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CONTINUED FRACTIONS AND THE DENSITY OF GRAPHS OF SOME FUNCTIONS

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Abstract. We consider some simple periodic functions on the field of rational numbers with values in $\mathbb{Q}/\mathbb{Z}$ which are defined in terms of lowest-term-expression of rational numbers. We prove the density of graphs of these functions by constructing explicitly points on the graphs close to a given point using continued fractions.

1. Introduction

Consider the following functions ψ_e for $e \in \mathbb{Z}$ of period 1 defined on $\mathbb{Q}$ with values in $\mathbb{Q}/\mathbb{Z}$: for relatively prime positive integers p, q,

$$\psi_e : \frac{p}{q} \mapsto \frac{p^e}{q} \mod \mathbb{Z}\ , \tag{1}$$

where p^{-1} denotes an inverse of p modulo q when $e < 0$. We will often identify $\mathbb{R}/\mathbb{Z}$ with $[0,1)$, a set of representatives. And we have $\psi_e(p/q) = \langle p^e/q\rangle$ where $\langle x\rangle = x - [x]$ denotes the fractional part of x. The goal of this paper is to show that the graphs of these functions for $e = 3, 2, -1, -2$ are dense in $[0,1)^2$ by constructing explicitly points on the graph arbitrarily close to a given point.

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The motivation for us to consider such problems comes from our study of Dedekind sums. The properties of the classical Dedekind sums

$$s(p,q) = \sum_{k=1}^{q} \left(\left(\frac{k}{q}\right)\right) \left(\left(\frac{pk}{q}\right)\right)$$

are well-documented in [5]. Here $((x)) = \langle x \rangle - 1/2$ if x is not an integer and $((x)) = 0$ if x is an integer. Using the reciprocity theorem for these sums

$$s(p,q) + s(q,p) = -\frac{1}{4} + \frac{1}{12}\left(\frac{p}{q} + \frac{1}{pq} + \frac{q}{p}\right) ,$$

Hickerson obtained an explicit formula for $s(p,q)$ in terms of continued fraction expansion of p/q and proved the density[1] in $\mathbb{R}^2$ of the graph of $p/q \mapsto s(p,q)$ in [4].

In [1] (and references therein), it is defined and proved some properties of generalized Dedekind sums of higher dimension. In particular, it is proved that these sums are *equidistributed* in $\mathbb{R}/\mathbb{Z}$. The equidistribution of sequences in $\mathbb{R}/\mathbb{Z}$ (or in higher dimensional tori) is a basic ingredient in the recent development of additive number theory in conjugation with ergodic theory and combinatorics [6]. The graph of Dedekind sums would be an interesting sequence in a torus. The equidistribution of the graph is certainly stronger than the equidistribution of Dedekind sums.

The equidistribution result of [1], proved by estimating exponential sums, is quite general and can be applied to prove the equidistribution, hence the density of the graph of these sums in a suitable product of $\mathbb{R}/\mathbb{Z}$.

In [2], an explicit formula for 2-dimensional Dedekind sums of higher degree is obtained. This may be seen as a generalization of the above mentioned formula in [4]. But it seems to be difficult to prove the density of the graph in $\mathbb{R}^2$.

This paper is our first attempt to extend the constructive proof in [4] of the density in $\mathbb{R}^2$ of the graph of the classical Dedekind sums to 2-dimensional Dedekind sums of higher degree, whose fractional parts are linear combinations of the above functions ψ_e for $e \in \mathbb{Z}$ with explicitly calculated coefficients [1].

[1] In this paper, we have used the term *density* for *denseness*, i.e. something being dense. We apologize for any confusion caused by this choice of words.

2. Continued fractions

We review quickly continued fractions. All the proofs and details can be found in any standard text on number theory. The (possibly infinite) *continued fraction*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

will be denoted by $\langle a_0; a_1, a_2, a_3, \cdots \rangle$. Unless otherwise stated, $a_0, a_1, a_2, \cdots$ are integers with $a_1, a_2, \cdots$ positive. The *k-th convergent* $C_k = p_k/q_k := \langle a_0; a_1, \cdots, a_k \rangle$ of the above continued fraction is given by sequences $\{p_k\}$ and $\{q_k\}$ given recursively: $p_0 = a_0,\ p_1 = a_0 a_1 + 1,\ q_0 = 1,\ q_1 = a_1$ and

$$p_k = a_k p_{k-1} + p_{k-2}\ , \qquad q_k = a_k q_{k-1} + q_{k-2}\ .$$

The sequence of convergents $\{C_k\}$ converges, whose limit will be represented by the continued fraction. Conversely, any real number can be expanded as a continued fraction. With the above notations, we have the following.

PROPOSITION 2.1. *(i) For each positive integer k, p_k and q_k are relatively prime. More precisely, we have: $p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}$.*
(ii) We have $q_k \geq f_k$ where $\{f_k\}$ is the Fibonacci sequence.
(iii) We have $q_{k-1}/q_k = \langle 0; a_k, a_{k-1}, \cdots, a_2, a_1 \rangle$.
(iv) Let $\alpha > 0$ and let $p = \alpha p_k + p_{k-1}$, $q = \alpha q_k + q_{k-1}$ (for a fixed k). Then

$$\frac{p}{q} = \langle a_0; a_1, \cdots, a_k, \alpha \rangle \quad \text{and} \quad \frac{p}{q} - \frac{p_k}{q_k} = \frac{(-1)^k}{(\alpha q_k + q_{k-1}) q_k}\ .$$

3. Density of graphs

In this section we prove our main result: the graph of ψ_e is dense in $[0,1)^2$ for $e = 3, 2, -1, -2$. To prove the case of $e = 3$, we need the following result on the distribution of quadratic (non) residues modulo a large prime in [3].

Proposition 3.1. *Let H_+ and H_- be the maximum numbers of consecutive quadratic residues and non-residues modulo a prime p, respectively. Then we have*

$$H_+ = O(\sqrt{p}) , \quad H_- = O(\sqrt{p}) .$$

More precisely, it follows from a formula on character sums [3, Lemma 1] that $h^3 \le ph - h^2$ where $h = [H_\pm/2]$. Hence the implied constants in the above proposition can be any number greater than 2. We also need the following simple lemma.

Lemma 3.2. *Let $a, b, c, d \in \mathbb{Z}$ be with $ad - bc = \pm 1$ and let m, n be relatively prime integers. Then m', n' given below are relatively prime.*

$$\begin{pmatrix} m' \\ n' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} .$$

Proof. Suppose not. Modulo the greatest common divisor of m' and n', the matrix in the right side is invertible. $\square$

Theorem 3.3. *Let $e = 3, 2, -1$ or -2. The part of the graph of ψ_e is dense in $[0, 1)^2$. More precisely, for any $(x, y) \in [0, 1)^2$ and $\epsilon > 0$, there exists a rational number p/q such that $||\ (p/q, \psi_e(p/q)) - (x, y)\ || < \epsilon$.*

Proof. We may assume both x and y are irrational. Let $\langle 0; a_1, a_2, \cdots \rangle$ and $\langle 0; b_1, b_2, \cdots \rangle$ be the expansions of x and y as infinite continued fractions, respectively. Their convergents will be denoted by $C_j = p_j/q_j$ and $C'_j = p'_j/q'_j$ $(j = 0, 1, \cdots)$, respectively.

Given $0 < \epsilon < 1$, choose $k \in \mathbb{Z}$ large enough so that $|C_k - x| < \epsilon$ and $q_k > 1/\epsilon$. (Of course, the second condition implies $|C_k - x| < \epsilon^2$ by the well-known property of continued fractions.) We may suppose k is sufficiently large that the similar conditions are also satisfied for convergents of the continued fraction of y.

($\mathbf{e = 2}$) Let $\alpha > 0$, which we will take as a variable, and let $p = \alpha p_k + p_{k-1}$, $q = \alpha q_k + q_{k-1}$ so that $p/q = \langle 0; a_1, a_2, \cdots, a_k, \alpha \rangle$. Then we have by Proposition 2.1 (iv) as $\alpha \to \infty$

$$\frac{p^2}{q} \approx \frac{p_k^2}{q_k}\alpha + \frac{p_k p_{k-1}}{q_k} + (-1)^k \frac{p_k}{q_k^2} , \tag{2}$$

which means that the difference of both sides tends to zero as α tends to the infinity. As a function of $\alpha \in \mathbb{Z}$ (actually a function of $\alpha \in \mathbb{Z}/q_k\mathbb{Z}$) with values in $\mathbb{R}/\mathbb{Z}$, the right side of (2) takes q_k distinct values since p_k is relatively prime to q_k. Since these values (in $\mathbb{R}/\mathbb{Z}$) are evenly spaced, one of these values corresponding to, say, α_0 is within the distance of $1/q_k$ from y. Choose $\alpha \in \mathbb{Z}$ with $\alpha \equiv \alpha_0 \mod q_k$ which is sufficiently large that the difference of two sides of (2) is less than ϵ. Then we have

$$|| \, (\frac{p}{q}, \psi_2(\frac{p}{q})) - (x,y) \, || \le |\frac{p}{q} - \frac{p_k}{q_k}| + |\frac{p_k}{q_k} - x| + |((\frac{p^2}{q})) - R| + |R - y| \le 4\epsilon \, ,$$

where R denote the right side of (2) modulo $\mathbb{Z}$. This completes the proof for $e = 2$.

($\mathbf{e = 3}$) By Dirichlet's theorem on primes in an arithmetic progression, there exists a (sufficiently large) positive integer a such that the denominator $s = aq_k + q_{k-1}$ of $r/s = \langle 0; a_1, \cdots, a_k, a \rangle$ is a prime. We may suppose that $s > 1/\epsilon^2$ and the maximum number of consecutive quadratic residues (and non-residues, respectively) modulo s is less than $3\sqrt{s}$ by Proposition 3.1. Let $r'/s' = p_k/q_k = \langle 0; a_1, \cdots, a_k \rangle$ and let $p = \alpha r + r'$, $q = \alpha s + s'$ so that $p/q = \langle 0; a_1, \cdots, a_k, a, \alpha \rangle$ where α is a positive integer which we will take as a variable as in the last paragraph. Then we have as $\alpha \to \infty$ with other choices fixed

$$\frac{p^3}{q} \approx \frac{r^3}{s}\alpha^2 + \frac{2r^2 r'}{s}\alpha + D + (-1)^{k+1}\frac{r^2}{s^2}\alpha \, , \tag{3}$$

where D is a rational number independent of α.

As in the proof for $e = 2$, consider the right side of (3) modulo $\mathbb{Z}$ as a function of $\alpha \in \mathbb{Z}/s^2\mathbb{Z}$. We claim that there exists $\alpha_1 \in \mathbb{Z}/s^2\mathbb{Z}$ such that the value at α_1 is within the distance of 4ϵ from y in $\mathbb{R}/\mathbb{Z}$ (or in $[0,1)$, to be more precise). Once this is proven, we can see as in the proof for $e = 2$ that for sufficiently large $\alpha \in \mathbb{Z}$ with $\alpha \equiv \alpha_1 \mod s^2$, $|| \, (p/q, \psi_3(p/q)) - (x,y) \, || < 7\epsilon$.

It remains to prove the claim. First, consider the first three terms of the right side of (3) modulo $\mathbb{Z}$ as a function of $\alpha \in \mathbb{Z}/s\mathbb{Z}$. In this case, the set of values of this function is not evenly spaced by $1/s$ in $\mathbb{R}/\mathbb{Z}$. But by completing the square in $r^3\alpha^2 + 2r^2r'\alpha$ modulo s and applying Proposition 3.1 (recall our choice of s above), we can see that there exists α_0 such that the value at α_0 is within the distance of $3/\sqrt{s}$ from y in $\mathbb{R}/\mathbb{Z}$. Fix α_0 and let $\alpha = \alpha_0 + s\alpha'$ in (3). By varying $\alpha' \in \mathbb{Z}/s\mathbb{Z}$, the last term of (3) can be made smaller than $1/s$ in $\mathbb{R}/\mathbb{Z}$ (in $[0,1)$, to be more precise). Suppose the minimum of the value is obtained at $\alpha' = \alpha'_0$. Then we can take $\alpha_1 = \alpha_0 + s\alpha'_0$. This completes the proof of the theorem for $e = 3$.

(**e** $= -$**1**) Let β be a positive integer and let r_i/s_i ($i = 1, \cdots, 2k+1$) be the convergents of the finite continued fraction

$$\frac{p}{q} = \langle 0; a_1, a_2, \cdots, a_k, \beta, b_k, b_{k-1}, \cdots, b_1 \rangle .$$

By Proposition 2.1 (i) and (iii), s_{2k} is an inverse of $p = r_{2k+1}$ modulo $q = s_{2k+1}$ and $p^{-1}/q = s_{2k}/q = \langle 0; b_1, b_2, \cdots, b_k, \beta, a_k, a_{k-1}, \cdots, a_1 \rangle$. By the choice of k and Proposition 2.1 (iv) with $\langle \beta; b_k, \cdots, b_1 \rangle$ and $\langle \beta; a_k, \cdots, a_1 \rangle$ in place of α, respectively, we have (for any positive integer β) both $|p/q - x|$ and $|p^{-1}/q - y|$ are less than 2ϵ. This complete the proof for $e = -1$.

(**e** $= -$**2**) By Dirichlet's theorem on primes in an arithmetic progression, there exists a positive integer a such that the denominator $aq_k + q_{k-1}$ of $\langle 0; a_1, \cdots, a_k, a \rangle$ is a prime. There also exits a positive integer b such that the denominator of $\langle 0; b_1, \cdots, b_k, b \rangle$ is a prime distinct from $aq_k + q_{k-1}$.

Let β a positive integer and let

$$\frac{p}{q} = \langle 0; a_1, a_2, \cdots, a_k, a, \beta, b, b_k, b_{k-1}, \cdots, b_1 \rangle .$$

Then as in the proof for $e = -1$, we have

$$\frac{p^{-1}}{q} = \langle 0; b_1, \cdots, b_k, b, \beta, a, a_k, \cdots, a_1 \rangle .$$

We will vary β with other components fixed. Let $m/n = \langle 0; a, a_k, \cdots, a_1 \rangle$. By Proposition 2.1 (iii) we have $n = aq_k + q_{k-1}$, which is a prime by the choice of a. Let $r/s = \langle 0; b_1, \cdots, b_k, b \rangle$ and $r'/s' = p'_k/q'_k = \langle 0; b_1, \cdots, b_k \rangle$. Recall b was chosen so that s is a prime distinct from n. We have

$$\frac{p^{-1}}{q} = \langle 0; b_1, \cdots, b_k, b, \beta + \frac{m}{n} \rangle = \frac{(n\beta + m)r + nr'}{(n\beta + m)s + ns'} .$$

The numerator and the denominator of the last quotient are relatively prime by Lemma 3.2. Hence,

$$\frac{p^{-2}}{q} = \frac{((n\beta + m)r + nr')^2}{(n\beta + m)s + ns'} .$$

As functions of $\beta \in \mathbb{Z}$ with values in $\mathbb{R}$, we have as $\beta \to \infty$

$$\frac{p^{-2}}{q} \approx \frac{nr^2}{s}\beta + \frac{r(mr + nr')}{s} + (-1)^{k+1}\frac{nr}{s^2} . \tag{4}$$

Since nr^2 and s are relatively prime, we can argue as in the proof for $e = 2$. There exists β_0 such that the right side of (4) with $\beta = \beta_0$ is within the distance of $1/s$ from y modulo $\mathbb{Z}$. Choose $\beta \in \mathbb{Z}$ with $\beta \equiv \beta_0$ mod s which is sufficiently large that the difference of the two sides of (4) is less than ϵ. Then we have $||(p/q, ((p^{-2}/q))) - (x, y)|| < 4\epsilon$ as before. This completes the proof for $e = -2$. $\square$

4. Remarks and examples

We hope to extend the constructive proof of this paper to other values of e. For $e \geq 4$, it may be proved by similar arguments as for $e = 3$ together with more precise estimate on the distribution of higher residues modulo a large prime.

In the following we give a simple example, in which we construct points on the graph of ψ_2 approximating arbitrary points (x, y) on the line $x = (\sqrt{5} - 1)/2$ with $0 \leq y < 1$. Recall that the Fibonacci sequence $\{f_n\}_{n=0,1,2,\cdots}$ is given by $f_0 = 0,\ f_1 = 1,\ f_k = f_{k-1} + f_{k-2}$.

Example 4.1. The k-th convergent of $\langle 0; 1, 1, 1, \cdots \rangle = (\sqrt{5} - 1)/2$ is f_k/f_{k+1}. The sequence $\psi_2(f_k/f_{k+1}) = f_k^2/f_{k+1} \mod \mathbb{Z}$ of values of ψ_2 converges to 0.

The first assertion is clear. The second one follows from the formula $f_k^2 - f_{k-1}f_{k+1} = (-1)^{k-1}$, which is a special case of Proposition 2.1 (i).

Example 4.2. For each positive integer n, let $\{x_k^{(n)}\}_{k=1,2,\cdots}$ be the sequence of rational numbers given by $x_k^{(n)} = \langle 0; 1, \cdots, 1, nf_k \rangle$ where the number of 1's is k. Then we have

$$\lim_{k\to\infty} x_k^{(n)} = \frac{\sqrt{5}-1}{2}\ ,\ \lim_{k\to\infty} \psi_2(x_{2k}^{(n)}) = n\frac{1-\sqrt{5}}{2}\ ,\ \lim_{k\to\infty} \psi_2(x_{2k+1}^{(n)}) = n\frac{\sqrt{5}-1}{2}$$

where the values of last two equations are taken in $\mathbb{R}/\mathbb{Z}$ as usual. Recall that if γ is an irrational number, then the sequence $\{n\gamma\}_{n=1,2,\cdots}$ is *equidistributed* in $\mathbb{R}/\mathbb{Z}$. Hence we can choose n so that the limit of the sequence $\{\psi_2(x_{2k}^{(n)})\}$ is arbitrarily close to any given number in $\mathbb{R}/\mathbb{Z}$.

The first equation is clear from Proposition 2.1 (iv). As for the others, we have

$$\begin{aligned}
\psi_2(x_k^{(n)}) &= \frac{(nf_k^2 + f_{k-1})^2}{nf_k f_{k+1} + f_k} = \frac{(f_{k-1}(nf_{k+1}+1) + (-1)^{k-1}n)^2}{f_k(nf_{k+1}+1)} \\
&= n\frac{f_{k-1}^2 f_{k+1}}{f_k} + \frac{f_{k-1}^2}{f_k} + (-1)^{k-1}2n\frac{f_{k-1}}{f_k} + \frac{n^2}{f_k(nf_{k+1}+1)} \\
&= nf_{k-1}f_k + (-1)^k n\frac{f_{k-1}}{f_k} + \frac{f_{k-1}^2}{f_k} + (-1)^{k-1}2n\frac{f_{k-1}}{f_k} + \frac{n^2}{f_k(nf_{k+1}+1)}
\end{aligned}$$

where we have used the identity $f_k^2 - f_{k-1}f_{k+1} = (-1)^{k-1}$ twice. Taking the limit (in $\mathbb{R}/\mathbb{Z}$), we obtain the desired result.

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