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STRONG CONVERGENCE OF AN ITERATIVE ALGORITHM FOR A CLASS OF NONLINEAR SET-VALUED VARIATIONAL INCLUSIONS

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Abstract. In this communication, we introduce an Ishikawa type iterative algorithm for finding the approximate solutions of a class of nonlinear set valued variational inclusion problems. We also establish a characterization of strong convergence of this iterative techniques.

1. Introduction

The variational inequalities were initially studied by Kinderlehrer and Stampacchia [1]. Since then, they have been widely investigated. They cover partial differential equations, optimal controls, optimizations, mathematical programming, mechanics, economics, transportation and finances, see [2-9]. In 1994, Hassouni and Moudafi [10] introduced a class of variational inequalities which includes various classes of variational inequalities as special cases. Since then, there are a great number of numerical methods for solving various variational inequalities and variational inclusions. It is well known that monotonicity, accretivity of the underlying operators and their generalizations plays indispensable roles for solving the generalized variational inequalities and

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generalized variational inclusions. For example, see [11-41].

Let $\mathcal{X}$ be a real Banach space with norm $\|\cdot\|$, $\mathcal{X}^*$ be the topological dual space of $\mathcal{X}$ and $\langle\cdot,\cdot\rangle$ be the generalized duality pairing between $\mathcal{X}$ and $\mathcal{X}^*$. Let J denote the normalized duality mapping from $\mathcal{X}$ to $2^{\mathcal{X}^*}$ defined by

$$Jx = \{f \in \mathcal{X}^* : \langle x, f\rangle = \|x\| \cdot \|f\|, \|f\| = \|x\|\}, x \in \mathcal{X}.$$

In this work, we shall denote the single valued normalized duality mapping by j. It is well known that if $\mathcal{X}$ is a smooth then j is a single valued. Also we denote by $\mathcal{H}(\cdot,\cdot)$, the Hausdorff metric on $CB(\mathcal{X})$ defined by

$$\mathcal{H}(A,B) = \max\{\sup_{x\in B}\inf_{y\in A} d(x,y), \sup_{x\in A}\inf_{y\in B} d(x,y)\}, A, B \in CB(\mathcal{X})$$

where $CB(\mathcal{X})$ denote the family of all nonempty closed and bounded subsets of $\mathcal{X}$.

Definition 1.1 A set valued mapping $A : D(A) \subset \mathcal{X} \to 2^{\mathcal{X}^*}$ is said to be

(i) accretive, if for any $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that for all $u \in Ax, v \in Ay$

$$\langle u - v, j(x-y)\rangle \geq 0;$$

(ii) strictly accretive if A is accretive and

$$\langle u - v, j(x-y)\rangle = 0 \text{ if and only if } x = y;$$

(iii) m-accretive if A is accretive and $(I + \rho A)(D(A)) = \mathcal{X}$ for every $\rho > 0$ where I is an identity mapping;

(iv) ϕ-strongly accretive if for any $x, y \in D(A)$ there exists $j(x-y) \in J(x-y)$ such that for all $u \in Ax$ and $v \in Ay$

$$\langle u - v, j(x-y)\rangle \geq \phi(\|x-y\|)\|x-y\|,$$

where $\phi : [0,\infty) \to [0,\infty)$ is a strictly increasing continuous function with $\phi(0) = 0$;

(v) ϕ-expansive, if any $x, y \in D(A), u \in Ax$ and $v \in Ay$

$$\|u - v\| \geq \phi(\|x-y\|).$$

Remark 1.1

(i) If A is ϕ-strongly accretive then A is ϕ-expansive.

(ii) If $\mathcal{X} = H$ is a Hilbert space, then A is m-accretive if and only if A is maximal monotone. A is said to be maximal monotone if its graph is not properly contained in the graph of any other monotone mapping. In addition, the concepts of accretiveness and strong accretiveness reduce to the one of monotonicity and strong monotonicity, respectively.

Definition 1.2 A mapping $T : \mathcal{X} \to CB(\mathcal{X})$ is said to be

(i) $\mathcal{H}$-continuous if for any given $\epsilon > 0$ and each $\hat{x} \in \mathcal{X}$ there exists a $\delta = \delta(\epsilon, \hat{x}) > 0$ such that whenever $\|\hat{x} - y\| < \delta$ then

$$\mathcal{H}(T\hat{x}, Ty) < \epsilon,$$

(ii) $\mathcal{H}$-uniformly continuous if for any given $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that whenever $\|x - y\| < \delta$ then

$$\mathcal{H}(Tx, Ty) < \epsilon.$$

Definition 1.3 [34] A mapping $T : \mathcal{X} \to 2^{\mathcal{X}}$ is called

(i) lower semi continuous if $T^{-1} = \{x \in E : Tx \cap V \neq \emptyset\}$ is open in $\mathcal{X}$ whenever $V \subset \mathcal{X}$ is open;
(ii) $\gamma - \mathcal{H}$-generalized Lipschitz continuous if there exists a constant $\gamma > 0$ such that for all $x, y \in \mathcal{X}$

$$\mathcal{H}(Tx, Ty) \leq \gamma(1 + \|x - y\|).$$

Definition 1.4 A mapping $N : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is said to be ϕ-strongly accretive with respect to $T : \mathcal{X} \to CB(\mathcal{X})$ in the first argument if for any $x, y \in D(T)$ there exists $j(x - y) \in J(x - y)$ such that for all $z \in \mathcal{X}, u \in Tx$ and $v \in Ty$

$$\langle N(u, z) - N(v, z), j(x - y)\rangle \geq \phi(\|x - y\|)\|x - y\|,$$

where $\phi : [0, \infty) \to [0, \infty)$ is a strictly increasing continuous function with $\phi(0) = 0$.

Definition 1.5 Let $\mathcal{X}$ be a uniformly smooth Banach space and let $N : \underbrace{\mathcal{X} \times \mathcal{X} \cdots X}_{l} \to \mathcal{X}$ be a single valued mapping. N is said to be $(\xi_1, \xi_2, \cdots, \xi_l)$-mixed Lipschitz continuous if there exist constants $\xi_i >$

$0,\ i = 1, 2, \cdots, l$ such that

$$\|N(x_1, x_2, \cdots, x_l) - N(y_1, y_2, \cdots, y_l)\| \leq \sum_{i=1}^{l} \xi_i \|x_i - y_i\|,\ \forall x_i, y_i \in \mathcal{X}, (i = 1, 2, \cdots, l).$$

LEMMA 1.1 [33] *Let $\mathcal{X}$ be a real smooth Banach space, then for any $x, y \in \mathcal{X}$*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle,$$

where $j : \mathcal{X} \to \mathcal{X}^*$ is a normalized duality mapping on $\mathcal{X}$.

LEMMA 1.2 [30] *Let $\mathcal{X}$ be a real smooth Banach space. Let $T, F : \mathcal{X} \to 2^{\mathcal{X}}$ be the two set valued mappings and $N(\cdot, \cdot) : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be a nonlinear mapping satisfying the following conditions:*

- *(i) the mapping $x \to N(x, y)$ is ϕ-strongly accretive with respect to the mapping T,*
- *(ii) the mapping $y \to N(x, y)$ is accretive with respect to the mapping F.*

Then the mapping $S : \mathcal{X} \to 2^{\mathcal{X}}$ defined by $Sx = N(Tx, Fx)$ is ϕ-strongly accretive mapping.

LEMMA 1.3 [26] *Let $\mathcal{X}$ be a real Banach space and $S : \mathcal{X} \to 2^{\mathcal{X}} \backslash \{\phi\}$ be a lower semi continuous and ϕ-strongly accretive mapping. Then for any $x \in \mathcal{X}, Sx$ is a one point set, i.e., S is a single valued mapping.*

LEMMA 1.4 [41] *Let $\mathcal{X}$ be a metric space, $T : \mathcal{X} \to CB(\mathcal{X})$ be a set valued mapping. Therefore for any given $\epsilon > 0$ and for any given $x, y \in \mathcal{X}, u \in Tx$, there exists $v \in Ty$ such that,*

$$d(u, v) \leq (1 + \epsilon)\mathcal{H}(Tx, Ty)$$

where $\mathcal{H}(\cdot, \cdot)$ is a Hausdorff metric on $CB(\mathcal{X})$.

Let $T_1, T_2, \cdots, T_l : \mathcal{X} \to CB(\mathcal{X})$ be the set valued mappings and $N : \underbrace{\mathcal{X} \times \mathcal{X} \cdots \mathcal{X}}_{l} \to \mathcal{X}$ be the nonlinear mapping. Let $F : \mathcal{X} \to 2^{\mathcal{X}}$ be a set valued mapping and let $A : D(A) \subset \mathcal{X} \to 2^{\mathcal{X}}$ be a m-accretive mapping and let $g : \mathcal{X} \to D(A)$ and $h : \mathcal{X} \to \mathcal{X}$ be the single valued mappings. In this paper, we will consider the following nonlinear set valued variational inclusion problems: for any given $f \in \mathcal{X}, \lambda > 0$, finding $x \in D(A)$ such that $(x, w_1, \cdots w_l, v)$ is a solution set of

$$f \in h(v) + N(w_1, w_2, \cdots, w_l) + \lambda(A(g(x))) \tag{1.1}$$

for any $w_i \in T_i(x), i = 1, 2, \cdots, l$ and $v \in F(x)$.

2. Main Results

In this section, we define an Ishikawa type iterative algorithm for solving the nonlinear set valued variational inclusion problems (1.1). Moreover, we give a characterization of strong convergence of this algorithm to the solutions set of the nonlinear set valued variational inclusion problems (1.1)

Algorithm 2.1 Let $\epsilon > 0$ be a some given number. Let $\mathcal{X}$ be a real smooth Banach space. Let $A : D(A) \subset \mathcal{X} \to 2^{\mathcal{X}}$ be a m-accretive mapping, $g : \mathcal{X} \to D(A)$ and $h : \mathcal{X} \to \mathcal{X}$ be the single valued mappings. Let $F : \mathcal{X} \to CB(\mathcal{X})$ and $T_1, T_2, \cdots, T_l : \mathcal{X} \to CB(\mathcal{X})$ be the set valued mappings and $N : \underbrace{\mathcal{X} \times \mathcal{X} \cdots \mathcal{X}}_{l} \to \mathcal{X}$ be the nonlinear mapping. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be the two sequences in $[0, 1]$. For any given initial $x_0 \in D(A), \overline{w}_{0,i} \in T_i(x_0),\ i = 1, 2, \cdots, l, \overline{v}_0 \in F(x_0)$ and $\overline{u}_0 \in A(g(x_0))$ define by

$$y_0 = (1 - \beta_0)x_0 + \beta_0(f + x_0 - h(\overline{v}_o) - N(\overline{w}_{0,1}, \overline{w}_{0,2}, \cdots, \overline{w}_{0,l}) - \lambda\overline{u}_0).$$

Since $\overline{u}_0 \in A(g(x_0))$, by Lemma 1.4, there exists $u_0 \in A(g(y_0))$ such that

$$\|\overline{u}_0 - u_0\| \leq (1 + \epsilon)\ \mathcal{H}(A(g(x_0)), A(g(y_0))),$$

$\overline{w}_{0,i} \in T_i(x_0), w_{0,i} \in T_i(y_0)$

$$\|\overline{w}_{0,i} - w_{0,i}\| \leq (1 + \epsilon)\ \mathcal{H}(T_i(x_0), T_i(y_0)),\ \text{for } i = 1, 2, \cdots l,$$

and $\overline{v}_0 \in F(x_0), v_0 \in F(y_0)$

$$\|\overline{v}_0 - v_0\| \leq (1 + \epsilon)\ \mathcal{H}(F(x_0), F(y_0)).$$

For any given $w_{0,i} \in T_i(y_0),\ i = 1, 2, \cdots, l$ and $u_0 \in A(g(y_0)), v_0 \in F(y_0)$, define

$$x_1 = (1 - \alpha_0)x_0 + \alpha_0(f + y_0 - h(v_0) - N(w_{0,1}, \cdots, w_{0,l}) - \lambda u_0).$$

Since $\overline{u}_0 \in A(g(x_0))$, by Lemma 1.4, there exists $\overline{u}_1 \in A(g(x_1))$ such that

$$\|\overline{u}_0 - \overline{u}_1\| \leq (1+\epsilon)\ \mathcal{H}(A(g(x_0)), A(g(x_1))),$$

$\overline{w}_{0,i} \in T_i(x_0), \overline{w}_{1,i} \in T_i(x_1)$

$$\|\overline{w}_{0,i} - \overline{w}_{1,i}\| \leq (1+\epsilon)\ \mathcal{H}(T_i(x_0), T_i(x_1)),\ \text{for } i = 1, 2, \cdots l,$$

and $\overline{v}_0 \in F(x_0), \overline{v}_1 \in F(x_1)$

$$\|\overline{v}_0 - \overline{v}_1\| \leq (1+\epsilon)\ \mathcal{H}(F(x_0), F(x_1)).$$

For any given $\overline{w}_{1,i} \in T_i(x_1)$, define

$$y_1 = (1-\beta_1)x_1 + \beta_1(f + x_1 - h(\overline{v}_1) - N(\overline{w}_{1,1}, \overline{w}_{1,2}, \cdots, \overline{w}_{1,l}) - \lambda\overline{u}_1).$$

Since $\overline{u}_1 \in A(g(x_1))$, by Lemma 1.4, there exists $u_1 \in A(g(y_1))$ such that

$$\|\overline{u}_1 - u_1\| \leq (1+\epsilon)\ \mathcal{H}(A(g(x_1)), A(g(y_1))),$$

$\overline{w}_{1,i} \in T_i(x_1), w_{1,i} \in T_i(y_1)$

$$\|\overline{w}_{1,i} - w_{1,i}\| \leq (1+\epsilon)\ \mathcal{H}(T_i(x_1), T_i(y_1)),\ \text{for } i = 1, 2, \cdots l,$$

and $\overline{v}_1 \in F(x_1), v_1 \in F(y_1)$

$$\|\overline{v}_1 - v_1\| \leq (1+\epsilon)\ \mathcal{H}(F(x_1), F(y_1)).$$

For any given $w_{1,i} \in T_i(y_1),\ i = 1, 2, \cdots, l$ and $u_1 \in A(g(y_1)), v_1 \in F(y_1)$, define

$$x_2 = (1-\alpha_1)x_1 + \alpha_1(f + y_1 - h(v_1) - N(w_{1,1}, \cdots, w_{1,l}) - \lambda u_1).$$

Continuing in this way, we can compute the sequences $\{x_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ and $\{w_{n,i}\}_{n=0}^{\infty}$, $i = 1, 2, \cdots, l$ by the iterative schemes such that for $n = 0, 1, 2, 3 \cdots$

(i)

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(f + y_n - h(v_n) - N(w_{n,1}, \cdots, w_{n,l}) - \lambda u_n),$$

$$\text{for } w_{n,i} \in T_i(y_n),\ i = 1, 2, \cdots, l, v_n \in F(y_n), u_n \in A(g(y_n));$$

(ii)

$$y_n = (1-\beta_n)x_n + \beta_n(f + x_n - h(\overline{v}_n) - N(\overline{w}_{n,1}, \overline{w}_{n,2}, \cdots, \overline{w}_{n,l}) - \lambda\overline{u}_n),$$

for any $\overline{w}_{n,i} \in T_i(x_n)$, $i = 1, 2, \cdots, l, \overline{v}_n \in F(x_n)$ and $\overline{u}_n \in A(g(x_n))$;

(iii) $\overline{u}_n \in A(g(x_n)), u_n \in A(g(y_n))$

$$\|\overline{u}_n - u_n\| \leq (1+\epsilon)\ \mathcal{H}(A(g(x_n)), A(g(y_n)));$$

$\overline{w}_{n,i} \in T_i(x_n), w_{n,i} \in T_i(y_n)$

$$\|\overline{w}_{n,i} - w_{n,i}\| \leq (1+\epsilon)\ \mathcal{H}(T_i(x_n), T_i(y_n)),\ \text{for } i = 1, 2, \cdots l,$$

and $\overline{v}_n \in F(x_n), v_n \in F(y_n)$

$$\|\overline{v}_n - v_n\| \leq (1+\epsilon)\ \mathcal{H}(F(x_n), F(y_n));$$

(iv) $\overline{u}_n \in A(g(x_n)), \overline{u}_{n+1} \in A(g(x_{n+1}))$

$$\|\overline{u}_n - \overline{u}_{n+1}\| \leq (1+\epsilon)\ \mathcal{H}(A(g(x_n)), A(g(x_{n+1}))),$$

$\overline{w}_{n,i} \in T_i(x_n), \overline{w}_{n+1,i} \in T_i(x_{n+1});$

$$\|\overline{w}_{n,i} - \overline{w}_{n+1,i}\| \leq (1+\epsilon)\ \mathcal{H}(T_i(x_n), T_i(x_{n+1})),\ \text{for } i = 1, 2, \cdots l,$$

and $\overline{v}_n \in F(x_n), \overline{v}_{n+1} \in F(x_{n+1})$

$$\|\overline{v}_n - \overline{v}_{n+1}\| \leq (1+\epsilon)\ \mathcal{H}(F(x_n), F(x_{n+1})).$$

The sequences $\{x_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ and $\{w_{n,i}\}_{n=0}^{\infty}, i = 1, 2, \cdots, l$ defined by (i)-(iv) is called the Ishikawa type iterative sequences.

Theorem 2.1 *Let $\mathcal{X}$ be a real smooth Banach space. Let $T_1, T_2, \cdots, T_l : \mathcal{X} \to CB(\mathcal{X})$ and $A : D(A) \subset \mathcal{X} \to 2^{\mathcal{X}}$ be the set valued mappings. Let $g : \mathcal{X} \to D(A)$ and $h : \mathcal{X} \to \mathcal{X}$ be the single valued mappings. Let $F : \mathcal{X} \to CB(\mathcal{X})$ be the set valued mapping and let $N : \underbrace{\mathcal{X} \times \mathcal{X} \cdots \mathcal{X}}_{l} \to \mathcal{X}$ be a single valued continuous mapping satisfying the following conditions:*

(i) *$Aog : \mathcal{X} \to 2^{\mathcal{X}}$ is m-accretive and $\mathcal{H}$-uniformly continuous mapping;*
(ii) *$T_1, T_2, \cdots, T_l : \mathcal{X} \to CB(\mathcal{X})$ is lower semi continuous and $\gamma_i - \mathcal{H}$-Lipschitz continuous mapping for $i = 1, 2, \cdots, l$*
(iii) *the mapping $x_1 \to N(x_1, x_2, \cdots, x_l)$ is ϕ-strongly accretive and ξ_i-mixed Lipschitz continuous mapping with constants $\xi_i > 0,\ i = 1, 2, \cdots, l$ with respect to T_i where $\phi : [0, \infty) \to [0, \infty)$ is a strictly increasing function with $\phi(0) = 0$;*
(iv) *the mapping $x \to N(T_1 x, \cdots, T_l x)$ is uniformly continuous;*
(v) *h is δ-Lipschitz continuous mapping;*
(vi) *$F : \mathcal{X} \to CB(\mathcal{X})$ be the lower semi continuous and $\eta - \mathcal{H}$-Lipschitz continuous mapping;*

Suppose $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are the sequences in $[0,1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \beta_n = 0$. Then the sequences $\{x_n\}_{n=0}^{\infty}$, $\{w_{n,i}\}_{n=0}^{\infty}$, $i = 1, 2, \cdots, l$ and $\{v_n\}_{n=0}^{\infty}$ generated by Algorithm 2.1 converges strongly to $x^* \in X$, if and only if the sequences $\{x_n\}_{n=0}^{\infty}$, $\{w_{n,i}\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}, \{\mathcal{H}(A(g(x_n)), A(g(x^*)))\}_{n=0}^{\infty}, \{\mathcal{H}(T_i(x_n), T_i(x^*))\}_{n=0}^{\infty}$, $\{\mathcal{H}(F(x_n), F(x^*))\}_{n=0}^{\infty}$, $i = 1, 2, \cdots, l$ are bounded where for any $u^* \in A(g(x^*)), v^* \in F(x^*) \in \mathcal{X}, w_i^* \in T_i(x^*) \in \mathcal{X}$, (x^*, w_i^*, v^*), $i = 1, 2, \cdots l$ is a solution set of nonlinear set valued variational inclusion problems (1.1).

Proof. Let $x^* \in \mathcal{X}$ such that $f \in h(v) + N(w_1, \cdots, w_l) + \lambda Aog(x^*)$ for any $w_i \in T_i(x^*), v \in F(x^*), i = 1, 2, \cdots l$, whose existence is guaranted by [35, Theorem 3.1] without loss of generality, we may assume that $f \equiv 0$ and $\lambda = 1$. Take $w_i^* \in T_i(x^*), i = 1, 2, \cdots l$, and $u^* \in A(g(x^*)), v^* \in F(x^*)$ such that

$$0 = h(v^*) + N(w_1^*, \cdots, w_l^*) + u^*.$$

Necessity.
Suppose that

$$\|x_n - x^*\| \to 0 \text{ as } n \to \infty.$$

Therefore, from the $\mathcal{H}$-uniform continuity of Aog,

$$\mathcal{H}(A(g(x_n)), A(g(x^*))) \to 0 \text{ as } n \to \infty.$$

Again from

$$\|w_{n,i} - w_i^*\| \to 0 \text{ as } n \to \infty \text{ for } i = 1, 2, \cdots, l,$$
$$\|v_n - v^*\| \to 0 \text{ as } n \to \infty,$$

it follows from the $\mathcal{H}$-uniform continuity of T_i and F that

$$\mathcal{H}(T_i(x_n), T_i(x^*)) \to 0 \text{ as } n \to \infty, \text{ for } i = 1, 2, \cdots, l,$$
$$\mathcal{H}(F(x_n), F(x^*)) \to 0 \text{ as } n \to \infty.$$

Thus the sequences $\{x_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}, \{w_{n,i}\}_{n=0}^{\infty}, \{\mathcal{H}(A(g(x_n)), A(g(x^*)))\}_{n=0}^{\infty}$, $\{\mathcal{H}(T_i(x_n), T_i(x^*))\}_{n=0}^{\infty}$ for $i = 1, 2, \cdots, l$
and $\{\mathcal{H}(F(x_n), F(x^*))\}_{n=0}^{\infty}$ are all bounded.

Sufficiency.
Suppose that the sequences $\{x_n\}_{n=0}^{\infty}, \{w_{n,i}\}_{n=0}^{\infty}, \{\mathcal{H}(A(g(x_n)), A(g(x^*)))\}_{n=0}^{\infty}$, $\{\mathcal{H}(T_i(x_n), T_i(x^*))\}_{n=0}^{\infty}$, $\{\mathcal{H}(F(x_n), F(x^*))\}_{n=0}^{\infty}$ for $i = 1, 2, \cdots, l$, are all bounded. Then we divide the proof of the sufficiency into five steps:

Step 1. We claim that the sequences $\{\overline{w}_{n,i}\}_{n=0}^{\infty}, \{\overline{u}_n\}_{n=0}^{\infty}, \{\overline{v}_n\}_{n=0}^{\infty}$, $i = 1, 2, \cdots, l$ are bounded. Indeed by Nadler's Theorem for each $n \geq 0$ there exist $\overline{w}_i \in T_i(x^*), i = 1, 2, \cdots, l, \overline{v} \in F(x^*)$ and $\overline{u} \in A(g(x^*))$ such that

$$\begin{aligned} \|\overline{w}_{n,i} - \overline{w}_i\| &\leq (1+\epsilon_i)\ \mathcal{H}(T_i(x_n), T_i(x^*)),\ \ i = 1, 2, \cdots, l; \\ \|\overline{u}_n - \overline{u}\| &\leq (1+\epsilon)\ \mathcal{H}(A(g(x_n)), A(g(x^*))); \\ \|\overline{v}_n - \overline{v}\| &\leq (1+\epsilon)\ \mathcal{H}(F(x_n), F(x^*)). \end{aligned}$$

Since $T_i : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ is $\gamma_i - \mathcal{H}$-generalized Lipschitz continuous and $F : \mathcal{X} \to CB(\mathcal{X})$ is $\eta - \mathcal{H}$-generalized Lipschitz continuous mapping, we have

$$\begin{aligned} \|\overline{w}_{n,i} - w_i^*\| &\leq \|\overline{w}_{n,i} - \overline{w}_i\| + \|w_i^* - \overline{w}_i\| \\ &\leq (1+\epsilon_i)\ \mathcal{H}(T_i(x_n), T_i(x^*)) + \operatorname{diam}(T_i(x^*)),\ \ i = 1, 2, \cdots, l \\ &\leq (1+\epsilon_i)\gamma_i(1 + \|x_n - x^*\|) + \operatorname{diam}(T_i(x^*)),\ \ i = 1, 2, \cdots, l; \end{aligned} \tag{2.2}$$

$$\begin{aligned} \|\overline{v}_n - v^*\| &\leq \|\overline{v}_n - \overline{v}\| + \|v^* - \overline{v}\| \\ &\leq (1+\epsilon)\ \mathcal{H}(F(x_n), F(x^*)) + \operatorname{diam}(F(x^*)) \\ &\leq (1+\epsilon)\eta(1 + \|x_n - x^*\|) + \operatorname{diam}(F(x^*)). \end{aligned} \tag{2.3}$$

Thus the sequences $\{\overline{w}_{n,i}\}_{n=0}^{\infty}, \{\overline{v}_n\}_{n=0}^{\infty}$ are bounded. Also, we note that

$$\begin{aligned} \|\overline{u}_n - u^*\| &\leq \|\overline{u}_n - \overline{u}\| + \|u^* - \overline{u}\| \\ &\leq (1+\epsilon)\ \mathcal{H}(A(g(x_n)), A(g(x^*))) + \operatorname{diam}(A(g(x^*))) \\ &\leq (1+\epsilon)(1 + \|A(g(x_n)) - A(g(x^*))\|) + \operatorname{diam}(A(g(x^*))). \end{aligned} \tag{2.4}$$

Hence the sequence $\{u_n\}_{n=0}^{\infty}$ is a bounded.

Step 2. We claim that $\|y_n - x_n\| \to 0$ as $n \to \infty$. Indeed, since the mapping $x \to N(w_1, w_2 \cdots w_l)$ is $\xi_i - \mathcal{H}$-mixed Lipschitz continuous mapping with respect to $T_i, i = 1, 2, \cdots l$ and from Algorithm 2.1, we have

$$\|y_n - x_n\| = \| - \beta_n(h(\overline{v}_n) + N(\overline{w}_{n,1}, \cdots, \overline{w}_{n,l}) + \overline{u}_n)\|$$

$$\leq \beta_n\|h(\overline{v}_n) - h(v^*)\| + \beta_n\|N(w_1^*, \cdots, w_l^*) - N(\overline{w}_{n,1}, \cdots \overline{w}_{n,l})\| + \beta_n\|\overline{u}_n - u^*\|$$

$$\leq \beta_n\delta\|\overline{v}_n - v^*\| + \beta_n(\|N(\overline{w}_{n,1}, \cdots \overline{w}_{n,l}) - N(w_1^*, \overline{w}_{n,2} \cdots \overline{w}_{n,l})\|$$

$$+ \cdots + \|N(w_1^*, \cdots \overline{w}_{n,l}) - N(w_1^*, \cdots, w_l^*)\|) + \beta_n\|\overline{u}_n - u^*\|$$

$$\leq \beta_n\delta\|\overline{v}_n - v^*\| + \beta_n(\xi_1\|\overline{w}_{n,1}, -w_1^*\| + \cdots + \xi_l\|\overline{w}_{n,l} - w_l^*\|) + \beta_n\|\overline{u}_n - u^*\|,$$

which implies that $\|y_n - x_n\| \to 0$ as $n \to \infty$.

STEP 3. We claim that the sequences $\{v_n\}_{n=0}^{\infty}, \{w_{n,1}\}_{n=0}^{\infty}, \cdots \{w_{n,l}\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ are bounded. Indeed by Algorithm 2.1(iii), we have

$$\begin{aligned}\|u_n\| &\leq \|\overline{u}_n - u_n\| + \|\overline{u}_n\| \\ &\leq (1+\epsilon)\ \mathcal{H}(A(g(x_n)), A(g(y_n))) + \|\overline{u}_n\|. \quad (2.5)\end{aligned}$$

$$\begin{aligned}\|w_{n,i}\| &\leq \|\overline{w}_{n,i} - w_{n,i}\| + \|\overline{w}_{n,i}\| \\ &\leq (1+\epsilon_i)\ \mathcal{H}(T_i(x_n), T_i(y_n)) + \|\overline{w}_{n,i}\|, \text{ for } i = 1, 2, \cdots, l. \quad (2.6)\end{aligned}$$

$$\begin{aligned}\|v_n\| &\leq \|\overline{v}_n - v_n\| + \|\overline{v}_n\| \\ &\leq (1+\epsilon)\ \mathcal{H}(F(x_n), F(y_n)) + \|\overline{v}_n\|. \quad (2.7)\end{aligned}$$

From step 2 and $\mathcal{H}$-uniform continuity of Aog, T_i, and F, we have

$$\begin{aligned}&\mathcal{H}(A(g(x_n)), A(g(y_n))) \to 0 \text{ as } n \to \infty; \\ &\mathcal{H}(T_i(x_n), T_i(y_n)) \to 0 \text{ as } n \to \infty, \text{ for } i = 1, 2, \cdots, l; \\ &\mathcal{H}(F(x_n), F(y_n)) \to 0 \text{ as } n \to \infty.\end{aligned}$$

Hence the sequences $\{\mathcal{H}(A(g(x_n)), A(g(y_n)))\}_{n=0}^{\infty}, \{\mathcal{H}(T_i(x_n), T_i(y_n))\}_{n=0}^{\infty}$, $\{\mathcal{H}(F(x_n), F(y_n))\}_{n=0}^{\infty}$ for $i = 1, 2 \cdots, l$ are bounded. Thus it follows from the boundedness of $\{\overline{u}_n\}_{n=0}^{\infty}, \{\overline{w}_{n,i}\}_{n=0}^{\infty}, \{\overline{v}_n\}_{n=0}^{\infty}$ and (2.1) that $\{u_n\}_{n=0}^{\infty}$, $\{w_{n,i}\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ are bounded for $i = 1, 2, \cdots, l$.
On there hand, by Nadler's Theorem for each $n \geq 0$ there exists $\widehat{w}_i \in T_i(x^*), i = 1, 2, \cdots l, \widehat{v} \in F(x^*)$ such that

$$\begin{aligned}\|v_n - \widehat{v}\| &\leq (1+\epsilon)\ \mathcal{H}(F(y_n), F(x^*)); \\ \|w_{n,i} - \widehat{w}_i\| &\leq (1+\epsilon_i)\ \mathcal{H}(T_i(y_n), T_i(x^*)),\ i = 1, 2, \cdots l.\end{aligned}$$

Observe that

$$\begin{aligned}\|w_{n,i} - w_i^*\| &\leq \|w_{n,i} - \widehat{w}_i\| + \|\widehat{w}_i - w_i^*\| \\ &\leq (1+\epsilon_i)\ \mathcal{H}(T_i(y_n), T_i(x^*)) + \mathrm{diam}(T_i(x^*)) \\ &\leq (1+\epsilon_i)\gamma_i(1 + \|y_n - x^*\|) + \mathrm{diam}(T_i(x^*)), \text{ for } i = 1, 2, \cdots, l.\end{aligned}$$

$$\begin{aligned}\|v_n - v^*\| &\leq \|v_n - \widehat{v}\| + \|\widehat{v} - v^*\| \\ &\leq (1+\epsilon)\ \mathcal{H}(F(y_n), F(x^*)) + \mathrm{diam}(F(x^*)) \\ &\leq (1+\epsilon)\eta(1 + \|y_n - x^*\|) + \mathrm{diam}(F(x^*)).\end{aligned}$$

Since the sequence $\{x_n\}_{n=0}^{\infty}$ is bounded and $\|y_n - x_n\| \to 0$ as $n \to \infty$, so $\{y_n\}_{n=0}^{\infty}$ is bounded. Hence $\{w_{n,i}\}_{n=0}^{\infty},\ i = 1, 2, \cdots l$ and $\{v_n\}_{n=0}^{\infty}$ are bounded.

STEP 4. We claim that

$$\|h(v_n)+N(w_{n,1},\cdots,w_{n,l})+u_n-h(\overline{v}_{n+1})-N(\overline{w}_{n+1,1},\cdots,\overline{w}_{n+1,l})-\overline{u}_{n+1}\| \to 0$$

as $n \to \infty$. Since

$$\|h(v_n) - h(v^*)\| \leq \delta\|v_n - v^*\|$$

and

$$\|N(w_{n,1},\cdots,w_{n,l})-N(w_1^*,\cdots,w_l^*)\| \leq \xi_1\|w_{n,1}-w_1^*\|+\cdots+\xi_l\|w_{n,l}-w_l^*\|.$$

It follows from the boundedness of $\{v_n\}_{n=0}^{\infty}, \{w_{n,i}\}_{n=0}^{\infty}$, $i = 1, 2\cdots l$, that $\{h(v_n)\}_{n=0}^{\infty}$ and $\{N(w_{n,1},\cdots,w_{n,l})\}_{n=0}^{\infty}$ are bounded. Furthermore, from Algorithm 2.1 (i) we deduce that as $n \to \infty$,

$$\|x_{n+1} - x_n\| = \alpha_n\|y_n - x_n - h(v_n) - N(w_{n,1},\cdots w_{n,l}) - u_n\|$$

$$\leq \alpha_n\|y_n - x_n\| + \alpha_n\|h(v_n) + N(w_{n,1},\cdots w_{n,l}) + u_n\| \to 0$$

and hence

$$\|y_n - x_{n+1}\| \leq \|y_n - x_n\| + \|x_n - x_{n+1}\| \to 0.$$

From [36], the mapping $x \to h(F(x))$ and $x \to N(T_1(x),\cdots,T_l(x))$ are single valued. Thus from condition (vii) and $\|y_n - x_{n+1}\| \to 0$ as $(n \to \infty)$, we conclude that as $n \to \infty$

$$\|N(w_{n,1},\cdots w_{n,l}) - N(\overline{w}_{n+1,1},\cdots\overline{w}_{n+1,l})\| \to 0. \tag{2.8}$$

On the other hand, Algorithm 2.1 and the $\mathcal{H}$-uniform continuity of Aog, we conclude that as $n \to \infty$,

(2.9)

$$\begin{aligned}
&\|u_n - \overline{u}_{n+1}\| \\
&\leq \|u_n - \overline{u}_n\| + \|\overline{u}_n - \overline{u}_{n+1}\| \\
&\leq (1+\epsilon)\ \mathcal{H}(A(g(y_n)), A(g(x_n))) + (1+\epsilon)\ \mathcal{H}(A(g(x_n)), A(g(x_{n+1}))) \to 0.
\end{aligned}$$

Therefore from (2.7) and (2.8), the assertion is valid.

STEP 5. We claim that the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to x^*. Indeed from [34], the set valued mapping

$$x \to h(F(x)) + N(T_1(x),\cdots,T_l(x)) + A(g(x))$$

is ϕ-strongly accretive mapping. Hence, we have

$$\begin{aligned}
&\langle x_{n+1} - h(\overline{v}_{n+1}) - N(\overline{w}_{n+1,1}, \cdots, \overline{w}_{n+1,l}) - \overline{u}_{n+1} - (x^* - h(v^*) \\
&\qquad - N(w_1^*, \cdots, w_l^*) - u^*), j(x_{n+1} - x^*)\rangle \\
&\leq \|x_{n+1} - x^*\|^2 - \langle h(\overline{v}_{n+1}) + N(\overline{w}_{n+1,1}, \cdots, \overline{w}_{n+1,l}) + \overline{u}_{n+1} \\
&\qquad - (h(v^*) + N(w_1^*, \cdots, w_l^*) + u^*), j(x_{n+1} - x^*)\rangle \\
&\leq \|x_{n+1} - x^*\|^2 - \phi(\|x_{n+1} - x^*\|)\|x_{n+1} - x^*\|.
\end{aligned} \tag{2.10}$$

We observe that from Algorithm 2.1 (i), Lemma 1.6 and (2.9)

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|(1 - \alpha_n)(x_n - x^*) + \alpha_n(y_n - x^* - h(v_n) - N(w_{n,1}, \cdots, w_{n,l}) - u_n)\|^2 \\
&\leq (1 - \alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\langle y_n - x^* - h(v_n) - N(w_{n,1}, \cdots, w_{n,l}) \\
&\quad - u_n, j(x_{n+1} - x^*)\rangle \\
&\leq (1 - \alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\langle y_n - h(v_n) - N(w_{n,1}, \cdots, w_{n,l}) \\
&\quad - u_n - (x_{n+1} - h(\overline{v}_{n+1}) - N(\overline{w}_{n+1,1}, \cdots, \overline{w}_{n+1,l}) - \overline{u}_{n+1}), j(x_{n+1} - x^*)\rangle \\
&\quad + 2\alpha_n\langle x_{n+1} - h(\overline{v}_{n+1}) - N(\overline{w}_{n+1,1}, \cdots, \overline{w}_{n+1,l}) - \overline{u}_{n+1} - (x^* - h(v^*) \\
&\quad - N(w_1^*, \cdots, w_l^*) - u^*), j(x_{n+1} - x^*)\rangle \\
&\leq (1 - \alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\|y_n - h(v_n) - N(w_{n,1}, \cdots, w_{n,l}) - u_n - (x_{n+1} \\
&\quad - h(\overline{v}_{n+1}) - N(\overline{w}_{n+1,1}, \cdots, \overline{w}_{n+1,l}) - \overline{u}_{n+1})\|\|x_{n+1} - x^*\| + 2\alpha_n\|x_{n+1} - x^*\|^2 \\
&\quad - 2\alpha_n\phi(\|x_{n+1} - x^*\|)\|x_{n+1} - x^*\|.
\end{aligned} \tag{2.11}$$

Since $\{x_n\}_{n=0}^{\infty}$ is bounded, we have

$$M = \sup\{\|x_n - x^*\| : n \geq 0\} < \infty.$$

Note that $\alpha_n \to 0$ as $n \to \infty$. Thus without loss of generality, we may assume that $1 - 2\alpha_n > 0$ for all $n \geq 0$. Hence it follows from (2.10) that

(2.12)

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\le \frac{(1-\alpha_n)^2}{1-2\alpha_n}\|x_n-x^*\|^2+\frac{2\alpha_n}{1-2\alpha_n}\|y_n-h(v_n)-N(w_{n,1},\cdots,w_{n,l}) \\
&\quad -u_n-(x_{n+1}-h(\overline{v}_{n+1})-N(\overline{w}_{n+1,1},\cdots,\overline{w}_{n+1,l})-\overline{u}_{n+1})\|\|x_{n+1}-x^*\| \\
&\quad -\frac{2\alpha_n}{1-2\alpha_n}\phi(\|x_{n+1}-x^*\|)\|x_{n+1}-x^*\| \\
&\le \|x_n-x^*\|^2+\frac{2\alpha_n}{1-2\alpha_n}\{\frac{M^2\alpha_n}{2}+M\|y_n-h(v_n)-N(w_{n,1},\cdots,w_{n,l}) \\
&\quad -u_n-(x_{n+1}-h(\overline{v}_{n+1})-N(\overline{w}_{n+1,1},\cdots,\overline{w}_{n+1,l})-\overline{u}_{n+1})\|\} \\
&\quad -\frac{2\alpha_n}{1-2\alpha_n}\phi(\|x_{n+1}-x^*\|)\|x_{n+1}-x^*\|.
\end{aligned}$$

Next, we discuss two possible cases:

Case(1).

$$\inf\{\|x_{n+1}-x^*\|:n\ge 0\}=\sigma>0.$$

From step 4, we have

$$\begin{aligned}
&M^2\alpha_n+2M\|y_n-h(v_n)-N(w_{n,1},\cdots,w_{n,l})-u_n-(x_{n+1}-h(\overline{v}_{n+1}) \\
&\qquad -N(\overline{w}_{n+1,1},\cdots,\overline{w}_{n+1,l})-\overline{u}_{n+1})\|\to 0 \text{ as } n\to\infty.
\end{aligned}$$

Hence there exists a possible integer N_0 such that for all $n\ge N_0$,

(2.13)

$$\begin{aligned}
&M^2\alpha_n+2M\|y_n-h(v_n)-N(w_{n,1},\cdots,w_{n,l})-u_n-(x_{n+1}-h(\overline{v}_{n+1}) \\
&\qquad -N(\overline{w}_{n+1,1},\cdots,\overline{w}_{n+1,l})-\overline{u}_{n+1})\| \\
&< \phi(\sigma)\sigma.
\end{aligned}$$

Thus, it follows from (2.11) and (2.12) that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \|x_n-x^*\|^2+\frac{\alpha_n}{1-2\alpha_n}\phi(\sigma)\sigma-\frac{2\alpha_n}{1-2\alpha_n}\phi(\sigma)\sigma \\
&\le \|x_n-x^*\|^2-\frac{\alpha_n}{1-2\alpha_n}\phi(\sigma)\sigma,
\end{aligned}$$

which implies that

$$\phi(\sigma)\sigma\sum_{n=N_0}^{\infty}\alpha_n\|x_{N_0}-x^*\|^2<\infty.$$

This contradicts $\sum_{n=0}^{\infty} \alpha_n = \infty$. Therefore case (1) is false.

CASE(2).

$$\inf\{\|x_{n+1} - x^*\| : n \geq 0\} = 0.$$

In this case, there exists a subsequence $\{x_{n_{j+1}}\} \subset \{x_n\}$ such that

$$\|x_{n_{j+1}} - x^*\| \to 0 \text{ as } j \to \infty.$$

Hence, for any given $\epsilon > 0$ there exists a positive integer n_j such that for all $n \geq n_j$

$$\|x_{n_{j+1}} - x^*\| \leq \varepsilon$$

(2.14)

$$\begin{aligned} M^2\alpha_n + 2M\|y_n - h(v_n) - N(w_{n,1}, \cdots, w_{n,l}) - u_n - (x_{n+1} - h(\overline{v}_{n+1}) \\ -N(\overline{w}_{n+1,1}, \cdots, \overline{w}_{n+1,l}) - \overline{u}_{n+1})\| < \phi(\varepsilon)\varepsilon. \end{aligned}$$

We claim that

$$\|x_{n_{j+m}} - x^*\| \leq \varepsilon, \ \forall\, m \geq 1.$$

Indeed, we prove that

$$\|x_{n_{j+2}} - x^*\| \leq \varepsilon.$$

If this false then

$$\|x_{n_{j+2}} - x^*\| \geq \varepsilon.$$

Hence, we have

$$\phi(\|x_{n_{j+2}} - x^*\|) \geq \phi(\varepsilon).$$

Thus, it follow from (2.11) that

$$\begin{aligned} \varepsilon^2 < \|x_{n_{j+2}} - x^*\|^2 &\leq \|x_{n_{j+1}} - x^*\|^2 + \frac{\alpha_{n_{j+1}}}{1 - 2\alpha_{n_{j+1}}}\phi(\varepsilon)\varepsilon - \frac{2\alpha_{n_{j+1}}}{1 - 2\alpha_{n_{j+1}}}\phi(\varepsilon)\varepsilon \\ &= \|x_{n_{j+1}} - x^*\|^2 - \frac{\alpha_{n_{j+1}}}{1 - 2\alpha_{n_{j+1}}}\phi(\varepsilon)\varepsilon \\ &\leq \|x_{n_{j+1}} - x^*\|^2 \leq \varepsilon^2. \end{aligned}$$

This contradict to show that

$$\|x_{n_{j+2}} - x^*\| \leq \varepsilon.$$

By induction we can show that

$$\|x_{n_{j+m}} - x^*\| \leq \varepsilon, \ \forall\, m \geq 1.$$

Hence

$$\|x_n - x^*\| \to 0 \text{ as } n \to \infty.$$

This completes the proof. □

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