

## GEOMETRIC RESULT FOR THE ELLIPTIC PROBLEM WITH NONLINEARITY CROSSING THREE EIGENVALUES

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ABSTRACT. We investigate the number of the solutions for the elliptic boundary value problem. We obtain a theorem which shows the existence of six weak solutions for the elliptic problem with jumping nonlinearity crossing three eigenvalues. We get this result by using the geometric mapping defined on the finite dimensional subspace. We use the contraction mapping principle to reduce the problem on the infinite dimensional space to that on the finite dimensional subspace. We construct a three dimensional subspace with three axis spanned by three eigenvalues and a mapping from the finite dimensional subspace to the one dimensional subspace.

### 1. Introduction

Let  $\Omega$  be a bounded, connected open subset of  $R^n$  with smooth boundary  $\partial\Omega$  and  $\Delta$  be the Laplace operator. Let  $0 < \lambda_1 < \lambda_2 \leq \dots \leq \lambda_k \leq \dots$  be the eigenvalues and  $\phi_k$  be the eigenfunctions belonging to the eigenvalue  $\lambda_k$ ,  $k \geq 1$ , of the eigenvalue problem for the elliptic problem  $-\Delta u = \lambda u$  in  $\Omega$ ,  $u = 0$  on  $\partial\Omega$ . We see that  $\phi_1(x)$  is the positive normalized eigenfunction associated to  $\lambda_1$ . In this paper we consider the

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number of the solutions of the following piecewise linear elliptic problem with Dirichlet boundary condition

$$(1.1) \quad \begin{aligned} \Delta u + f(u) &= s\phi_1 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{aligned}$$

where we consider the case  $f(u) = bu^+ - au^-$ . That is,

$$(1.2) \quad \begin{aligned} \Delta u + bu^+ - au^- &= s\phi_1 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned}$$

This type jumping problems for the wave equations are considered by the authors in [1, 2, 3, 4]. In [5, 6, 8, 9, 10] the authors considered this type jumping problems for the elliptic equations. In [7] the authors considered this type jumping problems for the suspension bridge equation.

McKenna and Walter [8] proved that if  $a < \lambda_1 < \lambda_2 < b$ , there exist three weak solutions by the Leray-Schauder degree theory. In this paper we improve this result to the case  $a < \lambda_1 < \lambda_2 < \lambda_3 < b < \lambda_4$  by the geometric method.

Our main result is the following:

**THEOREM 1.1.** *Assume that  $a < \lambda_1 < \lambda_2 < \lambda_3 < b < \lambda_4$  and  $s > 0$ . Then (1.2) has at least six solutions, two of which are a positive solution  $s \frac{\phi_1}{b-\lambda_1}$  and a negative solution  $s \frac{\phi_1}{a-\lambda_1}$ .*

The outline of the proof of Theorem 1.1 is as follows: In section 2, we use the contraction mapping principle to reduce the problem on the infinite dimensional space to that on a three-dimensional subspace. We construct a three-dimensional subspace spanned by three eigenfunctions and a mapping from the three-dimensional subspace to the one-dimensional subspace spanned by the eigenfunction  $\phi_1$ . In section 3 we prove Theorem 1.1.

## 2. Geometric mapping on the finite dimensional subspace

Let  $H$  be the Sobolev space with the norm

$$\|u\| = \int_{\Omega} |\nabla u|^2 dx.$$

Then problem (1.2) is equivalent to the problem

$$(2.1) \quad \Delta u + bu^+ - au^- = s\phi_1 \text{ in } H.$$

Let  $V$  be the three dimensional subspace of  $H$  spanned by  $\{\phi_1, \phi_2, \phi_3\}$  and  $W$  be the orthogonal complement of  $V$  in  $H$ . Let  $P$  be an orthogonal projection from  $H$  onto  $V$ . Then any element  $u \in H$  can be expressed by  $u = v + w$ , where  $v = Pu$ ,  $w = (I - P)u$ . Hence (2.1) is equivalent to a system

$$(2.2) \quad \Delta v + P(b(v + w)^+ - a(v + w)^-) = s\phi_1,$$

$$(2.3) \quad \Delta w + (I - P)(b(v + w)^+ - a(v + w)^-) = 0.$$

LEMMA 2.1. Assume that  $a < \lambda_1 < \lambda_2 < \lambda_3 < b < \lambda_4$ . Then for fixed  $v \in V$ , (2.3) has a unique solution  $w = \theta(v)$ . Furthermore  $\theta(v)$  is Lipschitz continuous in terms of  $v$ .

*Proof.* We shall use the contraction mapping principle. Let  $\delta = \frac{a+b}{2}$ . Then (2.3) can be rewritten as

$$(-\Delta - \delta)w = (I - P)(b(v + w)^+ - a(v + w)^- - \delta(v + w))$$

or

$$(2.4) \quad w = (-\Delta - \delta)^{-1}(I - P)(b(v + w)^+ - a(v + w)^- - \delta(v + w)).$$

The operator  $(-\Delta - \delta)^{-1}(I - P)$  is a self adjoint compact map from  $(I - P)H$  into itself. The operator  $L^2$  norm of  $(-\Delta - \delta)^{-1}(I - P)$  is  $\|(-\Delta - \delta)^{-1}(I - P)\| = \frac{1}{\lambda_4 - \delta}(L^2 \text{ norm})$ . We note that

$$\begin{aligned} & |(b(v + w_1)^+ - a(v + w_1)^- - \delta(v + w_1)) - (b(v + w_2)^+ - a(v + w_2)^- - \delta(v + w_2))| \\ &= |((b - \delta)(v + w_1)^+ - (a - \delta)(v + w_1)^-) - ((b - \delta)(v + w_2)^+ - (a - \delta)(v + w_2)^-)| \\ &= |((b - \delta)(v + w_1)^+ - (b - \delta)(v + w_1)^-) - ((b - \delta)(v + w_2)^+ - (b - \delta)(v + w_2)^-)| \\ &\leq |b - \delta||w_1 - w_2|. \end{aligned}$$

Thus we have

$$\begin{aligned} & \|(b(v + w_1)^+ - a(v + w_1)^- - \delta(v + w_1)) - (b(v + w_2)^+ - a(v + w_2)^- - \delta(v + w_2))\| \\ &\leq |b - \delta|\|w_1 - w_2\|. \end{aligned}$$

Since  $|b - \delta| \leq \lambda_4 - \delta$ , the right hand side of (2.4) defines a Lipschitz mapping from  $W$  into itself with Lipschitz constant  $r < 1$ . By the

contraction mapping principle, for fixed  $v \in V$ , there is a unique  $w \in W$  which solves (2.4). If  $\theta(v)$  denotes the unique  $w \in (I - P)L^2(\Omega)$  which solves (2.4), we claim that  $\theta$  is Lipschitz continuous in terms of  $v$ . In fact, if  $w_1 = \theta(v_1)$  and  $w_2 = \theta(v_2)$ , then

$$\begin{aligned} & \|w_1 - w_2\|_{L^2(\Omega)} \\ &= \|(-\Delta - \delta)^{-1}(I - P)((b(v_1 + w_1)^+ - a(v_1 + w_1)^- - \delta(v_1 + w_1)) \\ &\quad - (b(v_2 + w_2)^+ - a(v_2 + w_2)^- - \delta(v_2 + w_2)))\|_{L^2(\Omega)} \\ &\leq r\|(v_1 + w_1) - (v_2 + w_2)\|_{L^2(\Omega)} \\ &\leq r(\|v_1 - v_2\|_{L^2(\Omega)} + \|w_1 - w_2\|_{L^2(\Omega)}) \leq r\|v_1 - v_2\| + r\|w_1 - w_2\|. \end{aligned}$$

Hence

$$\|w_1 - w_2\| \leq C\|v_1 - v_2\| \quad C = \frac{r}{1 - r}.$$

Thus  $\theta$  is Lipschitz continuous in terms of  $v$ .  $\square$

By Lemma 2.1, the study of the multiplicity of the solutions of (2.1) is reduced to that of the multiplicity of the solutions of the problem

$$(2.5) \quad \Delta v + P(b(v + \theta(v))^+ - a(v + \theta(v))^-) = s\phi_1$$

defined on a three-dimensional subspace  $V$  spanned by  $\{\phi_1, \phi_2, \phi_3\}$ .

We note that if  $v \geq 0$  or  $v \leq 0$ , then  $\theta(v) = 0$ . In fact, if  $v \geq 0$  and  $\theta(v) = 0$ , then (2.3) is reduced to

$$\Delta 0 + (I - P)(bv^+ - av^-) = 0,$$

which is possible since  $v^+ = v$ ,  $v^- = 0$  and  $(I - P)(bv^+ - av^-) = 0$ .

Let us construct six subspaces of  $V$  as follows: Since the subspace  $V$  is spanned by  $\{\phi_1, \phi_2, \phi_3\}$  and  $\phi_1(x) > 0$  in  $\Omega$ , there exist a cone  $C_1$ , a small number  $\epsilon_1 > 0$ ,  $\epsilon_2 > 0$  defined by

$$C_1 = \{v = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 \mid c_1 \geq 0, |c_2| \leq \epsilon_1 c_1, |c_3| \leq \epsilon_2 |c_1|\}$$

so that  $v \geq 0$  for all  $v \in C_1$ . Here  $(c_1, c_2)$  with  $|c_2| \leq \epsilon_1 |c_1|$  is a plane spanned by  $c_1\phi_1$  and  $c_2\phi_2$  satisfying  $|c_2| \leq \epsilon_1 |c_1|$ . Let us define

$$C_2 = \{v = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 \mid |c_2| \geq \epsilon_1 |c_1|, c_2 < 0, |c_3| \leq \epsilon_2 |c_1|\},$$

$$C_3 = \{v = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 \mid c_1 \leq 0, |c_2| \leq \epsilon_1 |c_1|, |c_3| \leq \epsilon_2 |c_1|\}$$

such that  $v \leq 0$  for all  $v \in C_3$ . Let

$$C_4 = \{v = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 \mid |c_2| \geq \epsilon_1 |c_1|, c_2 > 0, |c_3| \leq \epsilon_2 |c_1|\},$$

$$C_5 = \{v = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 \mid |c_2| \geq \epsilon_1 |c_1|, |c_3| \geq \epsilon_2 |c_1|, c_3 > 0\},$$

$$C_6 = \{v = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 \mid |c_2| \geq \epsilon_1 |c_1|, |c_3| \geq \epsilon_2 |c_1|, c_3 < 0\}.$$

We do not know  $\theta(v)$  for all  $v \in PH$ , but we know that  $\theta(v) = 0$  for  $v \in C_1 \cup C_3$ . We consider the map

$$T : v \longrightarrow T(v) = \Delta v + P((b(v + \theta(v)))^+ - a(v + \theta(v))^-).$$

If  $v \in C_1$ , then  $v \geq 0$  and

$$T(v) = (b - \lambda_1)c_1\phi_1 + (b - \lambda_2)c_2\phi_2 + (b - \lambda_3)c_3\phi_3.$$

The image of  $c_1\phi_1 + c_2\phi_2 \pm c_3\phi_3$ ,  $|c_2| \leq \epsilon_1 c_1$ ,  $c_1 > 0$ ,  $|c_3| \leq \epsilon_2 |(c_1, c_2)|$  can be explicitly calculated and they are

$$(b - \lambda_1)c_1\phi_1 + (b - \lambda_2)c_2\phi_2 \pm (b - \lambda_3)c_3\phi_3, \\ |c_2| \leq \epsilon_1 c_1, \quad c_1 > 0, \quad |c_3| \leq \epsilon_2 |(c_1, c_2)|$$

or

$$d_1\phi_1 + d_2\phi_2 \pm d_3\phi_3, \quad d_1 > 0, \quad |d_2| \leq \frac{b - \lambda_2}{b - \lambda_1} \epsilon_1 d_1, \\ |d_3| \leq (b - \lambda_3)\epsilon_2 \left| \left( \frac{d_1}{b - \lambda_1}, \epsilon_1 \frac{d_1}{b - \lambda_1} \right) \right|$$

. Thus  $T$  maps  $C_1$  into the cone

$$D_1 = \{d_1\phi_1 + d_2\phi_2 + d_3\phi_3 \mid d_1 > 0, \quad |d_2| \leq \frac{b - \lambda_2}{b - \lambda_1} \epsilon_1 d_1, \\ |d_3| \leq (b - \lambda_3)\epsilon_2 \left| \left( \frac{d_1}{b - \lambda_1}, \epsilon_1 \frac{d_1}{b - \lambda_1} \right) \right|\}.$$

Similarly  $T$  maps  $C_3$  into the cone

$$D_3 = \{d_1\phi_1 + d_2\phi_2 + d_3\phi_3 \mid d_1 < 0, \quad |d_2| \leq \left| \frac{a - \lambda_2}{a - \lambda_1} \epsilon_1 d_1 \right|, \\ |d_3| \leq \left| (a - \lambda_3)\epsilon_2 \left( \frac{d_1}{a - \lambda_1}, \epsilon_1 \frac{d_1}{a - \lambda_1} \right) \right|\}.$$

### 3. Proof of Theorem 1.1

$T(v) = s\phi_1$  has one solution  $\frac{s\phi_1}{b-\lambda_1}$  in  $C_1$  and has one solution  $\frac{s\phi_1}{a-\lambda_1}$  in  $C_3$ . We shall find the other solutions in the complements of  $C_1 \cup C_3$  of the map  $T(v) = s\phi_1$  for  $s > 0$ . We need a lemma.

LEMMA 3.1. *There exist  $p_1, p_2 > 0$  such that*

- (i)  $(T(c_1\phi_1 + c_2\phi_2 + c_3\phi_3), \phi_1) \geq p_1|c_2|$ .
- (ii)  $(T(c_1\phi_1 + c_2\phi_2 + c_3\phi_3), \phi_1) \geq p_2|c_3|$ .

*Proof.* (i)

$$\begin{aligned} & T(c_1\phi_1 + c_2\phi_2 + c_3\phi_3) \\ &= \Delta(c_1\phi_1 + c_2\phi_2 + c_3\phi_3) + P((b(c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + \theta(c_1\phi_1 + c_2\phi_2 + c_3\phi_3))^+ \\ & \quad - a(c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + \theta(c_1\phi_1 + c_2\phi_2 + c_3\phi_3))^-) \end{aligned}$$

If  $u = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + \theta(c_1\phi_1 + c_2\phi_2 + c_3\phi_3)$ , then

$$\begin{aligned} & (T(c_1\phi_1 + c_2\phi_2 + c_3\phi_3), \phi_1) \\ &= ((\Delta + \lambda_1)(c_1\phi_1 + c_2\phi_2 + c_3\phi_3) + P(bu^+ - au^- - \lambda_1u, \phi_1)). \end{aligned}$$

Since  $(\Delta + \lambda_1)\phi_1 = 0$  and  $\Delta$  is self adjoint,  $((\Delta + \lambda_1)(c_1\phi_1 + c_2\phi_2 + c_3\phi_3), \phi_1) = 0$ . We note that

$$bu^+ - au^- - \lambda_1u = (b - \lambda_1)u^+ - (a - \lambda_1)u^- \geq \gamma|u|,$$

where  $\gamma = \min\{b - \lambda_1, -a + \lambda_1\} > 0$ . Thus

$$(bu^+ - au^- - \lambda_1u, \phi_1) \geq \gamma \int_{\Omega} |u| \phi_1.$$

Thus there exists  $p_1 > 0$  such that  $\gamma\phi_1 > p_1|\phi_2|$ , so that

$$\gamma \int_{\Omega} |u| \phi_1 \geq p_1 \int_{\Omega} |u| |\phi_2| \geq p_1 \left| \int_{\Omega} u \phi_2 \right| = p_1 |(u, \phi_2)| = p_1 |c_2|.$$

(ii) We also have that

$$\gamma \int_{\Omega} |u| \phi_1 \geq p_2 \int_{\Omega} |u| |\phi_3| \geq p_2 \left| \int_{\Omega} u \phi_3 \right| = p_2 |(u, \phi_3)| = p_2 |c_3|,$$

for some  $p_2 > 0$  such that  $\gamma\phi_1 \geq p_2|\phi_3|$ .  $\square$

Now we are looking for the preimages of the mapping  $T(v) = s\phi_1$ , for  $s > 0$ , in the complement of  $C_1 \cup C_3$ . Let us consider the image under  $T$  of  $c_1\phi_1 + c_2\phi_2 + c_3\phi_3 \in C_4$ ,  $c_2 \geq \epsilon_1|c_1|$ ,  $c_2 = k$ ,  $k > 0$ ,  $|c_3| \leq \epsilon_2|(c_1, c_2)|$ . By (i) of Lemma 3.1, the image of

$$c_2 = k, \quad |c_1| \leq \frac{1}{\epsilon_1}k, \quad |c_3| \leq \epsilon_2|(c_1, k)|$$

must lie to the right of the line  $c_1 = p_1k$  and must cross the positive  $\phi_1$  axis in the image space. Thus if  $u = c_1\phi_1 + k\phi_2 + c_3\phi_3 + \theta(c_1\phi_1 + k\phi_2 + c_3\phi_3)$ ,  $k > 0$ ,  $|c_1| < \frac{k}{\epsilon_1}$ ,  $|c_3| \leq \epsilon_2|(c_1, k)|$ , then  $u$  satisfies

$$\Delta u + bu^+ - au^- = t\phi_1 \quad \text{for } t > p_1k, \quad k > 0.$$

If we set

$$\hat{u} = \frac{s}{t}u,$$

then  $\hat{u}$  is a solution of  $\Delta\hat{u} + b\hat{u}^+ - a\hat{u}^- = s\phi_1$ . Thus we obtain a solution  $\hat{u}$  in  $C_4$ . Similarly, the image under  $T$  of  $c_1\phi_1 + c_2\phi_2 + c_3\phi_3 \in C_2$ ,  $|c_2| \geq \epsilon_1|c_1|$ ,  $c_2 = k$ ,  $k < 0$ ,  $|c_3| \leq \epsilon_2|(c_1, c_2)|$ . By (i) of Lemma 3.1, the image of

$$c_2 = k, \quad k < 0, \quad |c_1| \leq \frac{1}{\epsilon_1}k, \quad |c_3| \leq \epsilon_2|(c_1, k)|$$

must lie to the right of the line  $c_1 = p_1|k|$  and must cross the positive  $\phi_1$  axis in the image space. Thus if  $u = c_1\phi_1 + k\phi_2 + c_3\phi_3 + \theta(c_1\phi_1 + k\phi_2 + c_3\phi_3)$ ,  $k < 0$ ,  $|c_1| < \frac{k}{\epsilon_1}$ ,  $|c_3| \leq \epsilon_2|(c_1, k)|$ , then  $u$  satisfies

$$\Delta u + bu^+ - au^- = t\phi_1 \quad \text{for } t > p_1|k|, \quad k < 0.$$

If we set

$$\check{u} = \frac{s}{t}u,$$

then  $\check{u}$  is a solution of  $\Delta\check{u} + b\check{u}^+ - a\check{u}^- = s\phi_1$ . Thus we obtain a solution  $\check{u}$  in  $C_2$ .

Now we consider the image under  $T$  of  $c_1\phi_1 + c_2\phi_2 + l\phi_3 \in C_5$ ,  $|c_2| \geq \epsilon_1|c_1|$ ,  $|l| \geq \epsilon_2|(c_1, c_2)|$ ,  $l > 0$ . By (ii) of Lemma 3.1, the image of

$$c_3 = l, \quad |c_2| \geq \epsilon_1|c_1|, \quad |l| \geq \epsilon_2|(c_1, c_2)|$$

must lie to the right of the line  $c_1 = p_2|l|$  and must cross the positive  $\phi_1$  axis in the image space. Thus if  $u = c_1\phi_1 + c_2\phi_2 + l\phi_3 + \theta(c_1\phi_1 + c_2\phi_2 + l\phi_3)$ ,  $l > 0$ ,  $|c_2| \geq \epsilon_1|c_1|$ ,  $|l| \geq \epsilon_2|(c_1, c_2)|$ , then  $u$  satisfies

$$\Delta u + bu^+ - au^- = t\phi_1 \quad \text{for } t > p_2l, \quad l > 0.$$

If we set

$$\bar{u} = \frac{s}{t}u,$$

then  $\bar{u}$  is a solution of  $\Delta\bar{u} + b\bar{u}^+ - a\bar{u}^- = s\phi_1$ . Thus we obtain a solution  $\bar{u}$  in  $C_5$  for given  $s > 0$ .

Now we consider the image under  $T$  of  $c_1\phi_1 + c_2\phi_2 + l\phi_3 \in C_6$ ,  $|c_2| \geq \epsilon_1|c_1|$ ,  $|l| \geq \epsilon_2|(c_1, c_2)|$ ,  $l < 0$ . By (ii) of Lemma 3.1, the image of

$$c_3 = l, \quad |c_2| \geq \epsilon_1|c_1|, \quad |l| \geq \epsilon_2|(c_1, c_2)|$$

must lie to the right of the line  $c_1 = p_2|l|$  and must cross the positive  $\phi_1$  axis in the image space. Thus if  $u = c_1\phi_1 + c_2\phi_2 + l\phi_3 + \theta(c_1\phi_1 + c_2\phi_2 + l\phi_3)$ ,  $l < 0$ ,  $|c_2| \geq \epsilon_1|c_1|$ ,  $|l| \geq \epsilon_2|(c_1, c_2)|$ , then  $u$  satisfies

$$\Delta u + bu^+ - au^- = t\phi_1 \quad \text{for } t > p_2|l|, \quad l < 0.$$

If we set

$$\tilde{u} = \frac{s}{t}u,$$

then  $\tilde{u}$  is a solution of  $\Delta\tilde{u} + b\tilde{u}^+ - a\tilde{u}^- = s\phi_1$ . Thus we also have a solution  $\tilde{u}$  in  $C_6$  for given  $s > 0$ .

For given  $s > 0$ , there exist six solutions, one in each of the six regions. There exist a positive solution  $\frac{s\phi_1}{b-\lambda_1}$  in  $C_1$ , a negative solution  $\frac{s\phi_1}{a-\lambda_1}$  in  $C_3$ , a solution  $\hat{u}$  in  $C_4$ , a solution  $\check{u}$  in  $C_2$ , a solution  $\bar{u}$  in  $C_5$ , a solution  $\tilde{u}$  in  $C_6$  of (1.2). Thus we complete the proof of Theorem 1.1.

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