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POSITIVE SOLUTIONS OF SUPERLINEAR AND SUBLINEAR BOUNDARY VALUE PROBLEMS

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Abstract. We study the existence of positive solutions of second order nonlinear separated boundary value problems of superlinear as well as sublinear type without imposing monotonicity restrictions on the problem. The type of problem investigated cannot be analyzed using the linearization about the trivial solution because either it does not exist (the sublinear case) or is trivial (the superlinear case). The results follow from a known fixed point theorem by noticing that the concavity of the solutions provides an important condition for the applicability of the fixed point result.

1. Introduction

We study boundary value problems of the form

$$\begin{cases} x'' + f(t,x) = 0 \\ \alpha x(0) - \beta x'(0) = 0 \\ \gamma x(1) + \delta x'(1) = 0, \end{cases}$$

The function $f : (0,1) \times [0,\infty) \to [0,\infty)$ satisfies growth conditions at both zero and infinity, but no monotonicity assumptions are imposed on f. This problem, imposing a monotonicity assumption, was studied

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in [1], as well as in references [5, 6], which contain many references and historical comments. Many interesting results for problems of this type are discussed in [7]. These problems have been surveyed by P. L. Lions [4] and J. S. W. Wong [7].

Our purpose here is to establish an existence result for positive solutions to the problem above, assuming that f is either superlinear or sublinear by a simple application of a fixed point theorem in cones. We point out that we do not require any monotonicity assumptions on f.

The rest of the paper is organized as follows. We first illustrate some assumptions on f and useful lemmas. Using these results, we are devoted to the main results.

2. Main Results

We consider the boundary value problem

$$\begin{cases} x'' + f(t,x) = 0 \\ \alpha x(0) - \beta x'(0) = 0 \\ \gamma x(1) + \delta x'(1) = 0, \end{cases} \tag{2.1}$$

where $\alpha, \beta, \gamma, \delta$ are nonnegative numbers such that $\gamma\beta + \alpha\gamma + \alpha\delta > 0$ and $f : (0,1) \times [0,\infty) \to [0,\infty)$ is a continuous function such that

(H1) For fixed $x \in [0,\infty)$, $f(\cdot, x)$ is integrable over $[0,1]$;

(H2) $f(t,0) = 0$ for $t \in (0,1)$, and $f(t,x) > 0$ for all $(t,x) \in (0,1) \times (0,\infty)$.

We study the existence of positive solutions to this problem, and to do so, we utilize the following results:

Lemma 2.1. ([2, 3]) *Let X be a Banach space, $K \subset X$ a cone, $0 < r < R$ and $T : D \to K$ be a compact map such that:*

(a) $x \in D$, $\|x\| = R$, $Tx = \lambda x \;\Rightarrow\; \lambda \geq 1$;

(b) $x \in D$, $\|x\| = r$, $Tx = \lambda x \;\Rightarrow\; \lambda \leq 1$;

(c) $\inf_{\|x\|=R} \|Tx\| > 0$,

where $D := \{x \in K : r \leq \|x\| \leq R\}$. Then T has a fixed point in $x_0 \in D$, with $r \leq \|x_0\| \leq R$.

This result remains valid if the conditions are reversed, namely:

LEMMA 2.2. ([2, 3]) *Let* X *be a Banach space,* $K \subset X$ *a cone,* $0 < r < R$ *and* $T : D \to K$ *be a compact map such that:*

(a) $x \in D,\ \|x\| = R,\ Tx = \lambda x \ \Rightarrow\ \lambda \leq 1$*;*
(b) $x \in D,\ \|x\| = r,\ Tx = \lambda x \ \Rightarrow\ \lambda \geq 1$*;*
(c) $\inf_{\|x\|=r} \|Tx\| > 0$,

where $D := \{x \in K : r \leq \|x\| \leq R\}$*. Then* T *has a fixed point in* $x_0 \in D$*, with* $r \leq \|x_0\| \leq R$*.*

In order to satisfy condition (c) of the above results, we make extensive use of the following observation made in [1]:

LEMMA 2.3. *Let* $\phi : [0,1] \to [0,\infty)$ *be a continuous function whose graph is concave down, and let* $\|\phi\| = \max\{\phi(x) : x \in [0,1]\}$*. Then, if* $0 < \alpha < 1/2$*, we have that*

$$x \in [\alpha, 1-\alpha] \Rightarrow \alpha\|\phi\| \leq \phi(x).$$

Throughout the paper, we denote $X := C([0,1])$ and $K := \{\phi \in X : \phi \geq 0,\ \phi \text{ is concave down}\}$.

2.1. Superlinear Case. Assume that f is satisfied the following conditions:

(H3) $\lim_{x\to 0} \frac{f(t,x)}{x} = 0$ uniformly on $(0,1)$;
(H4) $\lim_{x\to\infty} \frac{f(t,x)}{x} = \infty$ uniformly on compact subsets of $(0,1)$.

THEOREM 2.4. *Under the conditions* (H1)–(H4)*, the problem* (2.1) *has a nontrivial solution.*

Proof. Let $G : [0,1] \times [0,1] \to [0,\infty)$ be the Green's function for the following boundary value problem

$$\begin{cases} x'' = 0 \\ \alpha x(0) - \beta x'(0) = 0 \\ \gamma x(1) + \delta x'(1) = 0, \end{cases} \tag{2.2}$$

and we recall that

$$G(t,s) > 0 \quad \text{for} \quad (t,s) \in (0,1) \times (0,1).$$

We know that ϕ is a solution of (2.1) if and only if $T\phi = \phi$, where $T : K \to K$ is defined by

$$(T\phi)(t) = \int_0^1 G(t,s) f(s,\phi(s)) ds.$$

We also recall that $(T\phi)''(t) = -f(t,\phi(t))$ and thus the graph of $T\phi$ is always concave down. Notice that if $\phi : [0,1] \to [0,\infty)$ is a continuous function whose graph is concave down and $\alpha \in (0,1/2)$, then it follows from Lemma 2.3 that $\phi(t) \geq \alpha\|\phi\|$ for $t \in [\alpha, 1-\alpha]$. We will show that if $R > 0$, then

$$\inf\{\|T\phi\| : \phi \in K,\ \|\phi\| = R\} > 0.$$

To see this, let $\mu = \inf\{G(t,s) : (t,s) \in [1/4,3/4] \times [1/4,3/4]\}$ and observe that $f(t,x) > 0$ for $(t,x) \in [1/4,3/4] \times [R/4,R]$. Let

$$P = \inf\left\{f(t,x) : (t,x) \in \left[\frac{1}{4},\frac{3}{4}\right] \times \left[\frac{R}{4},R\right]\right\} > 0.$$

Then, if $t \in [1/4,3/4]$, we have

$$\begin{aligned}
(T\phi)(t) &= \int_0^1 G(t,s) f(s,\phi(s))\, ds \\
&= \int_0^{\frac{1}{4}} G(t,s) f(s,\phi(s))\, ds + \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s) f(s,\phi(s))\, ds \\
&\qquad + \int_{\frac{3}{4}}^{1} G(t,s) f(s,\phi(s))\, ds \\
&\geq \frac{1}{4}\mu P,
\end{aligned}$$

and so $\|T\phi\| \geq \frac{1}{4}\mu P$ for all $\phi \in K$ with $\|\phi\| = R$.

Now we show that there exists $r_0 > 0$ such that for $r \leq r_0$, the problem

$$\begin{cases} x'' + \lambda f(t,x) = 0,\ 0 < \lambda < 1 \\ \alpha x(0) - \beta x'(0) = 0 \\ \gamma x(1) + \delta x'(1) = 0 \end{cases} \tag{2.3}$$

has no solution of norm r. To see this, let

$$M = \max\{G(t,s) : (t,s) \in [0,1] \times [0,1]\} > 0.$$

Pick $k > 0$ such that $Mk < 1$. Then there is a positive constant r_1 such that $f(t, x) < kx$ for $t \in (0, 1)$ and $0 < x \leq r_1$. Set $r_0 = \min\{1, r_1\}$. Let $r \leq r_0$. Assume that there exist λ in $(0, 1)$ and ϕ in K with $\|\phi\| = r$ such that

$$\begin{cases} \phi''(t) + \lambda f(t, \phi(t)) = 0, \ 0 < \lambda < 1 \\ \alpha\phi(0) - \beta\phi'(0) = 0 \\ \gamma\phi(1) + \delta\phi'(1) = 0. \end{cases} \tag{2.4}$$

Then

$$\phi(t) = \lambda \int_0^1 G(t, s) f(s, \phi(s))\, ds.$$

Pick $t_0 \in [0, 1]$ such that $\phi(t_0) = r$. Then

$$\begin{aligned} r = \phi(t_0) &= \lambda \int_0^1 G(t_0, s) f(s, \phi(s))\, ds \\ &< \lambda M k \int_0^1 \phi(s)\, ds \\ &\leq \lambda M k r < r. \end{aligned}$$

This is impossible. Hence the problem (2.3) has no solution of norm r.

Our next step is to prove that there exists $R_0 > 0$ such that for $R \geq R_0$, the problem (2.3) has no solution of norm R. If this is not true, then there exist a sequence $\{R_n\}_{n=1}^{\infty}$ with $R_n \uparrow \infty$ and a sequence $\{\lambda_n\}_{n=1}^{\infty}$ of real numbers $\lambda_n > 1$, and a sequence of functions $\{\phi_n\}_{n=1}^{\infty}$ such that $\|\phi\| = R_n$ and

$$\lambda_n T\phi_n = \phi_n, \quad n \in \mathbb{N}.$$

Notice that the functions ϕ_n are continuous on $[0, 1]$ and have concave down graph. For each $n \in \mathbb{N}$, let $t_n \in [0, 1]$ be the (unique) point of maximum for ϕ_n. Then we have

$$\phi_n(t) \geq \frac{1}{4}\phi_n(t_n) \quad \text{for} \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

Let

$$S = \inf\left\{G(t, s) : (t, s) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times \left[\frac{1}{4}, \frac{3}{4}\right]\right\} > 0$$

and pick $k > 0$ such that $kS > 4$. Then there is a positive constant L such that $f(t, x) > kx$ for $t \in [1/4, 3/4]$ and $x \geq L$. Pick $n \in \mathbb{N}$ such that $\phi_n(t) > L$ for $t \in [1/4, 3/4]$. Then

$$\begin{aligned}\phi_n(t) &= \lambda_n \int_0^1 G(t,s) f(s, \phi_n(s))\, ds \\ &\geq \lambda_n \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s) f(s, \phi_n(s))\, ds \\ &\geq \lambda_n \int_{\frac{1}{4}}^{\frac{3}{4}} kS\phi_n(s)\, ds.\end{aligned} \tag{2.5}$$

Let t_n^* be the point of minimum of ϕ_n in $[1/4, 3/4]$. Then it follows from (2.5) that

$$\phi_n(t_n^*) \geq \frac{1}{2}\lambda_n kS\phi_n(t_n^*) \geq 2\lambda_n \phi_n(t_n^*) > 2\phi_n(t_n^*).$$

This is a contradiction. Now we pick $R > R_0$ and by Lemma 2.1, there exists $\phi \in K$ with $r_0 \leq \|\phi\| \leq R$ such that $\phi = T\phi$. □

Observation: *Our result implies that under the superlinearity assumption, all nonlinear eigenvalue problems of the form*

$$\begin{cases} x'' + \lambda f(t,x) = 0 \\ \alpha x(0) - \beta x'(0) = 0 \\ \gamma x(1) + \delta x'(1) = 0, \end{cases} \tag{2.6}$$

have a positive solution. This should be compared with Theorem 1.2 of [4], where it is shown that if the condition of superlinearity at the origin is not met, then there is an upper bound for the values of λ, for which the eigenvalue problem has a positive solution.

2.2. Sublinear Case. Assume that f satisfies the following conditions:

($\bar{H}3$) $\lim_{x\to 0} \frac{f(t,x)}{x} = \infty$ uniformly on $(0,1)$;

($\bar{H}4$) $\lim_{x\to \infty} \frac{f(t,x)}{x} = 0$ uniformly on compact subsets of $(0,1)$.

Using essentially the same argument as in Theorem 2.4 and applying Lemma 2.2, we get the following assertion.

Theorem 2.5. *Under the conditions* (H1), (H2), ($\bar{H}3$), *and* ($\bar{H}4$), *the problem* (2.1) *has a nontrivial solution.*

Observation: *Our result implies that under the sublinearity assumption, all nonlinear eigenvalue problems of the form* (2.6) *have a positive solution.*

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