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GRADED INTEGRAL DOMAINS AND NAGATA RINGS, II

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Abstract. Let D be an integral domain with quotient field K, X be an indeterminate over D, $K[X]$ be the polynomial ring over K, and $R = \{f \in K[X] \mid f(0) \in D\}$; so R is a subring of $K[X]$ containing $D[X]$. For $f = a_0 + a_1X + \cdots + a_nX^n \in R$, let $C(f)$ be the ideal of R generated by $a_0, a_1X, \dots, a_nX^n$ and $N(H) = \{g \in R \mid C(g)_v = R\}$. In this paper, we study two rings $R_{N(H)}$ and $\mathrm{Kr}(R, v) = \{\frac{f}{g} \mid f, g \in R,\ g \neq 0, \text{ and } C(f) \subseteq C(g)_v\}$. We then use these two rings to give some examples which show that the results of [4] are the best generalizations of Nagata rings and Kronecker function rings to graded integral domains.

1. Introduction

Let D be an integral domain with quotient field K, X be an indeterminate over D, $D[X]$ be the polynomial ring over D, and A_f be the fractional ideal of D generated by the coefficients of a polynomial $f \in K[X]$. There are three types of interesting overrings of $D[X]$ two of which are

$$\begin{aligned} D(X) &= \{\frac{f}{g} \mid f, g \in D[X], g \neq 0, \text{ and } A_g = D\} \\ D[X]_{N_v} &= \{\frac{f}{g} \mid f, g \in D[X], g \neq 0, \text{ and } (A_g)_v = D\}. \end{aligned}$$

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Clearly, $D[X] \subseteq D(X) \subseteq D[X]_{N_v} \subseteq K(X)$, and we say that $D(X)$ (resp., $D[X]_{N_v}$) is the Nagata ring (resp., (t-)Nagata ring) of D. (Definitions will be reviewed in the sequel.) For the third one, assume that D is integrally closed, and let $*$ be an *e.a.b.* star operation on D and

$$Kr(D,*) = \{\frac{f}{g} \mid f, g \in D[X], g \neq 0, \text{ and } A_f \subseteq (A_g)_*\}.$$

Then $Kr(D,*)$, called the *Kronecker function ring of* D (with respect to $*$), is a Bezout domain such that $Kr(D,*) \cap K = D$ and $D(X) \subseteq Kr(D,*) \subseteq K(X)$ [12, Theorem 32.7].

Nagata ring has many interesting ring-theoretic properties. For example, every invertible ideal of $D(X)$ is principal, i.e., $\mathrm{Pic}(D(X)) = \{0\}$ [1, Theorem 2]; $\mathrm{Max}(D(X)) = \{M(X) \mid M \in \mathrm{Max}(D)\}$ [12, Proposition 33.1]; and if b is the b-operation on an integrally closed domain D, then D is a Prüfer domain if and only if $D(X)$ is a Prüfer domain, if and only if $D(X) = \mathrm{Kr}(D, b)$ [5, Theorem 4], if and only if $D(X)$ is a Bezout domain [12, Theorem 33.4]. These results were generalized to t-Nagata rings via the t-operation as follows: $\mathrm{Pic}(D[X]_{N_v}) = Cl(D[X]_{N_v}) = \{0\}$; $\mathrm{Max}(D[X]_{N_v}) = \{P[X]_{N_v} \mid P \in t\text{-}\mathrm{Max}(D)\}$; D is a Prüfer v-multiplication domain (PvMD) if and only if $D[X]_{N_v}$ is a Prüfer domain, if and only if $D[X]_{N_v}$ is a Bezout domain, if and only if each ideal of $D[X]_{N_v}$ is extended from D [13]; and if D is a v-domain (i.e., the v-operation on D is an *e.a.b.* star operation), then D is a PvMD if and only if $D[X]_{N_v} = \mathrm{Kr}(D, v)$ [11, Theorem 2.5]. For more on Nagata rings and Kronecker function rings, the reader can refer to [12, Sections 32-34] or Fontana-Loper's interesting survey article [10].

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be an integral domain graded by an arbitrary torsionless grading monoid Γ, H be the set of nonzero homogeneous elements of R, $S(H) = \{f \in R \mid C(f) = R\}$, and $N(H) = \{f \in R \mid C(f)_v = R\}$. It is clear that if $R = D[X, X^{-1}]$, then $R_{S(H)} = D(X)$ and $R_{N(H)} = D[X]_{N_v}$ [4, Proposition 3.1]. Motivated by these two facts, in [4], the authors generalized the notions of (t-)Nagata rings and Kronecker function rings to graded integral domains. They proved several properties of graded integral domain analogs of (t-)Nagata rings and Kronecker function rings under assumptions that R satisfies property (#) or R has a unit of nonzero degree (see Section 2 for definition and results). In this paper, we give some examples which show that the two assumptions are best for graded integral domain analogs of (t-)Nagata rings and Kronecker function rings. More precisely, in Section

2, we review some known results on (t-)Nagata rings and Kronecker function rings of graded integral domains. Assume that $D \neq K$, and let $R = D + XK[X]$, i.e., $R = \{f \in K[X] \mid f(0) \in D\}$; so R is an $\mathbb{N}_0$-graded integral domain with $\deg(aX^n) = n$ for $0 \neq a \in K$ and $n \geq 0$ ($a \in D$ when $n = 0$), where $\mathbb{N}_0$ is the additive monoid of nonnegative integers. In Section 3, we use the ring $R = D + XK[X]$ to construct concrete examples why the results of [4] are the best generalizations of (t-)Nagata rings and Kronecker function rings to graded integral domains.

Definitions related to star operations and graded integral domains.
To facilitate the reading of this paper, we review some definitions on star operations and graded integral domains. Let $\mathbf{F}(D)$ be the set of nonzero fractional ideals of D. A map $* : \mathbf{F}(D) \to \mathbf{F}(D)$, $I \mapsto I_*$, is called a *star operation* on D if the following three conditions are satisfied for all $0 \neq a \in K$ and $I, J \in \mathbf{F}(D)$: (i) $(aD)_* = aD$ and $(aI)_* = aI_*$, (ii) $I \subseteq I_*$ and if $I \subseteq J$, then $I_* \subseteq J_*$, and (iii) $(I_*)_* = I_*$. Given a star operation $*$ on D, one can construct a new star operation $*_f$ by setting $I_{*_f} = \bigcup\{J_* \mid J \in \mathbf{F}(D)$ is finitely generated and $J \subseteq I\}$ for all $I \in \mathbf{F}(D)$. Clearly, $(*_f)_f = *_f$ and $I_* = I_{*_f}$ for all finitely generated $I \in \mathbf{F}(D)$. Examples of the most well-known star operations include the v-, t-, and d-operations. The v-operation is defined by $I_v = (I^{-1})^{-1}$, where $I^{-1} = \{x \in K \mid xI \subseteq D\}$, the t-operation is defined by $t = v_f$, and the d-operation is just the identity function on $\mathbf{F}(D)$, i.e., $I_d = I$ for all $I \in \mathbf{F}(D)$; so $d_f = d$.

An $I \in \mathbf{F}(D)$ is called a *$*$-ideal* if $I_* = I$. A $*$-ideal is called a *maximal $*$-ideal* if it is maximal among proper integral $*$-ideals. Let $*$-Max(D) be the set of maximal $*$-ideals of D. It may happen that $*$-Max$(D) = \emptyset$ even though D is not a field (e.g., v-Max$(D) = \emptyset$ if D is a rank-one nondiscrete valuation domain). However, it is well known that $*_f$-Max$(D) \neq \emptyset$ if D is not a field; each maximal $*_f$-ideal is a prime ideal; each proper integral $*_f$-ideal is contained in a maximal $*_f$-ideal; and each prime ideal minimal over a $*_f$-ideal is a $*_f$-ideal. An $I \in \mathbf{F}(D)$ is said to be *$*$-invertible* if $(II^{-1})_* = D$, and D is a *Prüfer $*$-multiplication domain* (P$*$MD) if each nonzero finitely generated ideal of D is $*_f$-invertible. Let $T(D)$ (resp., Inv(D), $Prin(D)$) be the group of t-invertible fractional t-ideals (resp., invertible fractional ideals, nonzero principal fractional ideals) of D under the t-multiplication $I * J = (IJ)_t$. It is obvious that $Prin(D) \subseteq Inv(D) \subseteq T(D)$. The *$t$-class group* of D

is the abelian group $Cl(D) = T(D)/Prin(D)$ and the *Picard group* (or *ideal class group*) of D is the subgroup $Pic(D) = Inv(D)/Prin(D)$ of $Cl(D)$. It is clear that if each maximal ideal of D is a t-ideal (e.g., D is one-dimensional or a Prüfer domain), then $Pic(D) = Cl(D)$. Also, if D is a Krull domain, then $Cl(D)$ is the usual divisor class group of D.

Let Γ be a nonzero torsionless grading monoid, that is, Γ is a nonzero torsionless commutative cancellative monoid (written additively). It is well known that a cancellative monoid Γ is torsionless if and only if Γ can be given a total order compatible with the monoid operation [15, page 123]. By a Γ-graded integral domain $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$, we mean an integral domain graded by Γ. That is, each nonzero $x \in R_\alpha$ has degree α, i.e., $\deg(x) = \alpha$, and $\deg(0) = 0$. Thus, each nonzero $f \in R$ can be written uniquely as $f = x_{\alpha_1} + \cdots + x_{\alpha_n}$ with $\deg(x_{\alpha_i}) = \alpha_i$ and $\alpha_1 < \cdots < \alpha_n$. An $x \in R_\alpha$ for every $\alpha \in \Gamma$ is said to be *homogeneous*. Let H be the saturated multiplicative set of nonzero homogeneous elements of R. i.e., $H = \bigcup_{\alpha\in\Gamma} R_\alpha \setminus \{0\}$. Then R_H, called the homogeneous quotient field of R, is a graded integral domain whose nonzero homogeneous elements are units. It is known that R_H is a completely integrally closed GCD-domain [2, Proposition 2.1]. For an ideal I of R, let I^* be the ideal of R generated by the homogeneous elements in I. We say that I is *homogeneous* if $I^* = I$ and a homogeneous ideal is a *maximal homogeneous ideal* if it is maximal among proper homogeneous ideals of R. Let h-Max(R) be the set of maximal homogeneous ideals of R. It is easy to see that each ideal in h-Max(R) is a prime ideal and each proper homogeneous ideal of R is contained in at least one maximal homogeneous ideal of R. For $f \in R_H$, let $C(f)$ denote the fractional ideal of R generated by the homogeneous components of f. For an ideal I of R, let $C(I) = \sum_{f\in I} C(f)$. Clearly, $C(f)$ and $C(I)$ are homogeneous ideals of R.

2. Review on the rings $R_{N(H)}$ and $\mathbf{Kr}(R, *)$

Let Γ be a nonzero torsionless grading monoid, $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a Γ-graded integral domain with $R_\alpha \neq \{0\}$ for all $\alpha \in \Gamma$, H be the set of nonzero homogeneous elements of R, $S(H) = \{f \in R \mid C(f) = R\}$, and $N(H) = \{0 \neq f \in R \mid C(f)_v = R\}$. Note that if $R = D[X, X^{-1}]$, then R has a unit of nonzero degree, $R_{S(H)} = D(X)$, and $R_{N(H)} = D[X]_{N_v}$ where $N_v = \{f \in D[X] \mid (A_f)_v = D\}$. Hence, all results in this section are true for the rings $D(X)$ and $D[X]_{N_v}$.

Proposition 1. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain and Ω be the set of maximal t-ideals Q of R with $Q \cap H \neq \emptyset$.*

1. *Every prime ideal in Ω is homogeneous.*
2. *$N(H) = R \setminus \bigcup_{Q\in\Omega} Q$.*
3. *$Max(R_{N(H)}) = \{Q_{N(H)} \mid Q \in \Omega\}$ if and only if R has property (#) that if I is a nonzero ideal of R with $C(I)_t = R$, then $I \cap N(H) \neq \emptyset$.*

Proof. (1) [3, Lemma 1.2]. (2) and (3) [4, Proposition 1.4]. □

We say that $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ is a *graded-Prüfer v-multiplication domain* (graded PvMD) (resp., *graded-Prüfer domain, graded-Bezout domain*) if every nonzero finitely generated homogeneous ideal of R is t-invertible (resp., invertible, principal). Hence, graded-Bezout domain ⇒ graded-Prüfer domain ⇒ graded PvMD ⇔ PvMD [2, Theorem 6.4], while a graded-Prüfer domain need not be Prüfer [4, Example 3.6].

Theorem 2. [4, Corollaries 1.10 and 1.11] *If R satisfies property (#), then R is a PvMD if and only if $R_{N(H)}$ is a Prüfer domain. In this case, each ideal of $R_{N(H)}$ is extended from a homogeneous ideal of R.*

It is easy to see that $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ satisfies property (#) if R is one of the following rings: (i) R has a unit of nonzero degree, (ii) $R = D[\Gamma]$ is the monoid domain of Γ over an integral doman D, or (iii) $R = D[\{X_\alpha\}]$ is the polynomial ring [4, Example 1.6].

Theorem 3. [4, Theorem 3.3] *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree. Then $Cl(R_{N(H)}) = Pic(R_{N(H)}) = \{0\}$.*

Theorem 4. [4, Theorem 3.4] *The following statements are equivalent for a graded integral domain $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ with a unit of nonzero degree.*

1. *R is a PvMD.*
2. *Every ideal of $R_{N(H)}$ is extended from a homogeneous ideal of R.*
3. *$R_{N(H)}$ is a Prüfer domain.*
4. *$R_{N(H)}$ is a Bezout domain.*

Let D be an integral domain with quotient field K. A star operation $*$ on D is said to be an *endlish arithmetisch brauchbar* (*e.a.b.*) star operation if $(AB)_* \subseteq (AC)_*$ implies $B_* \subseteq C_*$ for all nonzero finitely generated $A, B, C \in \mathbf{F}(D)$. Clearly, $*$ is an *e.a.b.* star operation if and only if $*_f$ is an *e.a.b.* star operation. We know that if D admits an *e.a.b.* star operation, then D is integrally closed [12, Corollary 32.8]. Conversely, suppose that D is integrally closed, and for a star operation $*$ on D, define

$$I^{*_c} = \bigcap\{IV \mid V \text{ is a } *\text{-linked valuation overring of } D\},$$

then $*_c$ is an *e.a.b.* star operation on D [6, Lemma 3.1]. (A subring T of K containing D is said to be $*$-*linked* over D if $I_* = D$ for a finitely generated $I \in \mathbf{F}(D)$ implies $(IT)_v = T$.) As in [12, Theorem 32.5], we say that $d_c = b$.

Let $*$ be an *e.a.b.* star operation on an integrally closed domain D and

$$Kr(D,*) = \{\frac{f}{g} \mid f, g \in D[X], g \neq 0, \text{ and } A_f \subseteq (A_g)_*\}.$$

Then $Kr(D,*)$ is a Bezout domain and $Kr(D,*) \cap K = D$; D is a Prüfer domain if and only if $Kr(D,b) = D(X)$; and D is a PvMD if and only if $Kr(D,v_c) = D[X]_{N_v}$ [6, Corollary 3.8]. In [4], the authors introduced and studied a graded integral domain analog as follows.

THEOREM 5. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be an integrally closed graded integral domain, $*$ be an e.a.b. star operation on R, and*

$$Kr(R,*) = \{\frac{f}{g} \mid f, g \in R, g \neq 0, \text{ and } C(f) \subseteq C(g)_*\}.$$

1. *$Kr(R,*)$ is an integral domain such that $Kr(R,*) \cap R_H = R$ and $fKr(R,*) = C(f)Kr(R,*)$ for all $f \in R$.*
2. *Assume that R has a unit of nonzero degree.*
   (a) *$Kr(R,*)$ is a Bezout domain.*
   (b) *If $*_f = t$, then R is a PvMD if and only if $R_{N(H)} = Kr(R,t)$.*
   (c) *R is a graded-Prüfer domain if and only if $R_{S(H)} = Kr(R,b)$, if and only if $C(fg) = C(f)C(g)$ for all $0 \neq f, g \in R$.*

Proof. See [4, Theorem 2.9] (resp., [4, Theorem 3.5]; [4, Theorem 3.7] and [16, Theorem 4.2]) for (1) (resp., (2) (a)-(b); (2) (c)). □

It is known that if $\Gamma = \mathbb{N}_0$, then $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ is a Prüfer domain if and only if R_0 is a Prüfer domain and $R \cong R_0 + yF[y]$, where F is the quotient field of R_0 and y is an indeterminate over F [9, Proposition 3.4]. The ring $T = R_0 + yF[y]$ is very helpful when we construct rings with prescribed ring-theoretic properties, and this type of integral domains was first introduced and studied by Costa, Mott, and Zafrullah [8]. In the next section, we use rings of the type $T = R_0 + yF[y]$ to give some examples which show that the results of this section are best for generalizations of Nagata rings and Kronecker function rings to graded integral domains.

3. The ring $R = D + XK[X]$

Let D be an integral domain with quotient field K and $D \neq K$, X be an indeterminate over D, $K[X]$ be the polynomial ring over K, and $R = D + XK[X]$, i.e., $R = \{f \in K[X] \mid f(0) \in D\}$; so $D[X] \subsetneq R \subsetneq K[X]$. Clearly, R is an $\mathbb{N}_0$-graded integral domain with $\deg(aX^n) = n$ for $0 \neq a \in K$ and $n \geq 0$ ($a \in D$ when $n = 0$). Let H be the set of nonzero homogeneous elements of R, $S(H) = \{f \in R \mid C(f) = R\}$ and $N(H) = \{0 \neq f \in R \mid C(f)_v = R\}$; so $S(H) \subseteq N(H)$, and equality holds when each maximal homogeneous ideal of R is a t-ideal.

Lemma 6. *Let $R = D + XK[X]$.*

1. *If $f = a_kX^k + a_{k+1}X^{k+1} + \cdots + a_{k+n}X^{k+n} \in R$, $a_k \neq 0$, then $C(f) = a_kX^kR$.*
2. *If $f, g \in R$, then $C(fg) = C(f)C(g)$.*
3. *$N(H) = \{f \in R \mid f(0)$ is a unit of $D\} = S(H)$.*
4. *h-Max$(R) = \{P + XK[X] \mid P \in$ Max$(D)\}$.*

Proof. (1) Note that if $i > k$, then $a_iX^i = a_kX^k(\frac{a_i}{a_k}X^{i-k})$ and $\frac{a_i}{a_k}X^{i-k} \in XK[X] \subseteq R$. Thus, $C(f) = (a_kX^k, a_{k+1}X^{k+1}, \cdots, a_{k+n}X^{k+n}) = a_kX^kR$.

(2) This follows directly from (1).

(3) Let $0 \neq a \in D$. Then $(aR)_v = aR$; and $aR = R$ if and only if a is a unit of D. Thus, the result follows from (1) and the fact that $bX^nR \subseteq XK[X] \subsetneq R$ for all $bX^n \in XK[X]$.

(4) This follows from the fact that if A is a homogeneous ideal of R, then either $A \subseteq XK[X]$ or $A \cap D \neq (0)$ and $A = (A \cap D) + XK[X]$ [8, Proposition 4.12]. □

Proposition 7. *For a star operation $*$ on $R = D + XK[X]$, let*

$$\mathrm{Kr}(R, *) = \{\tfrac{f}{g} \mid f, g \in R, g \neq 0 \text{ and } C(f) \subseteq C(g)_*\}.$$

1. *Kr$(R, *)$ is an integral domain such that Kr$(R, *) \cap R_H = R$ and fKr$(R, *) = C(f)$Kr$(R, *)$ for all $f \in R$.*
2. *Kr(R, d) = Kr$(R, *)$ = Kr(R, v); hence R has a unique Kronecker function ring of this type.*
3. *$R_{N(H)}$ = Kr$(R, *)$.*

4. *$\mathrm{Kr}(R,*)$ is integrally closed if and only if D is integrally closed, if and only if R is integrally closed.*

Proof. (1) Let $0 \neq f, g \in R$. Then $C(fg) = C(f)C(g)$ by Lemma 6(2), and hence $C(fg)_* = (C(f)C(g))_*$. Thus, the result can be proved by the same argument as the proof of [4, Theorem 2.9].

(2) If $0 \neq f \in R$, then $C(f)$ is a nonzero principal ideal of R by Lemma 6(1), and so $C(f) = C(f)_* = C(f)_v$. Thus, the result follows.

(3) Let $f, g \in R$ be such that $g \neq 0$ and $C(f) \subseteq C(g)$. Note that $C(g) = uR$ for some $u \in H$ by Lemma 6(1); so $C(\frac{f}{u}) \subseteq C(\frac{g}{u}) = R$ and $\frac{f}{g} = \frac{f}{u}/\frac{g}{u} \in R_{N(H)}$. Thus, $\mathrm{Kr}(R,v) \subseteq R_{N(H)}$. The reverse containment is clear.

(4) It is clear that R is integrally closed if and only if D is integrally closed (because $K[X]$ is integrally closed and $R \cap K = D$). Note that R_H is integrally closed and $\mathrm{Kr}(R,*) \cap R_H = R$. So if $\mathrm{Kr}(R,*)$ is integrally closed, then R is integrally closed. For the reverse, note that if R is integrally closed, then $R_{N(H)}$ is integrally closed. Thus, $\mathrm{Kr}(R,*)$ is integrally closed by (3). □

Corollary 8. *Let $R = D + XK[X]$.*

1. *Every ideal of $R_{N(H)}$ is extended from a homogeneous ideal of R.*
2. *$\mathrm{Max}(R_{N(H)}) = \{PR_{N(H)} \mid P \in \mathrm{Max}(D)\}$.*

Proof. (1) By Proposition 7, $fR_{N(H)} = C(f)R_{N(H)}$ for all $f \in R$. Thus, if A is an ideal of R, then $A_{N(H)} = \sum_{f\in A} fR_{N(H)} = \sum_{f\in A} C(f)R_{N(H)}$ $= (\sum_{f\in A} C(f))R_{N(H)}$ and $\sum_{f\in A} C(f)$ is a homogeneous ideal of R.

(2) Let M be a maximal ideal of $R_{N(H)}$. Then $M = QR_{N(H)}$ for some homogeneous ideal Q of R by (1). Also, since M is a maximal ideal of $R_{N(H)}$, $XK[X] \subsetneq Q$, and hence $Q \cap D$ is a maximal ideal of D. Thus, $Q = (Q \cap D) + XK[X] = (Q \cap D)R$ and $M = (Q \cap D)R_{N(H)}$. For the reverse containment, let P be a maximal ideal of D. Then $PR = P + XK[X]$ is a maximal ideal of R and $PR \cap N(H) = \emptyset$. Thus, $PR_{N(H)}$ is a maximal ideal of $R_{N(H)}$. □

It is easy to see that if I is a nonzero fractional ideal of D, then $(IR)^{-1} = I^{-1} + XK[X] = I^{-1}R$, $(IR)_v = I_v + XK[X] = (I_v)R$, and $(IR)_t = I_t + XK[X] = (I_t)R$ (cf. [8, Lemma 4.41]). From now on, we use this result without further comments.

Corollary 9. *$R = D + XK[X]$ satisfies property (#) if and only if each maximal ideal of D is a t-ideal.*

Proof. Let P be a nonzero prime ideal of D. Then $PR = P + XK[X]$ and $(P + XK[X])_t = P_t + XK[X]$, and thus PR is a maximal t-ideal of R if and only if P is a maximal t-ideal of D. Also, if Q is a maximal t-ideal of R that is homogeneous, then $Q \cap D \neq (0)$, and hence $Q = (Q \cap D) + XK[X]$. Hence, $\{PR \mid P \in t\text{-Max}(D)\}$ is the set of maximal t-ideals of R that are homogeneous. Thus, R satisfies property (#) if and only if $\mathrm{Max}(R_{N(H)}) = \{PR_{N(H)} \mid P \in t\text{-Max}(D)\}$ by Proposition 1, if and only if $\mathrm{Max}(D) = t\text{-Max}(D)$ by Corollary 8(2), if and only if each maximal ideal of D is a t-ideal. □

Note that if D is a Prüfer domain, then each maximal ideal of D is a t-ideal. Thus, $R = D + XK[X]$ satisfies property (#) by Corollary 9.

Proposition 10. *The following statements are equivalent for $R = D + XK[X]$.*

1. *R is a Prüfer (resp., Bezout) domain.*
2. *$R_{N(H)}$ is a Prüfer (resp., Bezout) domain.*
3. *D is a Prüfer (resp., Bezout) domain.*
4. *R is a graded-Prüfer (resp., graded-Bezout) domain.*

Proof. (1) ⇒ (2) and (4) Clear.

(2) ⇒ (3) Let $0 \neq a, b \in D$. If $R_{N(H)}$ is a Prüfer domain, then $(a, b)R_{N(H)}$ is invertible, and hence

$$\begin{aligned} R_{N(H)} &= ((a,b)R_{N(H)})((a,b)R_{N(H)})^{-1} \\ &= ((a,b)R_{N(H)})(((a,b)R)^{-1}R_{N(H)}) \\ &= (((a,b)R)((a,b)R)^{-1})R_{N(H)} \end{aligned}$$

(cf. [4, Proposition 1.3] for the second equality). Since (a, b) is an ideal of D,

$$\begin{aligned} ((a,b)R)((a,b)R)^{-1} &= ((a,b) + XK[X])((a,b)^{-1} + XK[X]) \\ &= (a,b)(a,b)^{-1} + XK[X]. \end{aligned}$$

Note also that $(P + XK[X]) \cap N(H) = \emptyset$ for all $P \in \mathrm{Max}(D)$ by Lemma 6(3). Hence, $(a, b)(a, b)^{-1} \nsubseteq P$ for all $P \in \mathrm{Max}(D)$, and thus (a, b) is invertible. Thus, D is a Prüfer domain.

Next, if R is a Bezout domain, then $(a, b)R_{N(H)} = gR_{N(H)}$ for some $g \in R$. Since $a, b \in D$ and $gR_{N(H)} = C(g)R_{N(H)}$ by Proposition 7, we may assume that $g \in D$. Then, it is clear that $(a, b) = gD$ by Lemma 6.

(3) ⇒ (1) [8, Corollary 4.15] (resp., [8, Corollary 4.13]).

(4) $\Rightarrow$ (3) See the proof of "(2) $\Rightarrow$ (3)" above because $(a,b)R$ is a homogeneous ideal of R. $\square$

Corollary 11. *Let $R = D + XK[X]$, and assume that D is a Prüfer domain. Then $Pic(R_{N(H)}) = \{0\}$ if and only if D is a Bezout domain.*

Proof. This follows directly from Proposition 10 because a Prüfer domain is a Bezout domain if and only if its Picard group is trivial. $\square$

We next give a PvMD (resp., GCD domain) analog of Proposition 10 and Corollary 11. Even though their proofs are word for word translations of their counterparts, we give them for completeness.

Proposition 12. *The following statements are equivalent for $R = D + XK[X]$.*

1. *R is a PvMD (resp., GCD domain).*
2. *$R_{N(H)}$ is a PvMD (resp., GCD domain).*
3. *D is a PvMD (resp., GCD domain).*

Proof. (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) For the PvMD property, it suffices to show that each nonzero two generated ideal of D is t-invertible. For this, let $0 \neq a, b \in D$. If $R_{N(H)}$ is a PvMD, then $(a,b)R_{N(H)}$ is t-invertible, and hence by [4, Proposition 1.3],

$$\begin{aligned} R_{N(H)} &= (((a,b)R_{N(H)})((a,b)R_{N(H)})^{-1})_t \\ &= (((a,b)R_{N(H)})(((a,b)R)^{-1}R_{N(H)}))_t \\ &= (((a,b)R)((a,b)R)^{-1})_t R_{N(H)}. \end{aligned}$$

Since (a,b) is an ideal of D,

$$\begin{aligned} (((a,b)R)((a,b)R)^{-1})_t &= (((a,b) + XK[X])((a,b)^{-1} + XK[X]))_t \\ &= ((a,b)(a,b)^{-1} + XK[X])_t \\ &= ((a,b)(a,b)^{-1})_t + XK[X]. \end{aligned}$$

Note that $(P + XK[X]) \cap N(H) = \emptyset$ for all $P \in t\text{-Max}(D)$. Hence, $(a,b)(a,b)^{-1} \nsubseteq P$ for all $P \in t\text{-Max}(D)$, and thus (a,b) is t-invertible.

Next, if $R_{N(H)}$ is a GCD domain, then

$$((a,b)_t R)R_{N(H)} = ((a,b)R)_t R_{N(H)} = ((a,b)R_{N(H)})_t = gR_{N(H)}$$

for some $g \in R$. Since $a, b \in D$ and $gR_{N(H)} = C(g)R_{N(H)}$, we may assume that $g \in D$. Then, it is clear that $(a,b)_t = gD$.

(3) $\Rightarrow$ (1) [8, Theorem 4.43] (resp., [8, Theorem 1.1 and Corollary 1.3]). $\square$

Corollary 13. *Let $R = D+XK[X]$, and assume that D is a PvMD. Then $Cl(R_{N(H)}) = \{0\}$ if and only if D is a GCD domain.*

Proof. This follows directly from Proposition 12 because a PvMD is a GCD domain if and only if its t-class group is trivial. $\square$

Let $T = \bigoplus_{\alpha\in\Gamma} T_\alpha$ be a nontrivial graded Krull domain; equivalently, every nonzero homogeneous ideal of T is t-invertible [3, Theorem 2.4]. Then T satisfies property (#), $T_{N(H)}$ is a PID, $Cl(T_{N(H)})$ =Pic$(T_{N(H)}) = \{0\}$, and $T_{N(H)} = \mathrm{Kr}(T, v)$ [4, Section 2]; so in this case, Theorems 3, 4 and 5 hold even though R does not contain a unit of nonzero degree. We next give a remark which shows that the results of Section 1 are the best generalizations of Nagata rings and Kronecker function rings to graded integral domains.

Remark 14. (1) (cf. Proposition 1) If $D = \mathbb{Z}[y]$ is the polynomial ring over $\mathbb{Z}$, then $(2, y)$ is a maximal ideal of D but not a t-ideal, and hence $R = D + XK[X]$ does not satisfy property (#) by Corollary 9. Thus, a graded integral domain does not satisfy property (#) in general.

(2) If $D = \mathbb{Z}[y]$ is the polynomial ring over $\mathbb{Z}$, then D is a PvMD but not a Prüfer domain. Hence, R is a non-Prüfer PvMD, and thus $R_{N(H)}$ is a non-Prüfer PvMD by Propositions 10 and 12. Thus, in Theorem 2, the assumption that R satisfies property (#) cannot be deleted.

(3) Let D be a non-Bezout Prüfer domain. Then R satisfies property (#) by Corollary 9 and $R_{N(H)}$ is a non-Bezout Prüfer domain by Proposition 10, while $Cl((R_{N(H)}))$ =Pic$(R_{N(H)}) \neq \{0\}$ by Corollary 11 and $\mathrm{Kr}(R, *) = R_{N(H)}$ by Proposition 7(3). Thus, in Theorems 3 and 5(2)(a), the assumption that R has a unit of nonzero degree cannot be deleted.

(4) By Corollary 8, every (principal) ideal of $R_{N(H)}$ is extended from a homogeneous ideal of R. But, if D is not a PvMD, then $R_{N(H)}$ is not a PvMD by Proposition 12. Thus, in Theorem 4, the assumption that R has a unit of nonzero degree cannot be deleted.

(5) It is well known that if an integral domain admits an *e.a.b.* star operation, then it is integrally closed [12, Corollary 32.8]. But, Proposition 7(1) shows that we can define a Kronecker function ring of graded integral domains even though it is not integrally closed (i.e., $*$ need not be an *e.a.b.* star operation for Kronecker function rings $\mathrm{Kr}(R, *)$). Note

also that $\mathrm{Kr}(R, *) = R_{N(H)}$ by Proposition 7(3); so Theorem 5(2)(b) does not hold in general.

(6) Let $D = \mathbb{Q}[y^2, y^3]$, where y is an indeterminate over $\mathbb{Q}$. Then D is a one-dimension non-Prüfer domain and R satisfies property (#) by Corollary 9. Hence, R is not a graded-Prüfer domain, while $C(fg) = C(f)C(g)$ for all $0 \neq f, g \in R$ by Lemma 6(2). Thus, Theorem 5(2)(c) does not hold in general.

We end this paper with an example which shows that $\mathrm{Pic}(R_{N(H)}) \neq \mathrm{Cl}(R_{N(H)})$ in general (cf. Remark 14(3)).

Example 15. Let D be a quasi-local non-factorial Krull domain (see [7, Corollary 2] for such a Krull domain); then $\mathrm{Pic}(D) = \{0\}$ but $\mathrm{Cl}(D) \neq \{0\}$.

Claim 1: $\mathrm{Pic}(R_{N(H)}) = \{0\}$. (Proof: Let A be an ideal of R such that $AR_{N(H)}$ is invertible. By Corollary 8, we may assume that A is homogeneous. Then $R_{N(H)} = (AR_{N(H)})(AR_{N(H)})^{-1} = (AR_{N(H)})(A^{-1}R_{N(H)}) = (AA^{-1})R_{N(H)}$ (see [4, Proposition 1.3] for the second equality), and hence $AA^{-1} \cap N(H) \neq \emptyset$. Note that $AA^{-1} = R$ by Lemma 6(3) because AA^{-1} is homogeneous; so we may assume that $A \cap D \neq (0)$, and hence $A = (A \cap D)R$ and $A \cap D$ is invertible. Since $\mathrm{Pic}(D) = \{0\}$, $A \cap D$ is principal. Thus, $AR_{N(H)}$ is principal.)

Claim 2. $\mathrm{Cl}(R_{N(H)}) \neq \{0\}$. (Proof: Let I be a nonzero non-principal t-invertible ideal of D. Then $IR_{N(H)}$ is t-invertible. If $IR_{N(H)}$ is principal, then there is an $x \in I$ such that $IR_{N(H)} = xR_{N(H)}$ by Lemma 6 because $IR \nsubseteq XK[X]$. Thus, $I = IR_{N(H)} \cap K = xR_{N(H)} \cap K = xD$, a contradiction.)

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