)$.

For the converse direction, let $f_n(z) := \frac{1}{2}n^{\alpha-1}z^n$ for all $n \in \mathbb{N}$, $n \geq 2$.

$$\begin{aligned} \|f\|_{\mathcal{B}^*_{p,\log,\alpha}} &= \frac{p}{2}\sup_{a\in\mathbb{D}} \frac{n^{\frac{\alpha p}{2}}|z|^{\frac{\alpha p}{2}-1}(1-|z|^2)^\alpha}{1-2^{-p}n^{p(\alpha-1)}|z|^{np}} \\ &\leq (2^{p-1}+1)\sup_{a\in\mathbb{D}} n^{\frac{\alpha p}{2}}|z|^{\frac{\alpha p}{2}-1}(1-|z|^2)^\alpha \end{aligned}$$

Then the sequence $(f_n)_{n=1}^\infty$ belongs to the ball $\overline{B}(0;(2^{p-1}+1)) \subset \mathcal{B}^*_{p,\log,\alpha}$ (see [3]). We are assuming that C_ϕ maps the closed ball $\overline{B}(0;(2^{p-1}+1)) \subset \mathcal{B}^*_{p,\log,\alpha}$ into a compact subset of $F^*_{p,\log}(p,q,s)$, hence, there exists an unbounded increasing subsequence $(n_j)_{j=1}^\infty$ such that the image subsequence $(C_\phi f_{n_j})_{n=1}^\infty$ converges with respect to the norm. Since, both $(f_n)_{n=1}^\infty$ and $(C_\phi f_{n_j})_{n=1}^\infty$ converge to the zero function uniformly on compact subsets of $\mathbb{D}$, the limit of the latter sequence must be 0. Hence,

$$\lim_{j\to\infty} \|n_j^{\alpha-1}\phi^{n_j}\|_{F^*_{p,\log}(p,q,s)} = 0. \tag{9}$$

Now let $r_j = 1 - \frac{1}{n_j}$. For all numbers a, $r_j \leq a < 1$, we have the following estimate

$$\frac{n_j^\alpha a^{n_j-1}}{1-a^{n_j}} \geq \frac{1}{e(1-a)^\alpha}. \qquad \text{(see [3,9])} \tag{10}$$

Using (10) we deduce

$$\begin{aligned} &\|n_j^{\alpha-1}\phi^{n_j}\|_{F^*_{p,\log}(p,q,s)} \\ &\geq \frac{p}{2}\sup_{a\in\mathbb{D}} \ell^p(a)\int_{|\phi(z)|\geq r_j} \left|\frac{n_j^\alpha(\phi(z))^{n_j-1}|\phi^{n_j}(z)|^{\frac{p}{2}-1}|\phi'(z)|}{1-|\phi^{n_j}(z)|^p}\right|^p \\ &\qquad \times(1-|z|^2)^q\, g^s(z,a)\, dA(z) \\ &\geq \frac{Cp}{2(2e)^p}\sup_{a\in\mathbb{D}} \ell^p(a)\int_{|\phi(z)|>r_j} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^p)^{p\alpha}}(1-|z|^2)^q g^s(z,a)dA(z). \end{aligned} \tag{11}$$

From (9) and (11), the condition (ii) follows. The proof is therefore completed □

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