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SUBNORMALITY OF THE WEIGHTED CESÀRO OPERATOR $C_h \in l^2(h)$

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Abstract. The subnormality of some classes of operators is a very interesting property. In this paper, we prove that the weighted Cesàro operator $C_h \in \ell^2(h)$ is subnormal and we described completely the set of the extended eigenvalues for the weighted Cesàro operator, some other important results are also given.

1. Introduction and preliminaries

In this paper we discuss the Cesàro operator in weighted ℓ^2 spaces. For a sequence $h = (h(n))_{n\in\mathbb{N}}$ of positive numbers, called weights and a sequence $a = (a(n))_{n\in\mathbb{N}}$ of complex numbers the discrete weighted Cesàro operator C_h is defined by

$$(C_h a)(n) = \frac{1}{H(n)} \sum_{k=0}^{n} h(k)a(k), with \quad H(n) = \sum_{k=0}^{n} h(k) \tag{1}$$

Let $1 < p < \infty$ and

$$\ell^p(h) = \left\{ a = (a(n))_{n\in\mathbb{N}_0} : a(n) \in \mathbb{C}, \|a\|_{p,h}^p = \sum_{n=0}^{\infty} h(n)|a(n)|^p < \infty \right\} \tag{2}$$

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It is well known that the Cesàro operator in $\ell^p(h)$ is bounded by $\|C_h\| \leq \frac{p}{p-1}$, see [9]. An easy computation shows that the dual operator C_h^* of C_h in $\ell^q(h), \frac{1}{p} + \frac{1}{q} = 1$, is

$$(C_h^* a)(n) = \sum_{k=n}^{\infty} \frac{h(k)a(k)}{H(k)}. \tag{3}$$

In the Hilbert space $\ell^2(h)$ the inner product is defined by

$$\langle a, b \rangle_h = \sum_{n=0}^{\infty} h(n)a(n)\overline{b(n)}, \quad a, b \in \ell^2(h). \tag{4}$$

THEOREM 1.1. [8] *Let $T \in B(H)$. The following conditions on T are equivalent.*

1. *T is subnormal.*
2. *There exists a unitary operator $U \in B(H \oplus H)$ such that for $n = 0, 1, ...,: T^{*n} = P_H U^n T^n$, where P_H is the orthogonal projection of $H \oplus H$ onto $H \oplus 0$.*
3. *For $n = 0, 1, ...,:$*

$$T^{*n} = [\int_{\partial D} e^{int} dQ(t)] T^n \tag{5}$$

where Q is a positive operator measure (denoted by POM) defined on the boundary of the unit disc, ∂D.

4. *There exists a sequence of operators $K_n \in B(H)$ satisfying $T^{*n} = K_n T^n$ for $n = 0, 1, ...,$ Moreover if we define*

$$L_n = \begin{cases} K_n & \text{if } n \geq 0 \\ K_n^* & \text{if } n < 0. \end{cases}$$

then for any finite set $\{x_0, x_1, ..., x_n\}$ contained in H,

$$\sum_{j,k \geq 0}^{n} \langle L_{j-k} x_j, x_k \rangle \geq 0.$$

5. *There exists a sequence of operators $K_n \in B(H)$ satisfying $T^{*n} = K_n T^n$ for $n = 0, 1, ...,$ Moreover if we define*

$$L_n = \begin{cases} K_n & \text{if } n \geq 0 \\ K_n^* & \text{if } n < 0. \end{cases}$$

then for each $x \in H$ *and each* $n = 0, 1, ...,$ *the matrix* $[\langle L_{j-k}x, x\rangle]_{j,k\geq 0}^n$ *is positive definite.*

LEMMA 1.2. [6] *Let* T *be a bounded linear operator on a complex Banach space* E *and let us suppose that there is an analytic mapping* $f : int\sigma_p(T) \longrightarrow E$ *with* $f(z) \in ker(T - z)/\{0\}$ *for all* $z \in int\sigma_p(T)$ *and such that* $f(z) : z \in int\sigma_p(T)$ *is a total subset of* E*. Then* T *has rich point spectrum.*

THEOREM 1.3. [6] *Let* T *be a bounded linear operator with rich point spectrum and such that* $\sigma_p(T) = D(r, r)$ *for some* $r > 0$*. If* λ *is an extended eigenvalue for* T *then* λ *is real and* $0 < \lambda \leq 1$*.*

2. Main Results

2.1. Subnormality of C_h. As C_h and C_h^* are operators in a sequence space, they have matrix representations with respect to the basis $(e_j)_{j\in\mathbb{N}}$ of $\ell^2(h)$ (in the following also denoted by C_h and C_h^*, respectively). From (1) and (3) we can infer that

$$C_h = \begin{pmatrix} \frac{h(0)}{H(0)} & 0 & 0 & \cdots \\ \frac{h(0)}{H(1)} & \frac{h(1)}{H(1)} & 0 & \cdots \\ \frac{h(0)}{H(2)} & \frac{h(1)}{H(2)} & \frac{h(2)}{H(2)} & \cdots \\ \frac{h(0)}{H(3)} & \frac{h(1)}{H(3)} & \frac{h(2)}{H(3)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{6}$$

$$C_h^* = \begin{pmatrix} \frac{h(0)}{H(0)} & \frac{h(1)}{H(1)} & \frac{h(2)}{H(2)} & \cdots \\ 0 & \frac{h(1)}{H(1)} & \frac{h(2)}{H(2)} & \cdots \\ 0 & 0 & \frac{h(2)}{H(2)} & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{7}$$

Direct computation yields the matrix representations of C_h^{-1} and C_h^{*-1} with respect to $(e_j)_{j\in\mathbb{N}}$:

$$C_h^{-1} = \begin{pmatrix} \frac{H(0)}{h(0)} & 0 & 0 & \dots \\ \frac{-H(0)}{h(1)} & \frac{H(1)}{h(1)} & 0 & \dots \\ 0 & \frac{-H(1)}{h(2)} & \frac{H(2)}{h(2)} & \dots \\ 0 & 0 & \frac{-H(2)}{h(3)} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{8}$$

and

$$C_h^{*-1} = \begin{pmatrix} \frac{H(0)}{h(0)} & -\frac{H(0)}{h(0)} & 0 & \dots \\ 0 & \frac{H(1)}{h(1)} & -\frac{H(1)}{h(1)} & \dots \\ 0 & 0 & \frac{H(2)}{h(2)} & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{9}$$

THEOREM 2.1. *The weighted Cesàro operator $C_h \in \ell^2(h)$ is subnormal*

Proof. Let $C_h^* = K_1 C_h$, where $K_1 = C_h^* C_h^{-1}$.
Matrix K_1 knowing as follows

$$K_1 = \begin{pmatrix} \frac{h(1)}{H(1)} & \frac{h(2)}{H(2)} & \frac{h(3)}{H(3)} & \dots & 1 \\ \frac{-H(0)}{H(1)} & \frac{h(2)}{H(2)} & \frac{h(3)}{H(3)} & \dots & 1 \\ 0 & \frac{-H(1)}{H(2)} & \frac{h(3)}{H(3)} & \dots & 1 \\ 0 & 0 & \frac{-H(2)}{H(3)} & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & 1 \end{pmatrix} \tag{10}$$

And for each natural number n:

$$K_n = C_h^{*n} C_h^{-n} = C_h^{*n-1} K_1 C_h^{1-n}. \tag{11}$$

In [4] the positivity of the matrix T acting on ℓ^2 was proved by considering the determinants of its finite sections. In order to include the case when the matrix T is positive semidefinite, we give a more detailed proof for the positivity of the operator T here.

The bilinear form $\langle ., K_1 . \rangle_h$ is defined for all sequences

$$a, b \in \ell^2(h).$$

Using the vector representations for a and b, the matrix representation for K_1 and the inner product as defined in (4), we obtain

$$\begin{aligned}
\langle a, K_1 b \rangle_h &= \begin{pmatrix} a(0) \\ a(1) \\ \vdots \end{pmatrix}^t \begin{pmatrix} h(0) & 0 & \\ 0 & h(1) & \\ & & \ddots \end{pmatrix} \\
&\quad \times \begin{pmatrix} \frac{h(1)}{H(1)} & \frac{h(2)}{H(2)} & \cdots \\ \frac{-H(0)}{H(1)} & \frac{h(2)}{H(2)} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} b(0) \\ b(1) \\ \vdots \end{pmatrix} \\
&= \begin{pmatrix} a(0) \\ a(1) \\ a(2) \\ \vdots \end{pmatrix}^t K_{1h} \begin{pmatrix} b(0) \\ b(1) \\ b(2) \\ \vdots \end{pmatrix},
\end{aligned}$$

with

$$K_{1h} = \begin{pmatrix} \frac{h(0)h(1)}{H(1)} & \frac{h(0)h(2)}{H(2)} & \frac{h(0)h(3)}{H(3)} & \cdots \\ \frac{-h(1)H(0)}{H(1)} & \frac{h(1)h(2)}{H(2)} & \frac{h(1)h(3)}{H(3)} & \cdots \\ 0 & \frac{-h(2)H(1)}{H(2)} & \frac{h(2)h(3)}{H(3)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In particular, for all $n \in \mathbb{N}$

$$\langle e_i, K_1 e_i \rangle_h = \frac{h(i-1)h(i)}{H(i)} \geq 0, \quad i \geq 1.$$

If $i = n$ then

$$\langle e_n, K_1 e_n \rangle_h = h(n) \geq 0.$$

As well as for all $a \in \ell^2(h)$, $\quad \langle a, K_1 a \rangle_h \geq 0$.
And so K_1 is positive.

We have also C_h is positive so C_h^{-1} is positive. And this for any natural n, C_h^{-n} is positive.
The same applies to C_h^* so C_h^{*n} is positive.
And since, for each natural n : $\quad K_n = C_h^{*n-1} K_1 C_h^{1-n}$ is positive.
By Theorem (1.1), the operator C_h is subnormal. □

2.2. Extended eigenvalues for The weighted Cesàro operator $C_h \in l^p$. We shall prove in this section that the set of the extended eigenvalues for the weighted Cesàro operator is the interval $[1, \infty)$ when $p = 2$ and that it is contained in the interval $[1, \infty)$ when $1 < p < \infty$. Let us recall that the weighted Cesàro operator C_h is defined on the complex Banach space ℓ^p by the sequence of arithmetic means (1)

THEOREM 2.2. *1. The point spectrum of C_h is empty.*
2. If $|1-\lambda| < 1$ then λ is a simple proper value of C_h^*.
3. The spectrum of C_h is the closed disc $\{\lambda : |1-\lambda| \leq 1\}$.
4. The point spectrum of C_h^* is the open disc $\{\lambda : |1-\lambda| < 1\}$.

Proof. 1. Observe first if $C_h f = g$ then $f(0) = g(0)$ and if $n \geq 1$ then

$$C_h f(n) = \frac{1}{H(n)} \sum_{k=0}^{n} h(k) f(k) = g(n)$$

And therefore

$$\frac{h(0)f(0)}{H(n)} + \frac{h(1)f(1)}{H(n)} + ... + \frac{h(n)f(n)}{H(n)} = g(n)$$

$$\frac{h(0)f(0)}{H(n-1)} + \frac{h(1)f(1)}{H(n-1)} + ... + \frac{h(n-1)f(n-1)}{H(n-1)} = g(n-1)$$

And it

$$h(n)f(n) = H(n)g(n) - H(n-1)g(n-1)$$

Which

$$f(n) = \frac{H(n)g(n)}{h(n)} - \frac{H(n-1)g(n-1)}{h(n)}$$

consequently, if $C_h f = \lambda f$ then

$$f(n) = \lambda \left(\frac{H(n)f(n)}{h(n)} - \frac{H(n-1)f(n-1)}{h(n)} \right)$$

Or

$$\left(\lambda \frac{H(n)}{h(n)} - 1 \right) f(n) = \lambda \frac{H(n-1)f(n-1)}{h(n)}$$

Whenever $n \geq 1$, if m is the smallest integer for which $f(m) \neq 0$, then $\lambda = \frac{h(m)}{H(m)}$ so that $0 < \lambda \leq 1$ It follows that if $n \geq 1$ then

$$\begin{aligned} |f(n)| &= \left| \frac{\lambda \frac{H(n-1)f(n-1)}{h(n)}}{\lambda \frac{H(n)}{h(n)} - 1} \right| \\ &= \left| \frac{\lambda H(n-1)f(n-1)}{\lambda H(n) - h(n)} \right| \geq f(n-1) \end{aligned}$$

Which, for a non zero $f \in \ell^2$; is imposible.

2. Observe first that $(C_h^* f)(n) = \sum_{k=n}^{\infty} \frac{h(k)f(k)}{H(k)}$
If $C_h^* f = g$ then

$$\frac{h(n)f(n)}{H(n)} + \frac{h(n+1)f(n+1)}{H(n+1)} + \ldots.. = g(n)$$

$$\frac{h(n+1)f(n+1)}{H(n+1)} + \frac{h(n+2)f(n+2)}{H(n+2)} + \ldots.. = g(n+1)$$

$$\Longrightarrow f(n) = \frac{H(n)g(n)}{h(n)} - \frac{H(n)g(n+1)}{h(n)}, \quad n = 0, 1, 2...$$

consequently; if $C_h^* f = \lambda f$ then

$$f(n) = \lambda \left(\frac{H(n)f(n)}{h(n)} - \frac{H(n)f(n+1)}{h(n)} \right)$$

And it

$$\lambda \frac{H(n)f(n+1)}{h(n)} = \left(\lambda \frac{H(n)}{h(n)} - 1 \right) f(n)$$

It follows that 0 is not a proper value of C_h^*; if $\lambda = 0$ then $f(n) = 0, n = 0, 1, 2...$ and it follows also that

$$f(n+1) = \left(1 - \frac{h(n)}{\lambda H(n)} \right) f(n).$$

This implies that if $n \geq 1$ then

$$f(n) = \prod_{j=1}^{n} \left(1 - \frac{h(j)}{\lambda H(j)} \right) f(0).$$

And we can conclude, even before we know which values of λ can be proper values of C_h^* that all the proper values are simple.

3. Since $\|1 - C_h\| \leq 1$, the spectrum of $1 - C_h$ is included in the closed disc $\{\lambda : |\lambda| \leq 1\}$, and consequently, the spectrum of C_h is included in the closed disc $\{\lambda : |1 - \lambda| \leq 1\}$.
4. The preceding paragraph implies that the spectrum of $1 - C_h^*$ includes the open disc $\{\lambda : |\lambda| < 1\}$, and hence that the same is true of the spectrum of $1 - C_h$. This, in turn implies that the spectrum of C_h includes the open disc $\{\lambda : |1 - \lambda| < 1\}$, and the proof of (4) is complete.

□

THEOREM 2.3. *The adjoint of the weighted Cesàro operator $C_h^* \in B(\ell_q)$ has rich point spectrum.*

Proof. Notice that $\sigma_p(C_h^*) = D(q/2, q/2)$ is open and connected. It is easy to see that the mapping $f : \sigma_p(C_h^*) \longrightarrow \ell_q$ defined by

$$f_0(z) = 1, \quad f_n(z) = \prod_{j=1}^{n} \left(1 - \frac{h(j)}{zH(j)}\right) \quad for \quad n \geq 1 \tag{12}$$

is analytic, and $f(z) \in ker(C_h^* - z)/0$. It is a standard fact that the family of eigenvectors $\{f(z) : z \in D(q/2, q/2)\}$ is total in ℓ_q. Then $C_h^* \in B(\ell_q)$ has rich point spectrum. □

LEMMA 2.4. *if λ is an extended eigenvalue for C_h on ℓ_p then λ is real and $\lambda \geq 1$.*

Proof. First of all, we have $\lambda \neq 0$ because C_h is injective. Also, notice that λ is an extended eigenvalue for C_h if and only if $1/\overline{\lambda}$ is an extended eigenvalue for C_h^*, and therefore it is enough to show that if λ is an extended eigenvalue for C_h^* then λ is real and $0 < \lambda \leq 1$. □

Our next goal is to show in the Hilbertian case $p = 2$ that if λ is real and $\lambda \geq 1$ then λ is an extended eigenvalue for C_h. In section 2.1 we showed that C_h is subnormal using the following construction. Let μ be a positive finite measure defined on the Borel subsets of the complex plane with compact support and let $H^2(\mu)$ be the closure of the polynomials on the Hilbert space $\ell^2(\mu)$. Consider the shift operator M_z defined on the Hilbert space $H^2(\mu)$ by the expression $(M_z f)(z) = zf(z)$ there is a is a positive finite measure defined on the Borel subsets of the complex plane and supported on $\overline{D}$, and there is a unitary operator $U : \ell^2 \longrightarrow H^2(\mu)$ such that $I - C_h = U^* M_z U$, or in other words

$$C_h = U^*(I - M_z)U$$

Then, the extended eigenvalues of C_h are the extended eigenvalues of $I - M_z$ and the corresponding extended eigenoperators of C_h are in one to one correspondence with the extended eigenoperators of $I - M_z$ under conjugation with U, that is, if a non-zero operator X satisfies $(I-M_z)X = \lambda X(I-M_z)$ then the operator $Y = U^*XU$ satisfies $C_hY = \lambda YC_h$.

THEOREM 2.5. *If $\lambda_k = \frac{H(k)}{h(k)} \geq 1$ then λ_k is an extended eigenvalue for $I-M_z$ and a corresponding extended eigenoperator is the composition operator X_k defined by the expression*

$$(X_k f)(z) = f\left(\frac{H(k) - h(k)}{H(k)} + \frac{zh(k)}{H(k)}\right). \tag{13}$$

Where $H(k), h(k)$ defined by (1) and $k \in \mathbb{N}$

Proof. Let $f_n = X_k z^n = \left(\frac{H(k)-h(k)}{H(k)} + \frac{zh(k)}{H(k)}\right)^n$.
We have $f_{n+1} = \left(\frac{H(k)-h(k)}{H(k)} + \frac{zh(k)}{H(k)}\right) f_n$ so that

$$\begin{aligned} \frac{H(k)}{h(k)} f_{n+1} &= \left(\frac{H(k)}{h(k)} - 1 + z\right) f_n \\ &= \left(\frac{H(k)}{h(k)} - 1 + M_z\right) f_n \\ &= \frac{H(k)}{h(k)} f_n - (I - M_z) f_n \end{aligned}$$

and it follows that

$$(I - M_z) f_n = \frac{H(k)}{h(k)} (f_n - f_{n+1})$$

So that

$$\begin{aligned} (I - M_z) X_k z^n &= \frac{H(k)}{h(k)} (f_n - f_{n+1}) \\ &= \frac{H(k)}{h(k)} (X_k z^n - X_k M_z z^n) \\ &= \frac{H(k)}{h(k)} X_k (I - M_z) z^n, \end{aligned}$$

and since the family of monomials $\{z^n : n \in \mathbb{N}\}$ is a total set in $H^2(\mu)$, and $\lambda_k = \frac{H(k)}{h(k)}$ it follows that $(I - M_z)X_k = \lambda_k X_k(I - M_z)$. $\square$

Corollary 2.6. *If* $\lambda_k = \frac{H(k)}{h(k)} \geq 1$ *then* λ_k *is an extended eigenvalue for the weighted Cesàro operator* $C_h \in \ell^2$.

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