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QUANTUM MODULARITY OF MOCK THETA FUNCTIONS OF ORDER 2

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Abstract. In [9], we computed shadows of the second order mock theta functions and showed that they are essentially same with the shadow of a mock theta function related to the Mathieu moonshine phenomenon. In this paper, we further survey the second order mock theta functions on their quantum modularity and their behavior in the lower half plane.

1. Introduction

In his last letter to Hardy, Ramanujan introduced the notion of a mock theta function and offered 17 q-hypergemetric series as examples. More examples of mock theta functions were found in Ramanujan's lost notebook. They attracted many mathematicians' interests but remained mystery until Zwegers [18] showed that a mock theta function is the holomorphic part of a harmonic weak Maass form. This discovery led to many applications of mock theta functions in number theory and beyond, even in quantum physics. More precisely, a mock theta function f can be completed to essentially a harmonic weak Mass form of weight $1/2$ by adding a period integral of a certain weight $3/2$ unary theta series, say g. This period integral is defined in the lower half plane and related to a

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partial theta function or a linear combination of partial theta functions. In addition, the weight $3/2$ unary theta series g is a holomorphic modular form and is called the shadow of the mock theta function f. A mock theta function and its associated partial theta function often coincide at a rational point and this phenomenon resulted in a notion of a quatum modular form by Zagier [17].

In [9], the author with Swisher gave direct computation of the shadows of mock theta functions of order 2. In this paper, we further survey the second order mock theta functions on their quantum modularity and their behavior in the lower half plane from the q-hypergeometric perspective. In Section 2, we first introduce the three well known mock theta functions of order 2 and give a brief presentation of partial theta functions and quantum modular forms associated to mock theta functions. In Section 3, we introduce universal mock theta functions which are direct two variable generalizations of the three 2nd order mock theta functions and Zwegers' universal mock theta function μ. In Section 4, we find an explicit form of partial theta function defined in the lower half plane corresponding to each of the three 2nd order mock theta functions. We end the section with a brief discussion on the quantum modularity of the three second order mock theta functions.

2. Mock Theta Functions of Order 2

Throughout, we use the notation $\mathbb{H}^+ := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ to denote the complex upper-half plane, $q := e^{2\pi i\tau}$, and $(a)_0 := (a;q)_0 = 1$, $(a)_n := (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k)$, $(a)_\infty := (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k)$, to denote the finite and infinite q-Pochhammer symbols. In the infinite case, we assume $|q| < 1$.

In [9], we discussed the computation of shadows of the three well-known second order mock theta functions:

$$\begin{aligned}
A(q) &:= \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}^2},\\
B(q) &:= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(q;q^2)_{n+1}^2}, \qquad (2.1)\\
K(q) &:= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q^2;q^2)_n^2}.
\end{aligned}$$

The functions in (2.1) appear in [1, 6, 11, 15]. See [9] for detailed history of these functions. The last function $K(q)$ is usually denoted by $\mu(q)$ as in [9], but we use different notation not to be confused with Zwegers's μ function that will be introduced later.

In [9], we computed the shadows of the second order mock theta functions in (2.1) and showed that each function, up to multiplication by a rational power of q, has shadow related to $\eta^3(\tau)$, where $\eta(\tau)$ is the Dedekind eta-function defined on $\mathbb{H}^+$ by

$$q^{-1/24}\eta(\tau) = \prod_{n=1}^{\infty}(1-q^n). \tag{2.2}$$

For example, by [9, (4.2)] and [18, Theorem 1.16 (1)],

$$K(q) - iq^{1/8}\int_{-\bar{\tau}}^{i\infty} \frac{\eta^3(z)}{\sqrt{-i(z+\tau)}}\,dz \tag{2.3}$$

is a harmonic weak Maass form of weight 1/2. By [9, (4.5)] and [18, Theorem 1.16 (1)] again,

$$B(q) + iq^{-1/2}\int_{-\bar{\tau}}^{i\infty} \frac{\eta^3(4z)}{\sqrt{-i(z+\tau)}}\,dz \tag{2.4}$$

is also a harmonic weak Maass form of weight 1/2.

It follows from (2.3) and (2.4) that $K(q^4)+2qB(q)$ is a harmonic weak Maass form which is holomorphic, and thus it is a weakly holomorphic modular form of weight 1/2. In fact, Gordon and McIntosh established three mock theta conjectures of order 2 [6, eq. (5.2)] and one of them can be written as

$$K(q^4) + 2qB(q) = \frac{(q^2;q^2)_\infty (q^4;q^4)_\infty^3 (q^8;q^8)_\infty}{(q)_\infty^2 (q^{16};q^{16})_\infty^2}. \tag{2.5}$$

As we expected, the right hand side is an eta quotient which is essentially a weakly holomorphic modular form of weight 1/2. Speaking of the other two mock theta conjectures of the 2nd order mock theta functions, we gave very simple proofs in [9] and they are given by

$$K(q) + 4A(-q) = \frac{(q)_\infty^5}{(q^2;q^2)_\infty^4} = (q;q)_\infty (q;q^2)_\infty^4 \tag{2.6}$$

and

$$\frac{B(q)+B(-q)}{2} = \frac{(q^4;q^4)_\infty^5}{(q^2;q^2)_\infty^4} = (q^4;q^4)_\infty (-q^2;q^2)_\infty^4. \tag{2.7}$$

The three mock theta conjectures above along with (2.3) or (2.4) imply that $B(q)$, $B(-q)$, $A(-q^4)$ and $K(q^4)$ have the same shadows $h^3(4\tau)$ up to a constant multiple of rational power of q.

Following [10], for any weight $2-k$ ($k \in \frac{1}{2}\mathbb{Z}$) cusp form $g(\tau) := \sum_{n\geq 1} a_g(n)q^n$, we define the former Eichler integral

$$\widetilde{g}(\tau) := \sum_{n\geq 1} a_g(n)n^{k-1}q^n. \tag{2.8}$$

The period integral of $g(\tau)$ is also given in [10] by

$$g^*(\tau) = (i/2)^{k-1}\int_{-\bar{\tau}}^{i\infty} \frac{g(z)}{(z+\tau)^k}\,dz = \sum_{n>0} n^{k-1}a_g(n)\Gamma(k,4ny)q^{-n} \tag{2.9}$$

where $\Gamma(a,x)$ is the incomplete gamma function $\int_x^\infty u^{-k}e^{-\pi u}\,du$. $g^*(\tau)$ is nearly modular of weight k in the lower half plane. It has been shown in [7, 10] that when g is a theta function, $\widetilde{g}(\tau)$ and $g^*(\tau)$ agree to infinite order at rational points.

Note that

$$\eta^3(\tau) = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{(2n+1)^2}{8}} = \sum_{n=-\infty}^{\infty}(4n+1)q^{\frac{(4n+1)^2}{8}} = \sum_{n=1}^{\infty}\left(\frac{-4}{n}\right)nq^{\frac{n^2}{8}}. \tag{2.10}$$

Hence by (2.8) the formal Eichler integral of $\eta^3(\tau)$ is

$$\widetilde{\eta}^3(\tau) = q^{-\frac{1}{8}}\sum_{n=0}^{\infty}(-1)^n q^{-\frac{n(n+1)}{2}} = \sum_{n=1}^{\infty}\left(\frac{-4}{n}\right)q^{-\frac{n^2}{8}}, \tag{2.11}$$

which agrees the period integral of $\eta^3(\tau)$ appearing in (2.3) to infinite order at rational points.

This was a motivation for Zagier [17] to define a quantum modular form. A quantum modular form is a complex function defined on an appropriate subset of the rational numbers, which transforms like a modular form up to the addition of an error function that is suitably continuous or analytic in $\mathbb{R}$. In [4, Proposition 1.4], it is shown that $\widetilde{\eta}^3(\tau)$ is a quantum modular form of weight 1/2 and also in [4, Theorem 1.2], it is proved that $q^{1/8}A(q)$ is a quantum modular form of weight 1/2. Under the notation in [4], $\widetilde{\eta}^3(\tau)$ and $q^{1/8}A(q)$ equal $\widetilde{E_1(\tau/8)}$ and $V_{21}(4\tau)$, respectively.

3. Universal Mock Theta Functions

We consider the following q-hypergeometric series:

$$g_2(w;q) := \sum_{n=0}^{\infty} \frac{(-q)_n q^{n(n+1)/2}}{(w;q)_{n+1}(q/w;q)_{n+1}}, \tag{3.1}$$

$$K(w;q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(wq^2;q^2)_n(q^2/w;q^2)_n}, \tag{3.2}$$

$$K_1(w;q) := \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(wq;q^2)_n(q/w;q^2)_n}, \tag{3.3}$$

which have been substantially studied in [6, 12]. See also [8, 15]. As all of Ramanujan's original mock theta functions can be written in terms of $g_2(w;q)$ and $K(w;q)$ and $K_1(w;q)$ are related by modular transformation to $g_2(w;q)$, we may call all of them universal mock theta functions.

These are direct generalizations of the 2nd order mock theta functions in (2.1):

$$K(q) = K(-1;q), \; A(-q) = K_1(-1,q), \text{ and } B(q) = g_2(q,q^2). \tag{3.4}$$

However, if we do not restrict ourselves to use an Eulerian form, there is a genuine universal mock theta function discovered by Zwegers [18]. We begin with a more general function. For $\tau \in \mathbb{H}$, $u, v \in \mathbb{C}\backslash(\mathbb{Z}\tau + \mathbb{Z})$, and positive integer ℓ, Zwegers [19] defined the level ℓ Appell function A_ℓ by

$$A_\ell(u,v;\tau) := w^{\ell/2} \sum_{n=-\infty}^{\infty} \frac{(-1)^{\ell n} q^{\ell n(n+1)/2} z^n}{1 - wq^n}, \tag{3.5}$$

where $w = e^{2\pi i u}$, $z = e^{2\pi i v}$, $q = e^{2\pi i \tau}$. By adding a suitable non-holomorphic correction term, Zwegers showed that A_ℓ can be completed to form essentially a real analytic Jacobi form, $\widehat{A_\ell}$. The normalized level 1 Appell function is the genuine universal mock theta function $\mu(u,v;\tau)$ of Zwegers, which is defined by

$$\begin{aligned} \mu(u,v) &:= \mu(u,v;\tau) := \vartheta(v;\tau)^{-1} A_1(u,v;\tau) \\ &:= \frac{w^{1/2}}{\vartheta(v;\tau)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2} z^n}{1 - wq^n}, \end{aligned} \tag{3.6}$$

where

$$\vartheta(v;\tau) := \sum_{\nu\in\mathbb{Z}+\frac{1}{2}} e^{\pi i\nu} z^{\nu} q^{\frac{\nu^2}{2}} = -iq^{\frac{1}{8}} z^{-\frac{1}{2}} (q)_\infty (z)_\infty (q/z)_\infty \tag{3.7}$$

is the Jacobi theta series. As noted in [9, Eq.(4.1) and (4.4)], the second order mock theta functions are known to be level 2 Apell functions:

$$K(q) = \frac{-4q^{1/8}}{\vartheta(-\frac{1}{2};\tau)} A_2\left(-\frac{1}{2}, -\tau; 2\tau\right) \text{ and } B(q) = \frac{-iq^{-1/2}}{\vartheta(2\tau;4\tau)} A_2\left(\tau, \frac{1}{2}; 2\tau\right). \tag{3.8}$$

There are two ways to reduce higher level Appell functions into linear combinations of level 1 Appell functions in [19], but one does not able to reduce the Appell functions above into Zwegers' μ-functions due to the occurrence of poles. Nevertheless, we here show that each of the three 2nd order mock theta functions is a single Zwegers' μ-function up to holomorphic modular forms. These representations of the second order mock theta functions would greatly help us to understand their modularity in the whole complex plane.

Proposition 3.1. *For* $|q| < 1$,

$$A(q) = -i\mu(\tau, 2\tau; 4\tau), \tag{3.9}$$

$$B(q) = -iq^{-1/2}\mu(\tau, \tau; 4\tau), \tag{3.10}$$

$$K(q) = -4\mu(\tau + \frac{1}{2}, 2\tau; 4\tau) + (q;q^2)_\infty^5 (q^2;q^2)_\infty. \tag{3.11}$$

Proof. By [4, Remark on p.15 and Table E2], $A(q) = -q^{1/8}V_{21}(4\tau) = -i\mu(\tau, 2\tau; 4\tau)$, which proves (3.9). Next, in [8, Theorem 1.1], it is shown that

$$g_2(w,q) = \frac{\eta(2\tau)^4}{iw\eta(\tau)^2\vartheta(2u;2\tau)} - iq^{-1/4}\mu(2u,\tau;2\tau).$$

McIntosh [13] showed that $\mu(u,\tau-u;2\tau) - \mu(2u,\tau;2\tau) = \dfrac{\eta(2\tau)^4}{w\eta(\tau)^2\vartheta(2u;2\tau)}$. Hence $g_2(w,q) = -iq^{-1/4}\mu(u,\tau-u;2\tau)$, and $B(q) = g(q,q^2)$ implies (3.10). Lastly, it follows from (2.6) and (3.9) that

$$\begin{aligned} K(q) &= -4A(-q) + (q;q^2)_\infty^5 (q^2;q^2)_\infty \\ &= -4\mu(\tau - \frac{1}{2}, 2\tau - 1; 4\tau - 2) + (q;q^2)_\infty^5 (q^2;q^2)_\infty. \end{aligned}$$

By applying transformation formula $\mu(u,v;\tau+1) = e^{-\frac{\pi i}{4}}\mu(u,v;\tau)$ ([18, Proposition 1.5 (1)]) twice, we obtain

$$K(q) = -4\mu(\tau - \frac{1}{2}, 2\tau - 1; 4\tau) + (q;q^2)_\infty^5 (q^2;q^2)_\infty.$$

Since $\mu(u+1,v) = -\mu(u,v) = \mu(u,v+1)$ ([18, Proposition 1.4]), we have (3.11). $\square$

Zwegers [18] constructed a non-holomorphic function $R(u;\tau)$ for $u \in \mathbb{C}$ so that

$$\widehat{\mu}(u,v;\tau) := \mu(u,v;\tau) + \frac{i}{2}R(u-v;\tau)$$

is a harmonic weak Maass form when two elliptic variables u and v are restricted to torsion points $\mathbb{Q}\tau + \mathbb{Q}$. In fact, for $a, b \in \mathbb{R}$, $R(a\tau - b;\tau)$ is the period integral of a unary theta series as given in (2.9). More precisely, for a, $b \in \mathbb{R}$, if we define a unary theta series of weight 3/2 by

$$g_{a,b}(\tau) := \sum_{\nu \in a+\mathbb{Z}} \nu e^{\pi i \nu^2 \tau + 2\pi i \nu b}, \tag{3.12}$$

then $-\sqrt{2}e^{2\pi i a(b+\frac{1}{2})}q^{-a^2/2}\mu(u,v;\tau)$ with $u - v = a\tau - b$ has the correction term [18, Theorem 1.11]

$$-\frac{i}{\sqrt{2}}e^{2\pi i a(b+\frac{1}{2})}q^{-a^2/2}R(a\tau - b;\tau). \tag{3.13}$$

Moreover, for $a \in (-\frac{1}{2}, \frac{1}{2})$ and $b \in \mathbb{R}$ [18, Theorem 1.16],

$$-e^{2\pi i a(b+\frac{1}{2})}q^{-a^2/2}R(a\tau - b;\tau) = \int_{-\bar{\tau}}^{i\infty} \frac{g_{a+\frac{1}{2},b+\frac{1}{2}}(z)}{\sqrt{-i(z+\tau)}}\,dz \tag{3.14}$$

and we call $\overline{g_{a+\frac{1}{2},b+\frac{1}{2}}(-\overline{\tau})}$ the shadow of $-\sqrt{2}e^{2\pi i a(b+\frac{1}{2})}q^{-a^2/2}\mu(u,v;\tau)$. For a full description of the notions of harmonic Maass forms, mock modular forms, and shadows, the reader is referred to [14, 16].

4. The second order mock theta functions in the lower half plane

For $|q|<1$, let

$$
\begin{aligned}
S_1(q) &:= \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}},\\
S_2(q) &:= \sum_{n=0}^{\infty} \left(\frac{-12}{n}\right) q^{\frac{n^2-1}{24}} = \sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}),\\
S_3(q) &:= \sum_{n=0}^{\infty} \left(\frac{n}{3}\right) q^{\frac{n^2-1}{3}} = \sum_{n=0}^{\infty} q^{n(3n+2)}(1-q^{2n+1}).
\end{aligned} \tag{4.1}
$$

S_1 is essentially a half of $0=\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n(n+1)}{2}}$ and S_2 is a wrong linear combination of partial theta functions in the sense that $q^{-1/24}\eta(\tau) = (q;q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2}$ by Euler's pentagonal number theorem. S_3 also has a wrong character to be a theta function. This is why series like in (4.1) are often called partial theta functions or false theta functions. Note that S_1 is the partial theta function corresponding to Eichler integral of $\eta^3(\tau)$ in (2.11), which is defined in the lower half plane.

The three 2nd order mock theta functions are defined in the lower half plane as well. From (2.1), we obtain for $|q|<1$ that

$$
\begin{aligned}
A^-(q) &:= A(q^{-1}) = \sum_{n=0}^{\infty} \frac{q^{2n+1}(-q;q^2)_n}{(q;q^2)_{n+1}^2},\\
B^-(q) &:= B(q^{-1}) = \sum_{n=0}^{\infty} \frac{q^{2n+2}(-q^2;q^2)_n}{(q;q^2)_{n+1}^2},\\
K^-(q) &:= K(q^{-1}) = \sum_{n=0}^{\infty} \frac{q^{2n}(q;q^2)_n}{(-q^2;q^2)_n^2}.
\end{aligned} \tag{4.2}
$$

Representations of the second order mock theta functions in (4.2), however do not help us much to understand their behavior in the lower half plane. In [2], the universal mock Jacobi forms in (3.1), (3.2) and (3.3) are discussed both in the upper and lower half planes.

THEOREM 4.1. *For* $|q| < 1$,

$$(4.3)\qquad \begin{aligned} A^-(q) &= \frac{1}{2}S_1(-q) + \frac{(-q;q^2)_\infty}{(q;q^2)_\infty^2}S_2(q^2). \\ B^-(q) &= -q^{-1}S_1(q^4) - \frac{q(1-q)(-q^2;q^2)_\infty}{(q;q^2)_\infty^2}S_3(q). \\ K^-(q) &= 2S_1(q) - \frac{(q;q^2)_\infty}{(-q^2;q^2)_\infty^2}S_3(q). \end{aligned}$$

Proof. From [2, Eqs. (5.3) and (3.1), Remark in Theorem 3.1], we find that

$$A^-(-q) = K_1(-1,q^{-1}) = \frac{1}{2}S_1(q) + \frac{(q;q^2)_\infty}{(-q;q^2)_\infty^2}S_2(q^2),$$

and this proves the first identity. Also by using [2, Eqs. (5.4) and (3.1)] and $K(q) = K(-1;q)$ in (3.4), we can easily deduce the third identity. For the second identity, note that $B(q) = g_2(q;q^2)$ where $g_2(w;q)$ is defined in (3.4). Hence we need to compute $B^-(q) = g_2(q^{-1};q^{-2})$. It follows from [2, Theorem 4.2 and Eqs. (4.2) and (4.3)] that for $|q| < 1$ and $w \neq q^{2\ell}$ ($\ell \notin \mathbb{Z}$),

(4.4)

$$\begin{aligned} g_2(w;q^{-2}) &= \sum_{n=1}^{\infty}(-1)^n w^{2n-1}q^{2n^2} \\ &\quad + \frac{(-q^2;q^2)_\infty}{(wq^2;q^2)_\infty(w^{-1}q^2;q^2)\infty}\sum_{n=1}^{\infty}w^{3n-2}q^{3n^2-n}(1-wq^{2n}). \end{aligned}$$

Substituting $w = q^{-1}$, we have

$$g_2(q^{-1};q^{-2}) = -qS_1(q^4) + \frac{(1-q)(-q^2;q^2)_\infty}{(q;q^2)_\infty^2}\sum_{n=1}^{\infty}q^{3n^2-4n+2}(1-q^{2n-1}),$$

from which we can easily derive the second identity in the theorem. $\square$

Any property a function holds in the upper half plane does not necessarily hold in the lower half plane although it is well defined in both planes. But by comparing (2.5) and (2.6) with (4.3), we see that the three 2nd order mock theta functions carry some properties from the upper half plane to lower. Also, they are essentially the same partial theta functions, the Eichler integrals of $\eta^3(\tau)$ up to products of theta quotients and partial theta functions in the lower half plane, because the three mock theta functions essentially have the same shadow $\eta^3(\tau)$.

The Eichler integral of $\eta^3(\tau)$ or partial theta function $S_1(q)$ is also defined on a subset of rational numbers. What happens is as follows: there is the mock theta function $q^{1/8}K(q)$ in the upper half plane and partial (or false) theta function $q^{1/8}S_1(q)$ in the lower half plane corresponding to the correction term of the mock theta function, which is the Eichler integral of $\eta^3(\tau)$. Moreover, the two are connected through rational points via $x^{1/8}K(x) = x^{1/8}S_1(x)$ where x is in some subset of $\mathbb{Q}$, because at certain sets of rational points, the second term in the representation of $K^-(q)$ in (4.3) vanishes. In [5, Theorem 1.3], it is proved that $q^{1/8}S_1(q)$ is a quantum modular form of weight 1/2 and hence $q^{1/8}K(q)$ is also a quantum modular form. Similarly, $A(q)$ and $B(q)$ are also quantum modular forms up to constant multiples of rational power of q.

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