

BOUNDEDNESS IN NONLINEAR PERTURBED DIFFERENTIAL SYSTEMS VIA t_∞ -SIMILARITY

DONG MAN IM AND YOON HOE GOO

ABSTRACT. This paper shows that the solutions to nonlinear perturbed differential system

$$y' = f(t, y) + \int_{t_0}^t g(s, y(s))ds + h(t, y(t), Ty(t))$$

have bounded properties. To show the bounded properties, we impose conditions on the perturbed part $\int_{t_0}^t g(s, y(s))ds$, $h(t, y(t), Ty(t))$, and on the fundamental matrix of the unperturbed system $y' = f(t, y)$ using the notion of h -stability.

1. Introduction and preliminaries

We consider the nonautonomous differential system

$$x' = f(t, x), \quad x(t_0) = x_0, \quad (1.1)$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and \mathbb{R}^n is the Euclidean n -space. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. Also, we consider the perturbed differential systems of (1.1)

$$y' = f(t, y) + \int_{t_0}^t g(s, y(s))ds + h(t, y(t), Ty(t)), \quad y(t_0) = y_0, \quad (1.2)$$

Received September 9, 2015. Revised November 17, 2016. Accepted December 20, 2016.

2010 Mathematics Subject Classification: 34D10, 34D20.

Key words and phrases: h -stability, t_∞ -similarity, bounded, nonlinear nonautonomous system.

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where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $h \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t, 0) = 0$, $h(t, 0, 0) = 0$, and $T : C(\mathbb{R}^+, \mathbb{R}^n) \rightarrow C(\mathbb{R}^+, \mathbb{R}^n)$ is a continuous operator.

The symbol $|\cdot|$ will be used to denote any convenient vector norm in \mathbb{R}^n . For an $n \times n$ matrix A , define the norm $|A|$ of A by $|A| = \sup_{|x| \leq 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (1.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (1.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \quad v(t_0) = v_0 \quad (1.3)$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \quad z(t_0) = z_0. \quad (1.4)$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (1.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (1.3).

We recall some notions of h -stability [16].

DEFINITION 1.1. The system (1.1) (the zero solution $x = 0$ of (1.1)) is called an h -system if there exist a constant $c \geq 1$, and a positive continuous function h on \mathbb{R}^+ such that

$$|x(t)| \leq c|x_0| h(t) h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0|$ small enough (here $h(t)^{-1} = \frac{1}{h(t)}$).

DEFINITION 1.2. The system (1.1) (the zero solution $x = 0$ of (1.1)) is called

(hS) h -stable if there exists $\delta > 0$ such that (1.1) is an h -system for $|x_0| \leq \delta$ and h is bounded.

Pinto [15, 16] introduced the notion of h -stability (hS) with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. That is, Pinto extended the study of exponential asymptotic stability to a variety of reasonable systems called h -systems. Choi, Ryu [5] and Choi et al. [6] investigated bounds of solutions for nonlinear perturbed systems. Also, Goo [8,9,10,11] and Choi and Goo [3,4] investigated boundedness of solutions for nonlinear perturbed systems.

In this paper, we investigate bounds for solutions of the nonlinear differential systems using the notion of t_∞ -similarity.

Let \mathcal{M} denote the set of all $n \times n$ continuous matrices $A(t)$ defined on \mathbb{R}^+ and \mathcal{N} be the subset of \mathcal{M} consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞ -similarity in \mathcal{M} was introduced by Conti [7].

DEFINITION 1.3. A matrix $A(t) \in \mathcal{M}$ is t_∞ -similar to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over \mathbb{R}^+ , i.e.,

$$\int_0^\infty |F(t)| dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{1.5}$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞ -similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on \mathbb{R}^+ , and it preserves some stability concepts [7, 12].

For the proof we prepare some related properties.

LEMMA 1.4. [16] *The linear system*

$$x' = A(t)x, \quad x(t_0) = x_0, \tag{1.6}$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h -system (respectively h -stable) if and only if there exist $c \geq 1$ and a positive and continuous (respectively bounded) function h defined on \mathbb{R}^+ such that

$$|\phi(t, t_0)| \leq ch(t)h(t_0)^{-1} \tag{1.7}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (1.6).

We need Alekseev formula to compare between the solutions of (1.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y), \quad y(t_0) = y_0, \tag{1.8}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (1.8) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

LEMMA 1.5. [2] Let x and y be a solution of (1.1) and (1.8), respectively. If $y_0 \in \mathbb{R}^n$, then for all $t \geq t_0$ such that $x(t, t_0, y_0) \in \mathbb{R}^n$, $y(t, t_0, y_0) \in \mathbb{R}^n$,

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^t \Phi(t, s, y(s)) g(s, y(s)) ds.$$

THEOREM 1.6. [5] If the zero solution of (1.1) is hS, then the zero solution of (1.3) is hS.

THEOREM 1.7. [6] Suppose that $f_x(t, 0)$ is t_∞ -similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (1.3) is hS, then the solution $z = 0$ of (1.4) is hS.

LEMMA 1.8. (Bihari – type inequality) Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u . Suppose that, for some $c > 0$,

$$u(t) \leq c + \int_{t_0}^t \lambda(s)w(u(s))ds, \quad t \geq t_0 \geq 0.$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t \lambda(s)ds \right],$$

where $t_0 \leq t < b_1$, $W(u) = \int_{u_0}^u \frac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t \lambda(s)ds \in \text{dom}W^{-1} \right\}.$$

LEMMA 1.9. [3] Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that for some $c > 0$,

$$\begin{aligned} u(t) \leq & c + \int_{t_0}^t \lambda_1(s)u(s)ds + \int_{t_0}^t \lambda_2(s)w(u(s))ds \\ & + \int_{t_0}^t \lambda_3(s) \int_{t_0}^s \lambda_4(\tau)w(u(\tau))d\tau ds \\ & + \int_{t_0}^t \lambda_5(s) \int_{t_0}^s \lambda_6(\tau)w(u(\tau))d\tau ds, \quad 0 \leq t_0 \leq t. \end{aligned}$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^s \lambda_4(\tau) d\tau + \lambda_5(s) \int_{t_0}^s \lambda_6(\tau) d\tau) ds \right],$$

where $t_0 \leq t < b_1$, W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^s \lambda_4(\tau) d\tau + \lambda_5(s) \int_{t_0}^s \lambda_6(\tau) d\tau) ds \in \text{dom} W^{-1} \right\}.$$

LEMMA 1.10. [3] Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s) u(s) ds + \int_{t_0}^t \lambda_2(s) \int_{t_0}^s (\lambda_3(\tau) u(\tau) + \lambda_4(\tau) \int_{t_0}^\tau \lambda_5(r) w(u(r)) dr) d\tau ds + \int_{t_0}^t \lambda_6(s) \int_{t_0}^s \lambda_7(\tau) w(u(\tau)) d\tau ds.$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^\tau \lambda_5(r) dr) d\tau + \lambda_6(s) \int_{t_0}^s \lambda_7(\tau) d\tau) ds \right],$$

where $t_0 \leq t < b_1$, W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^\tau \lambda_5(r) dr) d\tau + \lambda_6(s) \int_{t_0}^s \lambda_7(\tau) d\tau) ds \in \text{dom} W^{-1} \right\}.$$

For the proof we need the following corollary from Lemma 1.10.

COROLLARY 1.11. Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that for some $c > 0$

and $0 \leq t_0 \leq t$,

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s) \int_{t_0}^s (\lambda_2(\tau)u(\tau) + \lambda_3(\tau) \int_{t_0}^{\tau} \lambda_4(r)w(u(r))dr)d\tau ds \\ + \int_{t_0}^t \lambda_5(s) \int_{t_0}^s \lambda_6(\tau)w(u(\tau))d\tau ds.$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t (\lambda_1(s) \int_{t_0}^s (\lambda_2(\tau) + \lambda_3(\tau) \int_{t_0}^{\tau} \lambda_4(r)dr)d\tau \right. \\ \left. + \lambda_5(s) \int_{t_0}^s \lambda_6(\tau)d\tau) ds \right],$$

where $t_0 \leq t < b_1$, W , W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) \int_{t_0}^s (\lambda_2(\tau) + \lambda_3(\tau) \int_{t_0}^{\tau} \lambda_4(r)dr)d\tau \right. \\ \left. + \lambda_5(s) \int_{t_0}^s \lambda_6(\tau)d\tau) ds \in \text{dom}W^{-1} \right\}.$$

LEMMA 1.12. [8] Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s)w(u(s))ds + \int_{t_0}^t \lambda_2(s) \int_{t_0}^s (\lambda_3(\tau)u(\tau) \\ + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r)u(r)dr)d\tau ds + \int_{t_0}^t \lambda_6(s) \int_{t_0}^s \lambda_7(\tau)w(u(\tau))d\tau ds.$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r)dr)d\tau \right. \\ \left. + \lambda_6(s) \int_{t_0}^s \lambda_7(\tau)d\tau) ds \right],$$

where $t_0 \leq t < b_1$, W , W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r)dr)d\tau \right. \\ \left. + \lambda_6(s) \int_{t_0}^s \lambda_7(\tau)d\tau) ds \in \text{dom}W^{-1} \right\}.$$

We prepare the following corollary from Lemma 1.12 that is used in proving the theorem.

COROLLARY 1.13. *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,*

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s)w(u(s))ds + \int_{t_0}^t \lambda_2(s) \int_{t_0}^s \lambda_3(\tau)u(\tau)d\tau ds + \int_{t_0}^t \lambda_4(s) \int_{t_0}^s \lambda_5(\tau)w(u(\tau))d\tau ds.$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s \lambda_3(\tau)d\tau + \lambda_4(s) \int_{t_0}^s \lambda_5(\tau)d\tau) ds \right],$$

where $t_0 \leq t < b_1$, W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s \lambda_3(\tau)d\tau + \lambda_4(s) \int_{t_0}^s \lambda_5(\tau)d\tau) ds \in \text{dom}W^{-1} \right\}.$$

2. Main Results

In this section, we investigate boundedness for solutions of nonlinear perturbed differential systems via t_∞ -similarity.

To obtain the bounded result, the following assumptions are needed:

(H1) $f_x(t, 0)$ is t_∞ -similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$.

(H2) The solution $x = 0$ of (1.1) is hS with the increasing function h .

(H3) $w(u)$ is nondecreasing in u such that $u \leq w(u)$ and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$.

THEOREM 2.1. *Let $a, b, c, k, w \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (1.2) satisfies*

$$|g(t, y(t))| \leq a(t)w(|y(t)|) + b(t) \int_{t_0}^t k(s)w(|y(s)|)ds \tag{2.1}$$

and

$$|h(t, y(t), Ty(t))| \leq \int_{t_0}^t c(s)|y(s)|ds, \tag{2.2}$$

where $a, b, c, k \in L^1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t \int_{t_0}^s (a(\tau) + c(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr) d\tau ds \right],$$

where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^t \int_{t_0}^s (a(\tau) + c(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr) d\tau ds \in \text{dom}W^{-1} \right\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By Theorem 1.6, since the solution $x = 0$ of (1.1) is hS, the solution $v = 0$ of (1.3) is hS. Using (H1), by Theorem 1.7, the solution $z = 0$ of (1.4) is hS. Using the nonlinear variation of constants formula Lemma 1.5, Lemma 1.4, together with (2.1), and (2.2), we have

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))| \left(\int_{t_0}^s |g(\tau, y(\tau))| d\tau + |h(s, y(s), Ty(s))| \right) ds \\ &\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 h(t) h(s)^{-1} \left(\int_{t_0}^s (a(\tau) w(|y(\tau)|) \right. \\ &\quad \left. + b(\tau) \int_{t_0}^{\tau} k(r) w(|y(r)|) dr) d\tau + \int_{t_0}^s c(\tau) |y(\tau)| d\tau \right) ds. \end{aligned}$$

Applying (H2) and (H3), we obtain

$$\begin{aligned} |y(t)| &\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 h(t) \left(\int_{t_0}^s c(\tau) \frac{|y(\tau)|}{h(\tau)} d\tau \right. \\ &\quad \left. + \int_{t_0}^s (a(\tau) w\left(\frac{|y(\tau)|}{h(\tau)}\right) + b(\tau) \int_{t_0}^{\tau} k(r) w\left(\frac{|y(r)|}{h(r)}\right) dr) d\tau \right) ds. \end{aligned}$$

Set $u(t) = |y(t)| h(t)^{-1}$. Then, by Corollary 1.11, we have

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t \int_{t_0}^s (a(\tau) + c(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr) d\tau ds \right],$$

where $c = c_1 |y_0| h(t_0)^{-1}$. From the above estimation, we obtain the desired result. Thus, the theorem is proved. \square

REMARK 2.2. Letting $c(s) = 0$ in Theorem 2.1, we obtain the similar result as that of Theorem 3.6 in [9].

THEOREM 2.3. Let $a, b, c, k, w \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (1.2) satisfies

$$\int_{t_0}^t |g(s, y(s))| ds \leq a(t)w(|y(t)|) + b(t) \int_{t_0}^t k(s)|y(s)| ds, \quad t \geq t_0 \geq 0, \tag{2.3}$$

and

$$|h(t, y(t), Ty(t))| \leq \int_{t_0}^t c(s)w(|y(s)|) ds, \tag{2.4}$$

where $a, b, c, k \in L^1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t (a(s) + \int_{t_0}^s c(\tau) d\tau + b(s) \int_{t_0}^s k(\tau) d\tau) ds \right],$$

where $t_0 \leq t < b_1$ and W, W^{-1} are the same functions as in Lemma 1.8 and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^t (a(s) + \int_{t_0}^s c(\tau) d\tau + b(s) \int_{t_0}^s k(\tau) d\tau) ds \in \text{dom}W^{-1} \right\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.1, the solution $z = 0$ of (1.4) is hS. Applying Lemma 1.4, lemma 1.5, together with (2.3) and (2.4) we have

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))| \left(\int_{t_0}^s |g(\tau, y(\tau))| d\tau + |h(s, y(s), Ty(s))| \right) ds \\ &\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 h(t) h(s)^{-1} \left(a(s)w(|y(s)|) \right. \\ &\quad \left. + b(s) \int_{t_0}^s k(\tau)|y(\tau)| d\tau + \int_{t_0}^s c(\tau)w(|y(\tau)|) d\tau \right) ds. \end{aligned}$$

It follows from (H2) and (H3), we obtain

$$\begin{aligned} |y(t)| &\leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^t c_2h(t)a(s)w\left(\frac{|y(s)|}{h(s)}\right)ds \\ &\quad + \int_{t_0}^t c_2h(t)\left(\int_{t_0}^s c(\tau)w\left(\frac{|y(\tau)|}{h(\tau)}\right)d\tau + b(s)\int_{t_0}^s k(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau\right)ds. \end{aligned}$$

Set $u(t) = |y(t)||h(t)^{-1}|$. Now an application of Corollary 1.13 yields

$$|y(t)| \leq h(t)W^{-1}\left[W(c) + c_2\int_{t_0}^t(a(s) + \int_{t_0}^s c(\tau)d\tau + b(s)\int_{t_0}^s k(\tau)d\tau)ds\right],$$

where $c = c_1|y_0|h(t_0)^{-1}$. Thus, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$, and so the proof is complete. \square

REMARK 2.4. Letting $c(s) = 0$ in Theorem 2.3, we obtain the same result as that of Theorem 3.4 in [10].

THEOREM 2.5. Let $a, b, c, k, q, w \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (1.2) satisfies

$$|g(t, y(t))| \leq a(t)w(|y(t)|) + b(t)\int_{t_0}^t k(s)|y(s)|ds \quad (2.5)$$

and

$$|h(t, y(t), Ty(t))| \leq c(t)(w(|y(t)|) + |Ty(t)|), |Ty(t)| \leq \int_{t_0}^t q(s)|y(s)|ds, \quad (2.6)$$

where $t \geq t_0 \geq 0$ and $a, b, c, k, q \in L^1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and it satisfies

$$\begin{aligned} |y(t)| &\leq h(t)W^{-1}\left[W(c) + c_2\int_{t_0}^t[c(s) + \int_{t_0}^s(a(\tau) + b(\tau)\int_{t_0}^{\tau}k(r)dr)d\tau \right. \\ &\quad \left. + c(s)\int_{t_0}^sq(\tau)d\tau]ds\right], \end{aligned}$$

where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\left\{t \geq t_0 : W(c) + c_2\int_{t_0}^t[c(s) + \int_{t_0}^s(a(\tau) + b(\tau)\int_{t_0}^{\tau}k(r)dr)d\tau + c(s)\int_{t_0}^sq(\tau)d\tau]ds \in \text{dom}W^{-1}\right\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.1, the solution $z = 0$ of (1.4) is hS. From Lemma 1.4, Lemma 1.5, together with (2.5), and (2.6), we have

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))| \left(\int_{t_0}^s |g(\tau, y(\tau))| d\tau + |h(s, y(s), Ty(s))| \right) ds \\ &\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 h(t) h(s)^{-1} \left(\int_{t_0}^s (a(\tau) w(|y(\tau)|) \right. \\ &\quad \left. + b(\tau) \int_{t_0}^\tau k(r) |y(r)| dr) d\tau + c(s) (w(|y(s)|) + \int_{t_0}^s q(\tau) |y(\tau)| d\tau) \right) ds. \end{aligned}$$

Using the assumptions (H2) and (H3), we obtain

$$\begin{aligned} |y(t)| &\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 h(t) \left(c(s) w\left(\frac{|y(s)|}{h(s)}\right) \right. \\ &\quad \left. + \int_{t_0}^s (a(\tau) w\left(\frac{|y(\tau)|}{h(\tau)}\right) + b(\tau) \int_{t_0}^\tau k(r) \frac{|y(r)|}{h(r)} dr) d\tau + c(s) \int_{t_0}^s q(\tau) \frac{|y(\tau)|}{h(\tau)} d\tau \right) ds. \end{aligned}$$

Set $u(t) = |y(t)| |h(t)|^{-1}$. Then, by Lemma 1.12, we have

$$\begin{aligned} |y(t)| &\leq h(t) W^{-1} \left[W(c) + c_2 \int_{t_0}^t [c(s) + \int_{t_0}^s (a(\tau) + b(\tau) \int_{t_0}^\tau k(r) dr) d\tau \right. \\ &\quad \left. + c(s) \int_{t_0}^s q(\tau) d\tau] ds \right], \end{aligned}$$

where $c = c_1 |y_0| h(t_0)^{-1}$. The above estimation yields the desired result since the function h is bounded. Hence the proof is complete. □

REMARK 2.6. Letting $a(t) = b(t)$ and $c(t) = 0$ in Theorem 2.5, we obtain the similar result as that of Theorem 3.7 in [10].

THEOREM 2.7. Let $a, b, k, q, w \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (1.2) satisfies

$$\int_{t_0}^t |g(s, y(s))| ds \leq a(t) \left(|y(t)| + \int_{t_0}^t k(s) w(|y(s)|) ds \right) \tag{2.7}$$

and

$$|h(t, y(t), Ty(t))| \leq b(t) (w(|y(t)|) + |Ty(t)|), \quad |Ty(t)| \leq \int_{t_0}^t q(s) w(|y(s)|) ds, \tag{2.8}$$

where $t \geq t_0 \geq 0$ and $a, b, k, q \in L^1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t (a(s) + b(s) + a(s) \int_{t_0}^s k(\tau) d\tau + b(s) \int_{t_0}^s q(\tau) d\tau) ds \right],$$

where $t_0 \leq t < b_1$ and W, W^{-1} are the same functions as in Lemma 1.8 and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^t (a(s) + b(s) + a(s) \int_{t_0}^s k(\tau) d\tau + b(s) \int_{t_0}^s q(\tau) d\tau) ds \in \text{dom} W^{-1} \right\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.1, the solution $z = 0$ of (1.4) is hS. By Lemma 1.4, Lemma 1.5, together with (2.7), and (2.8), we have

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))| \left(\int_{t_0}^s |g(\tau, y(\tau))| d\tau + |h(s, y(s), Ty(s))| \right) ds \\ &\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 h(t) h(s)^{-1} \left(a(s) |y(s)| + b(s) w(|y(s)|) \right. \\ &\quad \left. + a(s) \int_{t_0}^s k(\tau) w(|y(\tau)|) d\tau + b(s) \int_{t_0}^s q(\tau) w(|y(\tau)|) d\tau \right) ds. \end{aligned}$$

By the assumptions (H2) and (H3), we obtain

$$\begin{aligned} |y(t)| &\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 h(t) \left(a(s) \frac{|y(s)|}{h(s)} + b(s) w\left(\frac{|y(s)|}{h(s)}\right) \right. \\ &\quad \left. + b(s) \int_{t_0}^s q(\tau) w\left(\frac{|y(\tau)|}{h(\tau)}\right) d\tau + a(s) \int_{t_0}^s k(\tau) w\left(\frac{|y(\tau)|}{h(\tau)}\right) d\tau \right) ds. \end{aligned}$$

Set $u(t) = |y(t)| h(t)^{-1}$. Now an application of Lemma 1.9 yields

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t (a(s) + b(s) + a(s) \int_{t_0}^s k(\tau) d\tau + b(s) \int_{t_0}^s q(\tau) d\tau) ds \right].$$

Thus any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$. This completes the proof. \square

REMARK 2.8. Letting $b(t) = 0$ in Theorem 2.7, we obtain the similar result as that of Theorem 3.3 in [9].

Acknowledgement. The authors are very grateful for the referee's valuable comments.

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Dong Man Im
Department of Mathematics Education
Cheongju University
Cheongju 360-764, Republic of Korea
E-mail: dmim@cheongju.ac.kr

Yoon Hoe Goo
Department of Mathematics
Hanseο University
Seosan 356-706, Republic of Korea
E-mail: yhgoo@hanseo.ac.kr