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$L(3,2,1)$-LABELING FOR CYLINDRICAL GRID: THE CARTESIAN PRODUCT OF A PATH AND A CYCLE

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Abstract. An $L(3,2,1)$-labeling for the graph $G = (V, E)$ is an assignment f of a label to each vertices of G such that $|f(u)-f(v)| \geq 4 - k$ when $\text{dist}(u, v) = k \leq 3$. The $L(3,2,1)$-labeling number, denoted by $\lambda_{3,2,1}(G)$, for G is the smallest number N such that there is an $L(3,2,1)$-labeling for G with span N.

In this paper, we compute the $L(3,2,1)$-labeling number $\lambda_{3,2,1}(G)$ when G is a cylindrical grid, which is the cartesian product $P_m \Box C_n$ of the path and the cycle, when $m \geq 4$ and $n \geq 138$. Especially when n is a multiple of 4, or $m = 4$ and n is a multiple of 6, then we have $\lambda_{3,2,1}(G) = 11$. Otherwise $\lambda_{3,2,1}(G) = 12$.

1. Introduction

A *channel assignment* in the wireless network is an assignment of channels to transmitters in the network. When we assign channels, there may exist interference between the channels assigned to two closely located transmitters. Therefore there should be proper differences between two channels according to their distances. The goal of the channel assignment problem is to find an efficient channel assignment to minimize the span of channels in order to avoid the existing interferences.

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Hale [10] and Griggs and Yeh [9] considered the channel assignment problem on a distance two labeling problem for a graph in such a way that the vertices of a graph represent the transmitters of the network and two vertices are adjacent if the corresponding transmitters are very closely located. Formally for two integers $j,\ k$, an $L(j,k)$-labeling problem of a graph $G=(V,E)$ is an assignment f of nonnegative integers to V such that $|f(u)-f(v)|\geq j$ if u,v are adjacent and $|f(u)-f(v)|\geq k$ if u,v are of distance two. The minimum span over all $L(j,k)$-labelings for a graph G is called the $L(j,k)$-number, $\lambda_{j,k}(G)$, of G. For surveys of $\lambda_{j,k}(G)$, see [4,5,7,9,17].

The distance three labeling problem is a generalization of not only the distance two labeling problem but also the distance three coloring problem. For a graph $G=(V,E)$, an $L(k_1,k_2,k_3)$-labeling for the graph G is an assignment f of a nonnegative integer to each vertices of G such that $|f(u)-f(v)|\geq k_l$ when $\mathrm{dist}(u,v)=k\leq 3$. There are some results on the distance three labelings for graphs. Especially $L(1,1,1)$-labeling problems and $L(2,1,1)$-labeling problems are computed when G is a path, a cycle, a grid, a complete binary tree or a cube [1–3, 18]. One of the important problems on distance three labeling is to find $L(3,2,1)$-labelings for classes of graph G [6, 8, 11–15]. The $L(3,2,1)$-labeling number, denoted by $\lambda_{3,2,1}(G)$, for G is the smallest number N such that there is an $L(3,2,1)$-labeling $f:V\to[0,N]$. Recently, Shao and Vesel determine $\lambda_{3,2,1}-$number for toroidal grids and triangular grids [16].

Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be two graphs. The *Cartesian product* $G=G_1\Box G_2=(V,E)$ of G_1 and G_2 is the graph such that $V=V_1\times V_2$ and two vertices (u_1,u_2) and (v_1,v_2) are adjacent if $u_1=v_1$ and $\{u_2,v_2\}\in E_2$, or $u_2=v_2$ and $\{u_1,v_1\}\in E_1$. A cylindrical grid is the cartesian product $P_m\Box C_n$ of the path P_m and the cycle C_n. Figure 1 shows the cylindrical grid $P_4\Box C_8$.

In [8], Chia et.al found $\lambda_{3,2,1}(P_m\Box P_n)$ for $m,n\geq 2$. They also found the sufficient and necessary condition such that $\lambda_{3,2,1}(P_2\Box C_n)$ has minimum value 9. When $m,n\geq 3$ the minimum of $\lambda_{3,2,1}(P_m\Box C_n)$ is 11 and they provided a sufficient condition under which $\lambda_{3,2,1}(P_m\Box C_n)=11$. In this paper we show that if $m\geq 4$ and $n\geq 138$, then $\lambda_{3,2,1}(P_m\Box C_n)\leq 12$. Moreover, if $4\nmid n$ and $6\nmid n$, then $\lambda_{3,2,1}(P_m\Box C_n)=12$. We also show that if $m\geq 5$ and $4\nmid n$, then $\lambda_{3,2,1}(P_m\Box C_n)=12$.

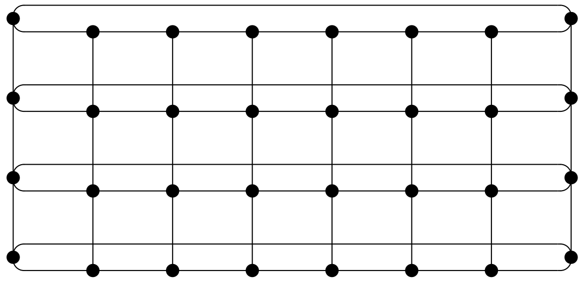

Figure 1. cylindrical grid graph $P_4\Box C_8$.

2. Some Lemmas and Main Theorem

Let $G = P_m\Box C_n$ be the Cartesian product of a path P_m and a cycle C_n.

Definition 1. Let $u = (i_0, j_0)$ with $1 \le i_0 \le n-1$, then the closed neighborhood $N[u]$ of u is the set $\{(i,j) \in v|\ \mathrm{dist}\{(i,j),(i_0,j_0)\} \le 1\}$ and the open neighborhood $N(u)$ of u is the set $\{(i,j) \in v|\ \mathrm{dist}\{(i,j),(i_0,j_0)\} = 1\}$.

For an $L(3,2,1)$-labeling f of $G = P_m\Box C_n = (V,E)$ with span 11, we have the following $L(3,2,1)$-labeling with span 11.

$$f_1 : V \to [0,11], f_1(i,j) = 11 - f(i,j),$$

$$f_2 : V \to [0,11], f_2(i,j) = f(i,-j),$$

$$f_3 : V \to [0,11], f_3(i,j) = f(m-i-1,j),$$

$$f_4 : V \to [0,11], f_4(i,j) = f(i,j'),\ \text{with } j' \equiv j+k \pmod{n}.$$

Note that f_1, f_2, f_3 and f_4 are inversion of labels, displacement by bilateral symmetry, reversing the top and bottom, and horizontal translation by k, respectively. We say that these labelings are equivalent to f.

Proposition 1. $\lambda_{3,2,1}(P_3\Box P_6) \ge 11$.

Proof. Suppose there is an $L(3,2,1)$-labeling $f : V \to [0,10]$ for G with span at most 10. Also suppose that $1 \le f(1,j), f(1,j+1) \le 9$ for some $j = 1, 2, 3, 4$. We may assume that $f(1, j) < f(1, j + 1)$. Consider $A = N[(1, j)] \cup N[(1, j+1)]$. Since each two elements of A are of distance at most three, $|f(A)| = 8$. Since each vertex $v \in A$ different from both $(1, j)$ and $(1, j + 1)$ is of distance at most two from both $(1, j)$ and $(1, j + 1)$, we have $|f(v) - f(1, j)| \geq 2$ and $|f(v) - f(1, j + 1)| \geq 2$. Thus $f(v) \neq f(1, j) \pm 1$ and $f(v) \neq f(1, j + 1) \pm 1$ for all $v \in A$ with $v \neq (1, j), (1, j + 1)$. Since $|f(1, j) - f(1, j + 1)| \geq 3$, we have

$$0 \leq f(1, j) - 1 < f(1, j) + 1 < f(1, j + 1) - 1 < f(1, j + 1) + 1 \leq 10.$$

Hence

$$11 = |[0, 10]| \geq |f(A) \cup \{f(1, j) \pm 1, f(1, j + 1) \pm 1\}| = |f(A)| + 4 = 12.$$

This is a contradiction. Thus $\{f(1, j), f(1, j + 1)\}$ contains 0 or 10 for all $j = 1, 2, 3, 4$.

We may assume that $f(1, 1) < f(1, 2)$. Then $f(1, 1) = 0$ or $f(1, 2) = 10$. If $f(1, 1) = 0$ and $f(1, 2) = 10$, then since f is an $L(3, 2, 1)$-labeling, $1 \leq f(1, 3) < f(1, 4) \leq 9$. This is a contradiction. Thus if $f(1, 1) = 0$, then $1 \leq f(1, 2) \leq 9$. If $f(1, 3) \neq 10$, then $1 \leq f(1, 2) < f(1, 3) \leq 9$. This is a contradiction. Thus $f(1, 3) = 10$. Consider the open neighborhood $N(1, 2)$ of the vertex $(1, 2)$. Let $f(N(1, 2)) = \{a_1, a_2, a_3, a_4\}$ with $a_1 < a_2 < a_3 < a_4$. Since each two elements of $N(1, 2)$ are of distance two, $a_{i+1} - a_i \geq 2$ for all $i = 1, 2, 3$. For all $v \in N(1, 2)$, since v is of distance one or three from $(1, 1)$, $f(v) \neq 0 = f(1, 1)$. As a result, $1 \leq a_1 < a_2 < a_3 < a_4 \leq 7$. Thus $a_1 = 1, a_2 = 3, a_3 = 5$ and $a_4 = 7$. Since $3 \leq f(1, 2) \leq 7$, $f(1, 2)$ is $3, 5$ or 7. If $f(1, 2) = 3$, then from the constraints $\{f(0, 2), f(2, 2)\} = \{6, 8\}$. We may assume that $f(0, 2) = 6$ and $f(2, 2) = 8$. Since $f(0, 3)$ satisfies $|f(0, 3) - f(0, 2)| = |f(0, 3)-6| \geq 3$, $|f(0, 3)-f(1, 3)| = |f(0, 3)-10| \geq 3$, $|f(0, 3)-f(1, 2)| = |f(0, 3)-3| \geq 2$ and $f(0, 3) \neq f(1, 1) = 0$, we have $f(0, 3) = 1$. Similarly $f(2, 3) = 5$. Then there is no number that satisfies all constraints for $f(2, 1)$. This is a contradiction. If $f(1, 2) = 5$, then from the constraints we have $\{f(0, 2), f(2, 2)\} = \{2, 8\}$. We may assume that $f(0, 2) = 2$ and $f(2, 2) = 8$. Then we have $f(0, 3) = 7$. It follows that $f(0, 1) = 9$, $f(2, 1) = 3$, $f(2, 3) = 1$, $f(1, 4) = 3$, $f(0, 4) = 0$, $f(2, 4) = 6$, $f(1, 5) = 8$ and $f(0, 5) = 5$. Then there is no number that satisfies all constraints for $f(2, 5)$. This is a contradiction. Similarly we can obtain a contradiction for $f(1, 2) = 7$. If $1 \leq f(1, 1) \leq 9$ and $f(1, 2) = 10$, then we have $f(1, 4) = 0$. By the same method as above $f(1, 3)$ is $3, 5$ or 7. Also we have a contradiction in each case by a similar way. Hence there is no $L(3,2,1)$-labeling for G with span at most 10 and thus $\lambda_{3,2,1}(G) \geq 11$. □

From Proposition 1, we have $\lambda_{3,2,1}(G) \geq 11$ when $G = P_m \square P_n$ for $m \geq 3, n \geq 6$ or $G = P_m \square C_n$ for $m, n \geq 3$. Proposition 1 was first proved by Chia et.al [8] in a different way, They also obtained $\lambda_{3,2,1}(G)$ for $G = P_m \square C_n$ when $m \geq 3$ and $4|n$.

PROPOSITION 2. [8] *If $m, n \geq 3$ and n is a multiple of 4, then $\lambda_{3,2,1}(P_m \square C_n) = 11$.*

PROPOSITION 3. *If n is a multiple of 6, then $\lambda_{3,2,1}(P_4 \square C_n) = 11$.*

Proof. Two patterns A and B in Table 1 are $L(3,2,1)$-labeling of $P_4 \square C_6$ with span 11. Thus if n is the multiple of 6, then we can obtain an $L(3,2,1)$-labeling of $P_4 \square C_n$ with span 11 by using one of the two patterns in Table 1 several times. □

3	8	5	10	1	6
0	11	2	7	4	9
7	4	9	0	11	2
10	1	6	3	8	5

A

3	10	5	8	1	6
0	7	2	11	4	9
11	4	9	0	7	2
8	1	6	3	10	5

B

Table 1. Two patterns of $L(3,2,1)-$labelings for $P_4 \square C_6$ with span 11.

PROPOSITION 4. *If $m \geq 4$ and $n \geq 138$, then $\lambda_{3,2,1}(P_m \square C_n) \leq 12$.*

Proof. It is known that if relatively prime positive integers a, b are given, then for all positive integer n larger than $ab - a - b$, the Frobenius number of a and b, there are non-negative integers x and y such that $n = ax + by$. Thus if $n > 59$, then $n = 6x + 13y$ for some integers $x, y \geq 0$. It follows that if n is even and $n \geq 120$, then $n = 12x + 26y$ for some integers $x, y \geq 0$. If n is odd and $n \geq 139$, then since $n - 19 \geq 120$, $n = 12x + 19 \cdot 1 + 26y$ for some integers $x, y \geq 0$. As a consequence if $n \geq 138$, then $n = 12x + 19y + 26z$ for some integers $x, y, z \geq 0$. In Table 2, a 4×12, a 4×19 and a 4×26 patterns of $L(3,2,1)$-labelings for $P_4 \square C_n$ where $n = 12, 19, 26$, are given. In fact these patterns are $L(3,2,1)$-labelings of $C_4 \times C_k$ for $k = 12, 19, 26$ respectively. The span s of these labelings are at most 12. Since the first two columns and last two columns of these three patterns are same, it is possible to expand these patterns in arbitrary order to construct an $L(3,2,1)$-labeling of $C_4 \times C_l$ for some l. We obtain an $L(3,2,1)$-labeling of $C_4 \times C_n$ by repeatedly using the 4×12 pattern x times, the 4×19 pattern y times and the 4×26 pattern z times. An $L(3,2,1)$-labeling of $G = P_{4k} \times C_n$ for $4k \geq m$ is obtained by repeatedly using this labeling vertically and an $L(3,2,1)$-labeling of $G = P_m \times C_n$ is thus obtained. As a consequence $\lambda_{3,2,1}(G) \leq 12$. □

LEMMA 1. *Suppose there is an $L(3,2,1)$-labeling of $G = P_4 \Box C_n$ such that the span of f is 11 and there are two adjacent vertices v and w of G satisfying $f(v) = 0$ and $f(w) = 11$. Then n is a multiple of 6 and f is equivalent to the labeling obtained by expanding one of two patterns given in Table 1 horizontally.*

0	5	10	3	8	1	6	11	4	9	2	7
3	8	1	6	11	4	9	2	7	0	5	10
6	11	4	9	2	7	0	5	10	3	8	1
9	2	7	0	5	10	3	8	1	6	11	4

4×12 pattern

0	5	11	3	9	1	7	12	5	10	3	8	1	6	11	4	9	2	7
3	8	1	6	12	4	10	2	8	0	6	11	4	9	2	7	0	5	10
6	12	4	10	2	8	0	6	11	4	9	2	7	0	5	10	3	8	1
9	2	7	0	5	11	3	9	1	7	12	5	10	3	8	1	6	11	4

4×19 pattern

0	5	11	3	9	1	7	12	5	10	3	8	1	6	12	4	10	2
3	8	1	6	12	4	10	2	8	0	6	11	4	9	2	7	0	5
6	12	4	10	2	8	0	6	11	4	9	2	7	0	5	11	3	9
9	2	7	0	5	11	3	9	1	7	12	5	10	3	8	1	6	12

8	0	6	11	3	9	2	7
11	3	9	1	7	12	5	10
1	7	12	5	10	2	8	1
4	10	2	8	0	6	11	4

4×26 pattern

Table 2. Three patterns of $L(3,2,1)-$labelings with span 11 or 12.

Proof. By taking a labeling equivalent to f if necessary we may assume that $v = (i_0, 0)$ such that i_0 is 0 or 1. Also we may assume that

w is $(i_0, 1)$ or $(i_0+1, 0)$. Let $u = (i_0+1, 1)$ and $N[u]$ be the closed neighborhood of u. If $f(N[u]) = \{a_1, a_2, \cdots, a_5\}$ with $a_i < a_{i+1}$ for all $i \in [0, 4]$, since every vertex in $N[u]$ is of distance at most three from v, $1 \le a_1 < a_2 < \cdots < a_5 \le 11$. Let $a_h = f(u)$. Then $a_{t+1} - a_t \ge 3$ if t is h or $h-1$ and otherwise $a_{t+1} - a_t \ge 2$. If $h \ne 1$, then since

$$10 = 11 - 1 \ge a_5 - a_1 = \sum_{t=1}^{4} (a_{t+1} - a_t) \ge 2 \cdot 2 + 2 \cdot 3 = 10,$$

we have $a_1 = 1$, $a_{t+1} - a_t = 3$ if t is h or $h-1$ and $a_{t+1} - a_t = 2$ if $t \ne h$, $h-1$. Thus $f(N[u])$ is one of $\{1, 4, 7, 9, 11\}$, $\{1, 3, 6, 9, 11\}$ and $\{1, 3, 5, 8, 11\}$. If $h = 1$, then $a_1 \ge 2$. Since

$$9 = 11 - 2 \ge a_5 - a_1 = \sum_{t=1}^{4} (a_{t+1} - a_t) \ge 3 \cdot 2 + 3 = 9,$$

we have $a_1 = 2$, $a_2 = 5$, $a_3 = 7$, $a_4 = 9$ and $a_5 = 11$. Assume $w = (i_0, 1)$. Thus we have

$$\text{(2.1)} \qquad \begin{cases} f(v) = f(i_0, 0) = 0 \\ f(w) = f(i_0, 1) = 11 \\ f(N[u]) = \{2, 5, 7, 9, 11\}. \end{cases}$$

For $x \in \{v\} \cup N[u]$, there are 18 cases which satisfying (2.1). We present them as (1)-(18) in Table 3.

We prove that the only possible cases are the patterns in Table 1 by a case by case consideration. We summarize the procedure of proof by tables of $f(i, j)$ for $m_0 \le i \le m_1, n_0 \le j \le n_1$ using some symbols since it is very lengthy and complicated to state all the proof. In each case we indicate the assumptions by numbers, and the numbers without marks are consequences of the deductions from the distance conditions and assumptions. We mark $*$ and $\#$ at the place where the label is not uniquely determined from given assumptions. We use another tables to consider each case for possible $*$ and $\#$, where the labels determined on previous tables are indicated by italic numbers. We use the notation "$\bigotimes$" at which it is impossible to find an adequate label satisfying distance conditions. It is indicated that $i_0 = 0$ or $i_0 = 1$ when it is needed. The sequence of decisions are given below the corresponding tables.

We also consider the case $w_0 = (i_0 + 1, 0)$. In this case, we have

$$
\tag{2.2}
\begin{cases}
f(v) = f(i_0, 0) = 0 \\
f(w) = f(i_0 + 1, 0) = 11 \\
f(N[u]) = \{2, 5, 7, 9, 11\}.
\end{cases}
$$

The possible cases for $f(x)$, $x \in \{v\} \cup N[u]$ are transposes of (1)-(18). The cases (1)-(18) of Table 3 are considered in (1-1)-(18-1) of the Appendix A respectively, and the transposes of (1)-(18) of Table 3 are considered in (1-2)-(18-2) of the Appendix A respectively. The cases (5-2), (7-2), (9-2), (10-2), (11-2), (12-2), (13-2), (18-2) are omitted since they are simply transposes of the patterns already considered.

(1)

0	11	
7	4	1
	9	

(2)

0	11	
7	4	9
	1	

(3)

0	11	
9	4	1
	7	

(4)

0	11	
9	4	7
	1	

(5)

0	11	
3	6	1
	9	

(6)

0	11	
3	6	9
	1	

(7)

0	11	
9	6	1
	3	

(8)

0	11	
9	6	3
	1	

(9)

0	11	
3	8	1
	5	

(10)

0	11	
3	8	5
	1	

(11)

0	11	
5	8	1
	3	

(12)

0	11	
5	8	3
	1	

(13)

0	11	
5	2	7
	9	

(14)

0	11	
5	2	9
	7	

(15)

0	11	
7	2	5
	9	

(16)

0	11	
7	2	9
	5	

(17)

0	11	
9	2	5
	7	

(18)

0	11	
9	2	7
	5	

Table 3. Cases in Lemma 1.

We explain the procedure of the proof in case (1-1) of the Appendix A with $i_0 = 1$ as an example. It is assumed that $f(1,0) = 0, f(1,1) = 11, f(2,0) = 7, f(2,1) = 4, f(2,2) = 1$ and $f(3,1) = 9$. Since

$$|f(2,0) - f(3,0)| = |7 - f(3,0)| \geq 3,$$

$$|f(3,1) - f(3,0)| = |9 - f(3,0)| \geq 3,$$

$$|f(1,0) - f(3,0)| = |0 - f(3,0)| \geq 2$$

and

$$|f(2,1) - f(3,0)| = |4 - f(3,0)| \geq 2,$$

we have $f(3,0) = 2$. Similarly $f(3,2) = 6$. As a consequence $f(1,2) = 8$. Since $f(2,3) = 1, f(1,3) = 8, f(3,3) = 6$ and $f(1,2) = 11$, we have $f(2,4) = 10$. Similarly $f(3,4) = 3, f(1,4) = 5, f(0,3) = 3$ and $f(0,2) = 6$. Also we have $f(0,1) = 9, f(2,0) = 10, f(3,0) = 5$ and $f(1,0) = 3$. Then there is no x such that $0 \leq x \leq 11$ and $|9-x| \geq 3, |3-x| \geq 3, |0-x| \geq 2$ and $|6-x| \geq 2$. Thus there is no x such that $f(-1,0) = x$. This is a contradiction as indicated in the table (1-1) of the Appendix A. The sequence of decision of this table is given below the corresponding table. Other cases are similar. The result of these lengthy consideration is that we have contradictions for all cases except four cases. They are

(1-2) with $i_0 = 1, * = 4$,
(2-1) with $i_0 = 1, * = 8$,
(8-2) with $i_0 = 0$,
(17-2) with $i_0 = 1, * = 1, \# = 2$.

In (8-2) with $i_0 = 0$, we obtain the same labels, indicated also bold faced numbers, as in the case (8-2) with $i_0 = 1$, in which case there is a contradiction. Thus (8-2) with $i_0 = 0$ has a contradiction. In the other cases, we obtain the labels same to the labels $f(x)$, $x \in \{v\} \cup N[u]$ satisfying $f(i,j) = f(3-i, j+3)$. They are also indicated by the bold faced numbers. Let $S = \{f(x) | x \in \{v\} \cup N[u]\}$. By the same method we also have a copy of these labels satisfying $f(i',j') = f(3-i', j'+3)$ for all $(i',j') \in S$. As a consequence for all $x \in \{v\} \cup N[u]$, we have $f(i,j) = f(3-i, j+3) = f(3-(3-i), j+3+3) = f(i, j+6)$. Thus we have an $L(3,2,1)$-labeling of $P_4 \square C_6$. It is the pattern B of Table 1. By the same method we have $(3,2,1)$-labelings from other two cases. From (2-1) with $i_0 = 1, * = 8$, we have the pattern A in Table 1, and from (17-2) with $i_0 = 1, * = 7, \# = 2$, we have an $L(3,2,1)$-labeling of $P_4 \square C_6$ isomorphic to the pattern B in Table 1. Thus n is a multiple of 6 and f is equivalent to a labeling obtained by expanding the patterns A or B in Table 1 horizontally. $\square$

LEMMA 2. *Let $m \geq 5$, f be an $L(3,2,1)$-labeling of $P_m \square C_n$ with span 11 and $f(v) = 0$ and $f(w) = 11$, then v and w are not adjacent.*

Proof. Suppose there is an $L(3,2,1)$-labeling f of $P_m \square C_n$ satisfying the conditions in the statement of this lemma. From Lemma 1, n is a multiple of 6 and the restriction of f to the subset $V_0 = \{(i,j) | 0 \leq i \leq 3, 0 \leq j \leq 5\}$ of V is isomorphic to the pattern A or pattern B in Table 1. We may assume that the restriction of f to V_0 is the pattern A, the pattern B or one of patterns obtained by reversing the top and bottom. Assume the restriction of f to the subset V_0 is the pattern A. Then, $f(0,0) = 3$, $f(0,1) = 8$, $\cdots$,$f(3,5) = 5$. Since f is an $L(3,2,1)$-labeling, we have $f(4,2) = 11$ and $f(4,3) = 10$, which is a contradiction. Thus it is impossible to extend this patterns to V. We can prove that it is also impossible to extend other patterns to V by a similar method. This is a contradiction. $\square$

LEMMA 3. *Let $m \geq 3$ and f be an $L(3,2,1)$-labeling of $P_m \square C_n$ with span 11, then there are no adjacent vertices v and w such that $f(v) = 0$ and $f(w) = 10$.*

Let f be an $L(3,2,1)$-labeling of $P_m \square C_n$ with span 11. If $m \geq 3$, then there are no adjacent vertices whose labels are 0 and 10 respectively.

Proof. We may assume $m = 3$. Let v and w be adjacent vertices in V such that $f(v) = 0$ and $f(w) = 10$. Suppose that there is j such that the vertex $u = (1, j) \in V$ is adjacent to w and not adjacent to v. Let $N[u] = \{u_1, u_2, u_3, u_4, u_5\}$ such that $f(u_1) \leq f(u_2) \leq \cdots \leq f(u_5)$. If $u_1 = u$, then since u and v are of distance two, $f(u_1) \geq 2$. Since $f(u_2) - f(u_1) \geq 3$ and $f(u_{t+1}) - f(u_t) \geq 2$ for all $t = 2, 3, 4$, we have

$$8 = 10 - 2 \geq f(u_5) - f(u_1) = \sum_{t=1}^{4} (f(u_{t+1}) - f(u_t)) \geq 3 + 2 \cdot 3 = 9.$$

This is a contradiction. Thus $u_1 \neq u$. Since u_1 and v are of distance 1 or 3, $f(u_1) \geq 1$. Since $f(u_{h+1}) - f(u_h) \geq 3$ and $f(u_h) - f(u_{h-1}) \geq 3$ where $u = u_h$, we have

$$9 = 10 - 1 \geq f(u_5) - f(u_1) = \sum_{t=1}^{4} (f(u_{t+1}) - f(u_t)) \geq 2 \cdot 3 + 2 \cdot 2 = 10.$$

This is a contradiction.

Suppose that there is no j such that a vertex $u = (1, j) \in V$ is adjacent to w and not adjacent to v. Then either $v = (1, s)$ and $w = (0, s)$ for some s or $v = (1, s)$ and $w = (2, s)$ for some s. We may assume that $v = (1, 0)$ and $w = (0, 0)$. Let $x = (1, 1)$ and $N[x] = \{x_1, x_2, x_3, x_4, x_5\}$ such that $f(x_1) \leq f(x_2) \leq \cdots \leq f(x_5)$. Since $v \in N[x]$ and $f(v) = 0$, we have $x_1 = v$. We also have $f(x_{t+1}) - f(x_t) \geq 2$ for all $t = 1, 2, 3, 4$ and $f(x_{t+1}) - f(x_t) \geq 3$ when $x_t = x$ or $x_{t+1} = x$. If $x_5 = x$, then

$$f(x_5) - f(x_1) = \sum_{t=1}^{4} (f(x_{t+1}) - f(x_t)) \geq 3 + 2 \cdot 3 = 9.$$

But w and x are of distance two and $|f(w) - f(x)| = |10 - f(x)| \le 1$. This is a contradiction. Thus $x_5 \ne x$. Since

$$f(x_5) - f(x_1) = \sum_{t=1}^{4}(f(x_{t+1}) - f(x_t)) \ge 3 \cdot 2 + 2 \cdot 2 = 10$$

and $f(x_5) \ne f(w) = 10$, we have $f(x_5) = 11$. It follows that x_5 and w are of distance two, and thus x_5 is $(1,2)$ or $(2,1)$. Hence $f(1,2) = 11$ or $f(2,1) = 11$. Let $x' = (1, n-1)$. By a similar method, we have $f(1, n-2) = 11$ or $f(2, n-1) = 11$. Since $(2,1)$ and $(2, n-1)$ are of distance two, we have $f(2,1) \ne f(2, n-1)$. Thus $f(1,2) = 11$ or $f(1, n-2) = 11$. We may assume that $f(1,2) = 11$. Let $y = (1,2)$. If $f(1,3) = 1$, then $(1,4)$ is adjacent to $(1,3)$ and not adjacent to $(1,2)$. Let $\tilde{f} = 11 - f$. Then $\tilde{f}$ is an $L(3,2,1)$-labeling of $P_m \square C_n$, $\tilde{f}(1,2) = 0$, $\tilde{f}(1,3) = 10$, and there is a vertex $(1,4)$ that is adjacent to $(1,3)$ and not adjacent to $(1,2)$. We have already a contradiction in this case. Thus $f(1,4) \ne 1$. If $f(0,2) = 1$, then $4 \le f(0,1) \le 7$. If $f(0,1) = 4$, then $f(1,1)$ is 7 or 8. It follows that $f(2,1) = 2$. Therefore $f(0,2)$ is 5 or 6. Then there is no suitable label $f(2,2)$ of $(2,2)$ satisfying all constraints. Similarly we have a contradiction when $f(0,1)$ is $5, 6$ or 7 respectively. We summarize these procedures in case 1 of Table 4. Basic assumptions $f(0,0) = 0, f(1,0) = 0, f(1,2) = 11$ and $f(0,2) = 1$ are indicated by bold faced letters. For numbers a and b, we use the notation $a(b)$ when the corresponding label to given vertex is a or b. As in Lemma 2, $\bigotimes$ means a contradiction, or means that there is no suitable label in this vertex. If $f(2,2) = 1$, then $3 \le f(1,1) \le 8$. We have a contradiction in each case. These procedures are also summarized in case 2 of Table 4. If $f(0,2) \ne 1$ and $f(12,2) \ne 1$, then the labels of four vertices adjacent to $y = (1,2)$ are all at least 2. Thus the labels of these four vertices are $2, 4, 6$ and 8. It follows that $f(0,1) \ne 2, 4, 6, 8$ since $(0,1)$ is of distance at most three from these vertices. Also $f(2,1) \ne 2, 4, 6, 8$ by the same reason. Since $f(1,0) = 0$, $f(1,1)$ is $4, 6$ or 8. If $f(1,1) = 8$, then $f(0,1)$ is 3 or 5. We have a contradiction in each case. These procedures are summarized in case 3 of Table 4. As a consequence there is no $L(3,2,1)$-labeling of $P_m \square C_n$ with span 11 when there are adjacent vertices whose labels are 0 and 10 respectively. □

By Lemma 3, we also have that if there is an $L(3,2,1)$-labeling f of $P_m \square C_n$ such that $m \ge 3$ and the span of f is at most 11, then there are no adjacent vertices whose labels are 1 and 11 respectively.

10	4	**1**
0	7(8)	**11**
5(6)	2	$\otimes$

(4,7(8),2,5(6))

10	5	**1**
0	8	**11**
6	2(3)	$\otimes$

(5,8,2(3),6)

10	6	**1**
0	3	**11**
5	8(9)	$\otimes$

10	7	**1**
0	3(4)	**11**
5(6)	9	$\otimes$

Case1) When $f(0,2) = 1$ and $f(0,1)$ is 4,5,6 or 7.

10	6(7)	$\otimes$
0	3	**11**
	8(9)	**1**

(3,6(7),8(9))

10	7	$\otimes$
0	4	**11**
	9	**1**

(4,7,9)

10	2	7	0	
0	5	**11**	3	$8'(9')$
	8(9)	**1**	6	$\otimes$

(5,2,8(9),7,3,0,6,$8'(9')$)

10	2(3)	8	0	
0	6	**11**	3(4)	9
	9	**1**	7	$\otimes$

(6,9,2(3),8,3(4),0,7,9)

10		
0	7	**11**
$\otimes$	4	**1**

(7,4)

10		
0	8	**11**
$\otimes$	4(5)	**1**

(8,4(5))

Case2) When $f(2,2) = 1$ and $f(1,1)$ is 3,4,5,6,7 or 8.

10	7	
0	4	**11**
$\otimes$	9	6

(4,7,9)

10	3	8	0(1)	
0	6	**11**	$4'$	$9'$
4	9	2	7	$\otimes$

(6,3,9,4,2,8,$4'$,7,0(1),$9'$)

10	3	
0	8	**11**
$\otimes$	5	

(8,3,5)

10	5	
0	8	**11**
6	2(3)	$\otimes$

(8,5,2(3),6)

Case3) When $f(0,2), f(2,2) \geq 2$ with $f(1,1)$ is 4,6 or 8.

Table 4. Labeling procedure when $f(0,0) = 10, f(1,2) = 11$ and $f(1,0) = 0$.

LEMMA 4. *Let f be an $L(3,2,1)$-labeling of $P_3 \square C_m$ such that the span of f is 11 and $|f(x) - f(y)| \leq 9$ for each two adjacent vertices $x, y \in V$. Then, we have*

$$\begin{cases} f(1,j) - f(1,j+1) \equiv 3,5,7,9 \pmod{12}, & \text{if } 0 \leq j \leq m-2 \\ f(1,m-1) - f(1,0) \equiv 3,5,7,9 \pmod{12}. \end{cases} \tag{2.3}$$

Proof. Let $v = (1,j)$ and $w = (1,j+1)$. Let $f(N(v)) = \{a_1, a_2, a_3, a_4\}$ such that $a_1 < a_2 < a_3 < a_4$. Since f is an $L(3,2,1)$-labeling, $a_{i+1} - a_i \geq 2$ for all $i = 1,2,3$. Thus $a_4 - a_1 \geq 6$ and if $a_4 - a_1 = 6$, then $a_{i+1} = a_i + 2$ for all $i = 1,2,3$. If $f(v) = 0$, then from the assumption, we have $3 \leq a_1 < a_2 < a_3 < a_4 \leq 9$. Thus $a_1 = 3, a_2 = 5, a_3 = 7$ and $a_4 = 9$. Hence $f(w)$ is $3, 5, 7$ or 9. It follows that (2.3) holds for the case $f(v) = 0$. If $f(v) = 1$, then since $|f(v) - a_4| \leq 9$, $4 \leq a_1 < a_4 \leq 10$. Since $f(w)$ is $4, 6, 8$ or 10, (2.3) holds for the cases $f(v) = 1$. Similarly we can prove (2.3) holds when $f(v)$ or $f(w)$ is $0, 1, 2, 9, 10$ or 11.

If $f(v) = 4$, then $0 \le a_1 \le 1$ and $7 \le a_2 < a_3 < a_4 \le 11$. Thus $a_2 = 7, a_3 = 9$ and $a_4 = 11$. If $f(w) = 0$, then by this lemma for $f(w) = 0$, $f(v)$ is $3, 5, 7$ or 9. This is a contradiction. Thus $a_1 = 1$. As a result, (2.3) holds for $f(v) = 4$. Similarly, (2.3) holds for $f(v)$ or $f(w)$ is 4 or 7. Suppose $f(v) = 3$. If $f(w) = 11$, then it is already verified that the labels of four vertices adjacent to w is $2, 4, 6$ and 8. This is a contradiction. Thus $f(w) \neq 11$. Similarly $f(w) \neq 7, 9$. Hence $f(w)$ is $0, 6, 8$ or 10. As a consequence, the lemma is true for $f(v) = 3$. Similarly (2.3) holds for $f(v) = 8$. If $f(v) = 5$, then $0 \le a_1 < a_2 \le 2$ and $8 \le a_4 < a_4 \le 11$. Thus $a_1 = 0, a_2 = 2$. it follows that $f(w) \neq 1$. If $f(w) = 11$, then since v is adjacent to w, $f(v)$ is $2, 4, 6$ or 8. This is a contradiction. Thus $f(w) \neq 11$. Similarly $f(w) \neq 9$. Thus $f(w)$ is $0, 2, 8$ or 11. Thus (2.3) holds for $f(v) = 5$, and thus also for $f(v) = 6$. $\square$

LEMMA 5. *Let $G = P_3 \square C_n$. Suppose that there is an $L(3,2,1)$-labeling f of G satisfying the following.*

1) The span of f is 11.
2) $f(1, j) = 0$ for some j.
3) $|f(v) - f(w)| \le 9$ for each adjacent vertices v, w of G.

Then n is a multiple of 4.

Proof. We will summarize the procedure of the proof in a similar way as in Lemma 1. We may assume that $j = 0$, or equivalently $f(1, 0) = 0$. Since $|f(v) - f(1, 0)| = |f(v)| \le 9$ for all $v \in N(1, 0)$, we have $f(N(1, 0)) = \{3, 5, 7, 9\}$. Thus it suffices to consider the six cases of $f(v)$ for $v \in N[(1, 0)]$ given in Table 5.

(1)			(2)			(3)			(4)			(5)			(6)		
	3			3			3			7			5			5	
7	0	9	5	0	9	5	0	7	3	0	5	3	0	7	3	0	9
	5			7			9			9			9			7	

Table 5. Cases in Lemma 5.

In Table 6, these cases are considered. In each of (1), (2), (4), (5) we have a contradiction. In cases (3) and (6), we have the identities $f(i, j) = f(i, j + 12)$ and $f(i, j) = f(i, j + 4)$ respectively by achieving the same condition boldfaced label as the assumptions. As a consequence n is 4 or 12, and thus n is a multiple of 4. $\square$

(1)

10	**3**	6
7	**0**	**9**
2	**5**	$\otimes$

(2)

	3	6
5	**0**	**9**
	7	$*$

11	8	**3**	6
2	**5**	**0**	**9**
$\otimes$	10	**7**	2

$* = 2$

	3	6	11
5	**0**	**9**	2
	7	4	$\otimes$

$* = 4$

(3)

8	**3**	10	5	0	7	2	9	4	11	6	1	8	**3**	
5	**0**	**7**	2	9	4	11	6	1	8	3	10	**5**	**0**	**7**
2	**9**	4	11	6	1	8	3	10	5	0	7	2	**9**	

(4)

10	**7**	2
3	**0**	**5**
6	**9**	$\otimes$

(5)

$*$	**5**	
3	**0**	**7**
6	**9**	

$\otimes$	8	**5**	
10	**3**	**0**	**7**
1	6	**9**	

$* = 8$

10	**5**	2	$\otimes$
3	**0**	**7**	10
6	**9**	4	1

$* = 10$

(6)

8	**5**	2	11	8	**5**	
3	**0**	**9**	6	**3**	**0**	**9**
10	**7**	4	1	10	**7**	

Table 6. Labeling Procedure.

LEMMA 6. *Let $G = (V, E) = P_4 \square C_n$. If there is an $L(3, 2, 1)$-labeling f of G such that $|f(v) - f(w)| \leq 9$ for all adjacent vertices v and w of G, then n is a multiple of 4.*

Proof. If $f(1, j) = 0$ or $f(2, j) = 0$ for some $j = 0, 1, \cdots n-1$, then by Lemma 6, n is a multiple of 4. If $f(1, j) = 11$ or $f(2, j) = 11$ for some j, then since the inversion $\widehat{f} = 11 - f$ of f is also an $L(3, 2, 1)$-labeling and $|\widehat{f}(v) - \widehat{f}(w)| \leq 9$ for all adjacent vertices v and w of G, n is a multiple of 4. We want to show that $f(i, j) = 0$ or $f(i, j) = 11$ for some $(i, j) \in V$ such that $i = 1, 2$. The proposition follows from this claim.
Since $\lambda_{3,2,1}(G) \geq 11 = \text{span}(f)$, these is $v \in V$ such that $f(v) = f(i, j) = 0$. If $i = 1, 2$, then our claim is already satisfied. By reversing the top and bottom, the case $i = 3$ is reduced to the case $i = 0$. We may assume that $v = (0, 0)$. Since f is an $L(3, 2, 1)$-labeling, we have $3 \leq f(1, 0) \leq 9$.

If $f(1,0)$ is 4, 6 or 8, then $f(w) = 11$ for some vertex w adjacent to $(0,1)$. Since $w \neq v = (0,0)$, $w \in \{(1,-1),(1,1),(2,0)\}$. Therefore our claim is proved when $f(1,0) = 4,6,8,10$. The cases $f(1,0) = 3,5,7,9$ are considered in Table 7 by the same manner as in Lemma 1. We can see that in any case there exist $(i,j) \in V$ such that $f(i,j) = 0, 11$ and $i = 1,2$. Thus our claim is proved. $\square$

(1)

5	**0**	7
8	3	10
11	6	1

(2)

9	**0**	
6	3	10
$*$	8	

(2-1)

	9	**0**	
11	6	3	10
	1	8	

$* = 1$

(2-2)

	0	
6	3	10
11	8	

$* = 11$

(3)

9	**0**	5
6	3	8
1	10	$\otimes$

(4)

3	**0**	7
8	5	10
11	2	

(5)

	0	
2	5	10
11	8	

(6)

	0	3	
2	5	8	11
	10	1	

(7)

9	**0**	
4	7	10
11	2	

(8)

	0	$*$
2	7	10
	4	1

(8-1)

	0	3	8	
2	7	10	5	0
	4	1	2	

$* = 3$

(8-2)

	0	5	8	
2	7	10	3	0
	4	1	6	

$* = 5$

(9)

5	**0**	9
2	7	4
$\otimes$	10	

(10)

7	**0**	
4	9	6
11	2	

(11)

0	3	
2	9	6
	4	$*$

(11-1)

	0	3	
2	9	6	11
	4	1	

$* = 1$

(11-2)

	0	3
2	9	6
	4	11

$* = 11$

(12)

5	**0**	7
2	9	4
11	6	

Table 7. Labeling Procedure.

PROPOSITION 5. *If n is neither a multiple of* 4 *nor a multiple of* 6*, then* $\lambda_{3,2,1}(P_4 \square C_n) \geq 12$.

Proof. Let f be an $L(3,2,1)-$labeling of G with span 11. If there are adjacent vertices v and w of G such that $|f(v) - f(w)| \geq 10$, then by Lemmas 1 and 3, n is a multiple of 6. If each pair of adjacent vertices v and w of G satisfies $|f(v) - f(w)| \leq 9$, then by Lemma 6, n is a multiple of 4. $\square$

PROPOSITION 6. *If $m \geq 5$ and 3 is not a multiple of 4, then we have* $\lambda_{3,2,1}(P_m \square C_n) \geq 12$.

Proof. Let f be an $L(3,2,1)-$labeling of G with span 11. Since $m \geq s$, by Lemmas 2 and 3, there are no adjacent vertices v and w of such that $|f(v) - f(w)| \geq 10$. By Lemma 6, n is a multiple of 4. Hence $\lambda_{3,2,1}(P_m \square C_n) \geq 12$ for $m \geq 5$ and $n \equiv 0 \pmod 4$. □

From Propositions $1 \sim 6$, we have the following theorem.

THEOREM 1.
1. *If $m \geq 3$ and n is a multiple of 4, then $\lambda_{3,2,1}(P_m \square C_n) = 11$.*
2. *If n is a multiple of 6, then $\lambda_{3,2,1}(P_4 \square C_n) = 11$.*
3. *If n is neither a multiple of 4 nor a multiple of 6, then $\lambda_{3,2,1}(P_4 \square C_m) \geq 12$. The equality holds when $n \geq 138$.*
4. *If $m \geq 5$ and n is not a multiple of 4, then $\lambda_{3,2,1}(P_m \square C_n) \geq 12$. The equality holds when $n \geq 138$.*

Appendix A.

In each table of Appendix A, the numbers and symbols obey the following rules.

(1) The bold faced numbers are used to indicate the basic assumptions (1) – (18) in Table 3.
(2) The italic numbers and symbols are used to indicate the labels deduced from the previous assumptions. The sequences of decisions are given below the corresponding tables. When a specific number a is repeated in a sequence of decisions, we indicate that in the order of a, a' and a''.
(3) If there are two choices of decision, a, b, c or a_1, b_1, c_1, then we use $a(a_1)$, $b(b_1)$, $c(c_1)$. When it becomes more complicated, we use $*$ and # and then those cases are handled in the subsequent tables.

(1-1)

	0	**11**	*8*	
$\otimes$	**7**	**4**	**1**	*10*
11′	*2*	**9**	*6*	*3*
	5	*0*	*11*	

$(2, 6, 8, 10, 3, 11, 0, 5, 11')$
$i_0 = 0$

$\otimes$	*9*	*6′*	*3′*	
3″	**0**	**11**	*8*	*5*
10′	**7**	**4**	**1**	*10*
5′	*2*	**9**	*6*	*3*

$(2, 6, 8, 10, 3, 5, 3', 6', 9, 10', 5', 3'')$
$i_0 = 1$

(1-2)

0	**7**	*2*	*5*	
11	**4**	**9**	*0*	$*$
8	**1**	*6*	*11*	
	10	*3*		

$(2,6,8,10,3,11,0,5)$
$i_0 = 0$

0	**7**	2	5	
11	**4**	**9**	0	3
8	**1**	6	11	*8*
	10	3	$\otimes$	

$(3,8)$
$i_0 = 0,\ * = 3$

0	**7**	2	5	*10*
11	**4**	**9**	0	7
8	**1**	6	11	$\#$
	10	3		

(10)
$i_0 = 0,\ * = 7$

0	**7**	2	5	10	*1*	*8′*	
11	**4**	**9**	0	7	*4*	*11*	$\otimes$
8	**1**	6	11	2	*9*	*6*	*1′*
	10	3	*8*	*5*	*0*	*3*	

$(8,5,9,4,1,11,0,6,8',3,1')$
$i_0 = 0, * = 7, \# = 2$

0	**7**	2	5	10	$\otimes$
11	**4**	**9**	0	7	*2*
8	**1**	6	11	*4*	*9*
	10	3	*8*	*1*	

$(8,1,9,2)$
$i_0 = 0, * = 7, \# = 4$

	10	*5*	*8*	
0	**7**	*2*	*11*	$*$
11	**4**	**9**	*0*	
	1	*6*		

$(2,10,5,11,6,0,8)$
$i_0 = 1$

	10	5	8	
0	**7**	2	11	6
11	**4**	**9**	0	*3*
	1	6	$\otimes$	

(3)
$i_0 = 1,\ * = 6$

	10	5	8	**1**	
0	**7**	2	**11**	**4**	**9**
11	**4**	**9**	**0**	**7**	
	1	6	*3*	*10*	

$(7,3,10,1,9)$
$i_0 = 1,\ * = 4$

(2-1)

0	**11**	*2*	
7	**4**	**9**	*0*
10	**1**	*6*	*3(11)*
	8	*11(3)*	$\otimes$

$(10,6,8,2,0,11(3),3(11))$
$i_0 = 0$

	$*$	
0	**11**	*2*
7	**4**	**9**
	1	*6*

$(6,2)$
$i_0 = 1$

	6	$\otimes$
0	**11**	2
7	**4**	**9**
	1	6

$i_0 = 1,\ * = 6$

	8	*5*	*10*	**1**	
0	**11**	2	**7**	**4**	**9**
7	**4**	**9**	**0**	**11**	
	1	6			

$(5,7,0,10,4,11,1,9)$
$i_0 = 1,\ * = 8$

(2-2)

0	**7**	*10*	
11	**4**	**1**	*8*
2	**9**	*6*	*11(3)*
	0	*3(11)*	$\otimes$

$(10,6,8,2,0,11(3),3(11))$
$i_0 = 0$

	2	*5*	*0*	
0	**7**	*10*	*3*	*6′*
11	**4**	**1**	*8*	*11*
	9	*6*	$\otimes$	

$(10,2,5,3,6,8,0,6',11)$
$i_0 = 1$

(3-1)

0	**11**	$*$
9	**4**	**1**
	7	*10*

(10)

0	**11**	8	*3*
9	**4**	**1**	*6*
	7	10	$\otimes$

$(6,3)$
$* = 8$

0	**11**	6	*3*
9	**4**	**1**	*8*
	7	10	*5*
	$\otimes$	*3′*	*0*

$(8,3,5,3',0)$
$i_0 = 0,\ * = 6$

		9	*0*	
0	**11**	6	*3*	*10*
9	**4**	**1**	*8*	*5*
	7	10	$\otimes$	

$(8,3,9,0,10,5)$
$i_0 = 1,\ * = 6$

(3-2)

0	**9**	
11	**4**	**7**
$*$	**1**	*10*

(10)
$i_0 = 0$

0	**9**	
11	**4**	**7**
8	**1**	10
3	*6*	$\otimes$

$(6,3)$
$i_0 = 0,\ * = 8$

0	**9**		
11	**4**	**7**	$\otimes$
6	**1**	10	*3′*
3	*8*	*5*	*0*

$(8,3,5,3',0)$
$i_0 = 0,\ * = 6$

	6	*11*	*8*	
0	**9**	*2*	*5*	*10′*
11	**4**	**7**	*0*	*3*
	1	*10*	$\otimes$	

$(2,6,11,5,10,0,8,10',3)$
$i_0 = 1$

(4-1)

0	**11**	*2*	
9	**4**	**7**	*0*
6	**1**	*10*	*3(5)*
	8	*5(3)*	$\otimes$

(2, 10, 0, 6, 8, 3(5), 5(3))
$i_0 = 0$

		9(5)	$\otimes$
0	**11**	*2*	*5(9)*
9	**4**	**7**	*0*
6	**1**	*10*	

(6, 10, 2, 0, 5(9), 9(5))
$i_0 = 1$

(4-2)

0	**9**	*6*	
11	**4**	**1**	*8*
2	**7**	10	*5(3)*
	0	*3(5)*	$\otimes$

(2, 10, 0, 6, 8, 3(5), 5(3))
$i_0 = 0$

	2	*11*	*0*	
0	**9**	*6*	*3*	$\otimes$
11	**4**	**1**	*8*	*11′*
	7	*10*	*5*	

(6, 2, 11, 3, 10, 8, 0, 5, 11′)
$i_0 = 1$

(5-1)

0	**11**	
3	**6**	**1**
$\otimes$	**9**	

0	**3**	$\otimes$
11	**6**	**9**
	1	

(6-1)

0	**11**	
3	**6**	**9**
$*$	**1**	

	7	**0**	**11**	
1	*10*	**3**	**6**	**9**
$\otimes$	*5*	8	**1**	

(10, 5, 7, 1)
$* = 8$

$\otimes$	**0**	**11**	
8	**3**	**6**	**9**
5	10	**1**	

(8, 5)
$* = 10$

(6-2)

0	**3**	
11	**6**	**1**
2	**9**	*4*
$*$	*0*	

(4, 2, 0)
$i_0 = 0$

	9	**0**	**3**	
1	*4*	**11**	**6**	**1**
$\otimes$	*7*	2	**9**	4
	10	5	0	

(7, 10, 4, 9, 1)
$i_0 = 0, * = 5$

$\otimes$	**0**	**3**	
8	**11**	**6**	**1**
5	2	**9**	4
10	7	0	

(7, 2, 10, 8, 5, 3, 0)
$i_0 = 0,\ * = 7$

	10(8)	*5*	
0	**3**	8(10)	*11(7)*
11	**6**	**1**	$\otimes$
	9	4	

(4, 8(10), 10(8), 5, 11(7))
$i_0 = 1$

(7-1)

0	**11**	
9	**6**	**1**
$\otimes$	**3**	

0	**9**	$\otimes$
11	**6**	**3**
	1	

(8-1)

	5	**0**	**11**	*8*
11′	*2*	**9**	**6**	**3**
$\otimes$	*7*	*4*	**1**	*10*
	0	*11*	*8′*	

(8, 10, 4, 8′, 11, 7, 2, 0, 5, 11′)
$i_0 = 0$

$*$	*10′*	*7*	*4′*	*1*	
8′	*5′*	**0**	**11**	*8*	*5*
	2	**9**	**6**	**3**	*0*
		4	**1**	*10*	

(4, 8.10, 0, 5, 1, 4′, 7, 2, 5′, 10′, 8′)
$i_0 = 1$

$\otimes$	1	10	7	4	1	
3	8	5	**0**	**11**	8	5
6	*11*	2	**9**	**6**	**3**	0
	0	*7*	4	**1**	10	

(11, 7, 0, 3, 6)
$i_0 = 1,\ * = 1$

$\otimes$	*6*	3	10	7	4	1	
10	*1*	8	5	**0**	**11**	8	5
	4	*11*	2	**9**	**6**	**3**	0
				4	**1**	10	

(11, 1, 6, 4, 10)
$i_0 = 1,\ * = 3$

(8-2)

	0	**9**	
$*$	**11**	**6**	**1**
	8	**3**	

(8)
$i_0 = 1$

	8	*1*	*10*			
$\otimes$	*11*	*4*	*7*	**0**	**9**	
1′	*6*	*9*	2	**11**	**6**	**1**
	3	*0*	*5*	8	**3**	

$(5, 7, 9, 0, 4, 10, 1, 11, 6, 8, 3, 1')$
$i_0 = 1,\ * = 2$

5	*10*	*3*	$\otimes$	
2	*7*	**0**	**9**	
9	4	**11**	**6**	**1**
6	*1*	8	**3**	

$(1, 7, 9, 6, 2, 10, 5, 3)$
$i_0 = 1,\ * = 4$

	2	*7*	**0**	**9**	
0′	**9**	*4*	**11**	**6**	**1**
11	**6**	**1**	*8*	**3**	*10*
	3	*10′*	*5*	*0*	

$(8, 10, 0, 5, 1, 10', 4, 7, 9, 6, 3, 11, 2, 0')$
$i_0 = 0$

(9-1)

0	**11**	
3	**8**	**1**
10	**5**	$\otimes$

(10)

0	**3**	10
11	**8**	**5**
	1	$\otimes$

(10)

(10-1)

(10-1)

	0	**11**	
10′	**3**	**8**	**5**
$\otimes$	*6*	**1**	*10*

$(10, 6, 10')$

(10-2)

0	**3**	*6*	
11	**8**	**1**	*4*
2	**5**	*10*	*7*
	0	*3*	$\otimes$

$(2, 10, 0, 6, 4, 7, 3)$
$i_0 = 0$

	10′	$\otimes$
0	**3**	*6*
11	**8**	**1**
	5	*10*

$(10, 6, 10')$
$i_0 = 1$

(11-1)

(11-1)

0	**11**	*4*	
5	**8**	**1**	*10′*
10	**3**	*6*	$\otimes$

$(10, 6, 4, 10')$

(11-2)

0	**5**	*10*
11	**8**	**3**
4	**1**	*6*
	10′	$\otimes$

$(10, 6, 4, 10')$
$i_0 = 0$

	2	*7*	
0	**5**	*10*	$\otimes$
11	**8**	**3**	*0*
	1	*6*	

$(10, 2, 7, 6, 0)$
$i_0 = 1$

(12-1)

0	**11**	$\otimes$
5	**8**	**3**
10	**1**	*6*

$(10, 6)$

0	**5**	*10*
11	**8**	**1**
$\otimes$	**3**	*6*

$(10, 6)$

(13-1)

0	**11**	
5	**2**	**7**
$\otimes$	**9**	

0	**5**	$\otimes$
11	**2**	**9**
	7	

(14-1)

3	**0**	**11**	*6*	
8	**5**	**2**	**9**	$0'$
1	*10*	**7**	*4*	*11*
	$3'$	*0*	$\otimes$	

$(10, 8, 3, 1, 3', 0, 4, 6, 0', 11)$
$i_0 = 0$

	$10'$	*7*	*4*	$1'$	$*$
6	*3*	**0**	**11**	$6'$	$3'$
11	*8*	**5**	**2**	**9**	
	1	*10*	**7**		

$(10, 8, 3, 1, 11, 6, 10', 7, 4, 6', 1', 3')$
$i_0 = 1$

	10	7	4	1	8	*5*	$\otimes$
6	3	**0**	**11**	6	3	*10*	*1*
11	8	**5**	**2**	**9**	*0*	*7*	
	1	10	**7**				

$(0, 10, 5, 7)$
$i_0 = 1, \; * = 8$

	10	7	4	1	10	*5*
6	3	**0**	**11**	6	3	*8*
11	8	**5**	**2**	**9**	*0*	*11*
	1	10	**7**	$\otimes$	$5'$	

$(0, 8, 5, 11, 5')$
$i_0 = 1, \; * = 10$

(14-2)

0	**5**	
11	**2**	**7**
$*$	**9**	

$i_0 = 0$

	0	**5**	*10*
$\otimes$	**11**	**2**	**7**
7	4	**9**	*0*
	1	*6*	

$(0, 6, 1, 7)$
$i_0 = 0, * = 4$

0	**5**	
11	**2**	**7**
6	**9**	#

$i_0 = 0, * = 6$

$\otimes$	**0**	**5**	10
8	**11**	**2**	**7**
3	6	**9**	0
	1	*4*	

$(4, 1, 3, 8)$
$i_0 = 0, * = 6, \# = 0$

0	**5**	10	
11	**2**	**7**	$\otimes$
6	**9**	4	*1*
	0	*11*	

$(0, 11, 1, 10)$
$i_0 = 0, * = 6, \# = 4$

	3	*8*	
$*$	**0**	**5**	*10*
	11	**2**	**7**
		9	

$(10, 8, 3)$
$i_0 = 1$

		3	8	
	7	**0**	**5**	10
9	*4*	**11**	**2**	**7**
$\otimes$	*1*	*6*	**9**	

$(4, 6, 1, 9)$
$i_0 = 1, * = 7$

	8	*11*	*6*	3	8	
$10'$	*5*	*2*	9	**0**	**5**	10
$\otimes$	*0*	*7*	*4*	**11**	**2**	**7**
	3	*10*	*1*	*6*	**9**	

$(6, 4, 6', 1, 7, 10, 2, 11, 5, 0, 3, 8, 10')$
$i_0 = 1, * = 9$

(15-1)

	0	**11**	8	
10	**7**	**2**	**5**	$10'$
1	*4*	**9**	*0*	*7*
	11	*6*	*3*	$\otimes$

$(8, 0, 4, 10, 1, 11, 6, 3, 10', 7)$
$i_0 = 0$

0	**11**	*8*	$*$
7	**2**	**5**	*10*
	9	*0*	

$(8, 0, 10)$
$i_0 = 1$

	$\otimes$	*3*	*6*
0	**11**	8	1
7	**2**	**5**	10
	9	0	

$(3, 6)$
$i_0 = 1, \; * = 1$

		1	*6*	
0	**11**	8	3	*0*
7	**2**	**5**	10	$\otimes$
	9	0		

$(1, 6, 0)$
$i_0 = 1, \; * = 3$

(15-2)

0	**7**	
11	**2**	**9**
8	**5**	*0*
$*$	*10*	

$(8, 0, 10)$
$i_0 = 0$

	0	**7**	
$\otimes$	**11**	**2**	**9**
3	8	**5**	0
6	1	10	

$(3, 6)$
$i_0 = 0, * = 1$

		0	**7**	
	4	**11**	**2**	**9**
10	*1*	8	**5**	0
$\otimes$	*6*	3	10	

$(1, 6, 4, 10)$
$i_0 = 0, * = 3$

	10	*1*	*8*	
0	**7**	*4*	*11*	$\otimes$
11	**2**	**9**	*6*	$1'$
	5	*0*	*3*	

$(4, 0, 6, 10, 1, 11, 8, 3, 1')$
$i_0 = 1$

(16-1)

	9	**0**	**11**	
11	*4*	**7**	**2**	**9**
8′	*1*	*10*	**5**	*0*
$\otimes$	*6*	*3*	*8*	

$(0, 10, 4, 9, 1, 3, 8, 6, 8'11)$
$i_0 = 0$

$\otimes$	6(2)			
2(6)	*9*	**0**	**11**	
11	*4*	**7**	**2**	**9**
	1	*10*	**5**	0

$(0, 10, 4, 9, 1, 11, 2(6), 6(2))$
$i_0 = 1$

(16-2)

0	**7**	*10*
11	**2**	**5**
$\ast$	**9**	*0*

$(10, 0)$
$i_0 = 0$

	0	**7**	10
$\otimes$	**11**	**2**	**5**
7	4	**9**	0
	1	*6*	

$(6, 1, 7)$
$i_0 = 0, \ast = 4$

	5	**0**	**7**	10
$\otimes$	*8*	**11**	**2**	**5**
0	*3*	6	**9**	0
	10	*1*	*4*	

$(4, 1, 3, 8, 5, 10, 0)$
$i_0 = 0, \ast = 6$

	4	*1*	*6*	
0	**7**	*10*	*3*	*0′*
11	**2**	**5**	*8*	$\otimes$
	9	*0*	*11*	

$(10, 4, 1, 3, 0, 8, 6, 0', 11)$
$i_0 = 1$

(17-1)

0	**11**	*8*
9	**2**	**5**
	7	$\ast$

(8)

0	**11**	8	*1*	
9	**2**	**5**	*10*	*7*
	7	0	*3*	$\otimes$

$(10, 3, 1, 7)$
$\ast = 0$

0	**11**	8	*3*
9	**2**	**5**	*0*
	7	10	$\otimes$

$(0, 3)$
$\ast = 10$

(17-2)

0	**9**	
11	**2**	**7**
8	**5**	$\ast$

(8)
$i_0 = 0$

	0	**9**	4
6	**11**	**2**	**7**
3′	8	**5**	0
$\otimes$	*1*	*10*	*3*

$(10, 3, 1, 3', 6)$
$i_0 = 0, \ast = 0$

0	**9**	
11	**2**	**7**
8	**5**	10
3	*0*	$\otimes$

$(0, 3)$
$i_0 = 0, \ast = 10$

	6	$\ast$
0	**9**	4
11	**2**	**7**
	5	

$(4, 6)$
$i_0 = 1$

	6	1	*8*	
0	**9**	4	*11*	#
11	**2**	**7**	*0*	
	5	*10*	*3*	

$(11, 8, 0, 10, 3)$
$i_0 = 1, \ast = 1$

	6	1	8	**5**	
0	**9**	4	**11**	**2**	**7**
11	**2**	**7**	**0**	**9**	
	5	10	3		

$(5, 9, 7)$
$i_0 = 1, \ast = 1, \# = 2$

	6	1	8	*3*
0	**9**	4	11	6
11	**2**	**7**	0	*9*
	5	10	3	$\otimes$

$(3, 9)$
$i_0 = 1, \ast = 1, \# = 6$

	6	11	8	
0	**9**	4	1	6
11	**2**	**7**	10	$\otimes$
	5	0	3	

$(1, 8, 10, 0, 3, 6)$
$i_0 = 1, \ast = 11$

(18-1)

0	**11**	
9	**2**	**7**
$\otimes$	**5**	

0	**9**	$\otimes$
11	**2**	**5**
	7	

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