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SHIFTING AND MODULATION FOR FOURIER-FEYNMAN TRANSFORM OF FUNCTIONALS IN A GENERALIZED FRESNEL CLASS

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Abstract. Time shifting and frequency shifting proprerties for the Fourier-Feynman transform of functionals in a generalized Fresnel class $\mathcal{F}_{A_1,A_2}$ are given. We discuss scaling and modulation propreties for the Fourier-Feynman transform. These properties help us to obtain Fourier-Feynman transforms of new functionals from the Fourier-Feynman transforms of old functionals which we know their Fourier-Feynman transforms.

1. Introduction

Let (H, B, ν) be an abstract Wiener space and let $\{e_j\}$ be a complete orthonormal system in H such that the e_j's are in B^*, the dual of B. For each $h \in H$ and $x \in B$, we define a stochastic inner product $(h, x)^\sim$ as follows:

$$(1.1) \qquad (h,x)^\sim = \begin{cases} \lim\limits_{n\to\infty} \sum\limits_{j=1}^{n} \langle h, e_j\rangle (x, e_j), & \text{if the limit exists} \\ 0, & \text{otherwise,} \end{cases}$$

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where $(\cdot,\cdot)$ denotes the natural dual pairing between B and B^*. It is well known [12, 13] that for each $h(\neq 0)$ in H, $(h,\cdot)^\sim$ is a Gaussian random variable on B with mean zero and variance $|h|^2$, that is,

$$\int_B \exp\{i(h,x)^\sim\}\,d\nu(x) = \exp\Big\{-\frac{1}{2}|h|^2\Big\}. \tag{1.2}$$

A subset E of a product abstract Wiener space B^2 is said to be scale-invariant measurable provided $\{(\alpha x_1, \beta x_2) : (x_1,x_2)\in E\}$ is abstract Wiener measurable for every $\alpha > 0$ and $\beta > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $(\nu\times\nu)(\{(\alpha x_1,\beta x_2) : (x_1,x_2)\in N\}) = 0$ for every $\alpha > 0$ and $\beta > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.) [11].

Let $\mathbb{C}$ denote the set of complex numbers and let

$$\Omega = \{\vec{\lambda} = (\lambda_1,\lambda_2)\in\mathbb{C}^2 : \operatorname{Re}\lambda_k > 0 \text{ for } k = 1,2\} \tag{1.3}$$

and

$$\tilde{\Omega} = \{\vec{\lambda} = (\lambda_1,\lambda_2)\in\mathbb{C}^2 : \lambda_k\neq 0, \operatorname{Re}\lambda_k \geq 0 \text{ for } k = 1,2\}. \tag{1.4}$$

Let F be a complex-valued function on B^2 such that the integral

$$J_F(\lambda_1,\lambda_2) = \int_{B^2} F(\lambda_1^{-1/2}x_1, \lambda_2^{-1/2}x_2)\,d(\nu\times\nu)(x_1,x_2) \tag{1.5}$$

exists as a finite number for all real numbers $\lambda_1 > 0$ and $\lambda_2 > 0$. If there exists a function $J_F^*(\lambda_1,\lambda_2)$ analytic on Ω such that $J_F^*(\lambda_1,\lambda_2) = J_F(\lambda_1,\lambda_2)$ for all $\lambda_1 > 0$ and $\lambda_2 > 0$, then $J_F^*(\lambda_1,\lambda_2)$ is defined to be the analytic Wiener integral of F over B^2 with parameter $\vec{\lambda} = (\lambda_1,\lambda_2)$, and for $\vec{\lambda}\in\Omega$ we write

$$\int_{B^2}^{\mathrm{anw}_{\vec{\lambda}}} F(x_1,x_2)\,d(\nu\times\nu)(x_1,x_2) = J_F^*(\lambda_1,\lambda_2). \tag{1.6}$$

Let q_1 and q_2 be nonzero real numbers and F be a functional on B^2 such that $\int_{B^2}^{\mathrm{anw}_{\vec{\lambda}}} F(x_1,x_2)\,d(\nu\times\nu)(x_1,x_2)$ exists for all $\vec{\lambda}\in\Omega$. If the following limit exists, then we call it the analytic Feynman integral of F over B^2 with parameter $\vec{q} = (q_1,q_2)$ and we write

$$\int_{B^2}^{\mathrm{anf}_{\vec{q}}} F(x_1,x_2)\,d(\nu\times\nu)(x_1,x_2) = \lim_{\vec{\lambda}\to -i\vec{q}} \int_{B^2}^{\mathrm{anw}_{\vec{\lambda}}} F(x_1,x_2)\,d(\nu\times\nu)(x_1,x_2), \tag{1.7}$$

where $\vec{\lambda} = (\lambda_1,\lambda_2)$ approaches $(-iq_1,-iq_2)$ through Ω.

Let $M(H)$ denote the space of complex-valued countably additive Borel measures on H. Under the total variation norm $\|\cdot\|$ and with convolution as multiplication, $M(H)$ is a commutative Banach algebra with identity [2].

Now we state a generalized Fresnel class $\mathcal{F}_{A_1,A_2}$ introduced by Kallianpur and Bromley [12]. Let A_1 and A_2 be bounded, non-negative self-adjoint operators on H. Let $\mathcal{F}_{A_1,A_2}$ be the space of all s-equivalence classes of functionals F on B^2 which have the form

$$F(x_1, x_2) = \int_H \exp\Big\{ i \sum_{j=1}^{2} (A_j^{1/2} h, x_j)^\sim \Big\}\, d\sigma(h) \tag{1.8}$$

for some complex-valued countably additive Borel measure σ on H.

As is customary, we will identify a functional with its s-equivalence class and think of $\mathcal{F}_{A_1,A_2}$ as a collection of functionals on B^2 rather than as a collection of equivalence classes. Moreover the map $\sigma \mapsto [F]$ defined by (1.8) sets up an algebra isomorphism between $M(H)$ and $\mathcal{F}_{A_1,A_2}$ if the range of $A_1 + A_2$ is dense in H. In this case, $\mathcal{F}_{A_1,A_2}$ becomes a Banach algebra under the norm $\|F\| = \|\sigma\|$ [12].

Remark 1.1. Let $\mathcal{F}(B)$ denote the Fresnel class of functions F on B of the form

$$F(x) = \int_H \exp\{ i(h, x)^\sim \}\, d\sigma(h) \tag{1.9}$$

for some $\sigma \in M(H)$. If A_1 is the identity operator on H and $A_2 = 0$, then $\mathcal{F}_{A_1,A_2}$ is essentially the Fresnel class $\mathcal{F}(B)$.

The concept of L_1 analytic Fourier-Feynman transform for functionals on Wiener space was introduced by Brue in [3]. In [4], Cameron and Storvick introduced L_2 analytic Fourier-Feynman transform. In [10], Johnson and Skoug developed L_p analytic Fourier-Feynman transform for $1 \le p \le 2$ that extended the results in [4].

In [8, 9], Huffman, Park and Skoug defined a convolution product for functionals on Wiener space and showed that the Fourier-Feynman transform of a convolution product is a product of Fourier-Feynman transforms. Recently, Chang, Kim, Song and Yoo [7, 15, 17] extended the above results for functionals on a product abstract Wiener space.

Also in [14], the author studied shifting, scaling, modulation and variational properties for Fourier-Feynman transform of functionals on

Wiener space. In this paper we extend the results in [14] for some functionals on a product abstract Wiener space.

In this paper, we develop shifting, scaling and modulation properties for the Fourier-Feynman transform of functionals in a generalized Fresnel class $\mathcal{F}_{A_1,A_2}$. Since the class $\mathcal{F}_{A_1,A_2}$ is a generalization of the Fresnel class $\mathcal{F}(B)$ which is an abstract Wiener space version of the Banach algebras $\mathcal{S}$ introduced by Cameron and Storvick [5], the results in Section 2 of [14] can be obtained as corollaries of our results.

2. Shifting for the Fourier-Feynman transform

In this section we develop some of important properties relevant to shifting (translating) and computational rules for Fourier-Feynman transform of functionals in the generalized Fresnel class $\mathcal{F}_{A_1,A_2}$. Let us begin with the definition of Fourier-Feynman transform of functionals on a product of abstract Wiener space.

Let $1 \le p < \infty$ and let $\vec{q} = (q_1, q_2)$, where q_1 and q_2 are nonzero real numbers throughout this paper.

Definition 2.1. Let F be a functional on B^2. For $\vec{\lambda} = (\lambda_1, \lambda_2) \in \Omega$ and $(y_1, y_2) \in B^2$, let

$$T_{\vec{\lambda}}[F](y_1, y_2) = \int_{B^2}^{\text{anw}_{\vec{\lambda}}} F(x_1 + y_1, x_2 + y_2)\, d(\nu \times \nu)(x_1, x_2). \tag{2.1}$$

For $1 < p < \infty$, we define the L_p analytic Fourier-Feynman transform $T_{\vec{q}}^{(p)}[F]$ of F on B^2 by the formula ($\vec{\lambda} \in \Omega$)

$$T_{\vec{q}}^{(p)}[F](y_1, y_2) = \underset{\vec{\lambda} \to -i\vec{q}}{\text{l.i.m.}}\, T_{\vec{\lambda}}[F](y_1, y_2), \tag{2.2}$$

whenever this limit exists; that is, for each $\alpha > 0$ and $\beta > 0$,

$$\lim_{\vec{\lambda} \to -i\vec{q}} \int_{B^2} |T_{\vec{\lambda}}[F](\alpha x_1, \beta x_2) - T_{\vec{q}}^{(p)}[F](\alpha x_1, \beta x_2)|^{p'}\, d(\nu \times \nu)(x_1, x_2) = 0$$

where $1/p + 1/p' = 1$. We define the L_1 analytic Fourier-Feynman transform $T_{\vec{q}}^{(1)}[F]$ of F by ($\vec{\lambda} \in \Omega$)

$$T_{\vec{q}}^{(1)}[F](y_1, y_2) = \lim_{\vec{\lambda} \to -i\vec{q}} T_{\vec{\lambda}}[F](y_1, y_2), \tag{2.3}$$

for s-a.e. $(y_1, y_2) \in B^2$, whenever this limit exists [4, 7–10, 15, 17].

Since $T_{\vec{\lambda}}$ is linear, obviously $T_{\vec{q}}^{(p)}$ is also linear, that is,

$$T_{\vec{q}}^{(p)}[aF+bG](y_1,y_2) = aT_{\vec{q}}^{(p)}[F](y_1,y_2) + bT_{\vec{q}}^{(p)}[G](y_1,y_2) \tag{2.4}$$

for all constants a, b and functionals F, G on B^2, whenever each transforms exist.

By the definitions (1.7) of the analytic Feynman integral and the L_1 analytic Fourier-Feynman transform (2.3), it is easy to see that

$$T_{\vec{q}}^{(1)}[F](y_1,y_2) = \int_{B^2}^{\text{anf}_{\vec{q}}} F(x_1+y_1, x_2+y_2)\, d(\nu\times\nu)(x_1,x_2). \tag{2.5}$$

In particular, if $F \in \mathcal{F}_{A_1,A_2}$, then F is analytic Feynman integrable and

$$T_{\vec{q}}^{(1)}[F](0,0) = \int_{B^2}^{\text{anf}_{\vec{q}}} F(x_1,x_2)\, d(\nu\times\nu)(x_1,x_2). \tag{2.6}$$

Now we will introduce a result by Chang, Kim and Yoo [7], on the existence of Fourier-Feynman transform of functionals in $\mathcal{F}_{A_1,A_2}$.

THEOREM 2.2 (Theorem 3.1 of [7]). *Let $F \in \mathcal{F}_{A_1,A_2}$ be given by* (1.8). *Then the Fourier-Feynman transform $T_{\vec{q}}^{(p)}[F]$ exists, belongs to $\mathcal{F}_{A_1,A_2}$ and is given by*

(2.7)

$$T_{\vec{q}}^{(p)}[F](y_1,y_2) = \int_H \exp\Big\{ i\sum_{j=1}^{2}\Big[(A_j^{1/2}h, y_j)^{\sim} - \frac{1}{2q_j}|A_j^{1/2}h|^2\Big]\Big\}\, d\sigma(h)$$

for s-a.e. $(y_1,y_2) \in B^2$.

In the classical Fourier analysis, the Fourier transform $\mathcal{F}$ turns a function f into a new function $\mathcal{F}[f]$. Because the transform is used in signal analysis, we usually use t for time as the variable of the function f, and ω as the variable of the transform $\mathcal{F}[f]$, that is,

$$\mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt.$$

Engineers refer to the variable ω in the transformed function as the frequency of the signal f [16].

We will use the same convention in this paper, that is, for a Fourier-Feynman transform $T_{\vec{q}}^{(p)}[F](y_1,y_2)$ of $F(x_1,x_2)$, we call the variable (x_1,x_2) as a time and the variable (y_1,y_2) as a freqeuency.

Our first result in this section shows that the time shifting of the Fourier-Feynman transform is equal to the frequency shifting of the Fourier-Feynman transform.

THEOREM 2.3. *Let F be a functional on B^2 and let $(w_1, w_2) \in B^2$. Then we have*

$$T_{\vec{q}}^{(p)}[F(\cdot - w_1, \cdot - w_2)](y_1, y_2) = T_{\vec{q}}^{(p)}[F](y_1 - w_1, y_2 - w_2) \tag{2.8}$$

if each sides exist.

Proof. For all $\lambda_1, \lambda_2 > 0$ and for s-a.e. $(y_1, y_2) \in B^2$,

$$\begin{aligned} &T_{\vec{\lambda}}[F(\cdot - w_1, \cdot - w_2)](y_1, y_2) \\ &= \int_{B^2} F(\lambda_1^{-1/2} x_1 - w_1 + y_1, \lambda_2^{-1/2} x_2 - w_2 + y_2)\, d(\nu \times \nu)(x_1, x_2) \\ &= T_{\vec{\lambda}}[F](y_1 - w_1, y_2 - w_2) \end{aligned}$$

if the abstract Wiener integral exists. Extending analytically each sides and taking limits as $\vec{\lambda} \to -i\vec{q}$, we have the result. □

In the Fourier analysis, if we shift time back t_0, then the Fourier transform of this shifted function is the transform of $f(t)$ multiplied by the exponential factor $e^{-i\omega t_0}$ [16], that is,

$$\mathcal{F}[f(t - t_0)](\omega) = e^{-i\omega t_0} \mathcal{F}[f(t)](\omega).$$

The following theorem is reminiscent of the time shifting theorem for the Fourier transform. Hence we call the following theorem as time shifting formula for Fourier-Feynman transform on a product abstract Wiener space. It says that if we shift back (w_1, w_2) and replace $F(x_1, x_2)$ by $F(x_1 - w_1, x_2 - w_2)$, then Fourier-Feynman transform of this shifted function is equal to the Fourier-Feynman transform of

$$F(x_1, x_2) \exp\Big\{ i \sum_{j=1}^{2} q_j (w_j, x_j)^{\sim} \Big\}$$

multiplied by an exponential factor.

Theorem 2.4 (time shifting). *Let* $F \in \mathcal{F}_{A_1,A_2}$ *be given by* (1.8) *and let* $(w_1, w_2) \in H^2$. *Then we have*

$$\begin{aligned} &T_{\vec{q}}^{(p)}[F(\cdot - w_1, \cdot - w_2)](y_1, y_2) \\ &= \exp\Big\{-i\sum_{j=1}^{2}\Big[q_j(w_j, y_j)^{\sim} - \frac{q_j}{2}|w_j|^2\Big]\Big\} \\ &\quad T_{\vec{q}}^{(p)}\Big[F(\cdot,\cdot)\exp\Big\{i\sum_{j=1}^{2} q_j(w_j,\cdot)^{\sim}\Big\}\Big](y_1,y_2) \end{aligned} \tag{2.9}$$

for s-a.e. $(y_1, y_2) \in B^2$.

Proof. Let

$$G(x_1, x_2) = F(x_1, x_2)\exp\Big\{i\sum_{j=1}^{2} q_j(w_j, x_j)^{\sim}\Big\}.$$

Using (1.8) we write $G(x_1, x_2)$ as

$$G(x_1, x_2) = \int_H \exp\Big\{i\sum_{j=1}^{2}(A_j^{1/2}h + q_j w_j, x_j)^{\sim}\Big\}\, d\sigma(h).$$

For all $\lambda_1 > 0, \lambda_2 > 0$ and s-a.e. $(y_1, y_2) \in B^2$,

$$\begin{aligned} T_{\vec{\lambda}}[G](y_1, y_2) &= \int_{B^2} G(\lambda_1^{-1/2}x_1 + y_1, \lambda_2^{-1/2}x_2 + y_2)\, d(\nu\times\nu)(x_1, x_2) \\ &= \int_{B^2}\int_H \exp\Big\{i\sum_{j=1}^{2}(A_j^{1/2}h + q_j w_j, \lambda_j^{-1/2}x_j + y_j)^{\sim}\Big\} \\ &\quad d\sigma(h)\, d(\nu\times\nu)(x_1, x_2). \end{aligned}$$

Using the Fubini theorem and (1.2), we obtain

$$\begin{aligned} T_{\vec{\lambda}}[G](y_1, y_2) = \int_H \exp\Big\{\sum_{j=1}^{2}\Big[&i(A_j^{1/2}h + q_j w_j, y_j)^{\sim} \\ &- \frac{1}{2\lambda_j}|A_j^{1/2}h + q_j w_j|^2\Big]\Big\}\, d\sigma(h). \end{aligned}$$

Extending analytically and using the dominated convergence theorem, we obtain

$$\begin{aligned} T_{\vec{q}}^{(p)}[G](y_1,y_2) &= \int_H \exp\Big\{ i\sum_{j=1}^{2}\Big[(A_j^{1/2}h+q_jw_j,y_j)^{\sim} \\ &\qquad -\frac{1}{2q_j}|A_j^{1/2}h+q_jw_j|^2\Big]\Big\}\,d\sigma(h) \\ &= \exp\Big\{ i\sum_{j=1}^{2}\Big[q_j(w_j,y_j)^{\sim}-\frac{q_j}{2}|w_j|^2\Big]\Big\} \\ &\qquad \int_H \exp\Big\{ i\sum_{j=1}^{2}\Big[(A_j^{1/2}h,y_j)^{\sim}-\langle A_j^{1/2}h,w_j\rangle \\ &\qquad -\frac{1}{2q_j}|A_j^{1/2}h|^2\Big]\Big\}\,d\sigma(h). \end{aligned}$$

But since $A_j^{1/2}h \in H$ and $w_j \in H$, we know that the inner product $\langle A_j^{1/2}h, w_j\rangle$ is equal to the stochastic inner product $(A_j^{1/2}h, w_j)^{\sim}$ and so by (2.7) we see that the last integral is equal to $T_{\vec{q}}^{(p)}[F](y_1-w_1, y_2-w_2)$. Finally by (2.8) the proof is completed. □

Cameron and Storvick [6] presented a new translation theorem for the analytic Feynman integral on Wiener space. Moreover Ahn, Chang, Kim and Yoo [1] gave a simple proof of an abstract Wiener space version of the translation theorem. Taking $p=1$ and $y_1=y_2=0$ in (2.9) and considering (2.6) we obtain Cameron and Storvick's translation theorem as follows. Hence Theorem 2.4 above can be viewed as a generalized Cameron and Storvick's translation theorem for the Fourier-Feynman transform for functionals in $\mathcal{F}_{A_1,A_2}$.

Corollary 2.5. *Let $F \in \mathcal{F}_{A_1,A_2}$ be given by (1.8) and let $(w_1,w_2) \in H^2$. Then we have*

$$\begin{aligned} &\int_{B^2}^{\mathrm{anf}_{\vec{q}}} F(x_1-w_1,x_2-w_2)\,d(\nu\times\nu)(x_1,x_2) \\ &= \exp\Big\{ i\sum_{j=1}^{2}\frac{q_j}{2}|w_j|^2\Big\} \int_{B^2}^{\mathrm{anf}_{\vec{q}}} F(x_1,x_2)\exp\Big\{ i\sum_{j=1}^{2}q_j(w_j,x_j)^{\sim}\Big\} \\ &\quad d(\nu\times\nu)(x_1,x_2). \end{aligned} \tag{2.10}$$

The Fourier transform of $e^{i\omega_0 t}f(t)$ is nothing more than the Fourier transform of $f(t)$ shifted ω_0 units to the right [16], that is,

$$\mathcal{F}[f(t)](\omega-\omega_0)=\mathcal{F}[e^{i\omega_0 t}f(t)](\omega).$$

Next thoerem is reminiscent of the frequency shifting theorem for the Fourier transform. Using Theorem 2.3 we have the following property for the frequency shifting of the Fourier-Feynman transform.

THEOREM 2.6 (frequency shifting). *Let* $F\in\mathcal{F}_{A_1,A_2}$ *be given by* (1.8) *and let* $(w_1,w_2)\in H^2$. *Then we have*

(2.11)

$$T^{(p)}_{\vec{q}}[F](y_1-w_1,y_2-w_2)=\exp\Big\{-i\sum_{j=1}^{2}\Big[q_j(w_j,y_j)^{\sim}-\frac{q_j}{2}|w_j|^2\Big]\Big\}$$
$$T^{(p)}_{\vec{q}}\Big[F(\cdot,\cdot)\exp\Big\{i\sum_{j=1}^{2}q_j(w_j,\cdot)^{\sim}\Big\}\Big](y_1,y_2)$$

for s-a.e. $(y_1,y_2)\in B^2$.

3. Scaling and modulation for the Fourier-Feynman transform

In this section we study scaling and modulation properties for the Fourier-Feynman transform.

The following theorem is called a scaling theorem because we want the transform not of $F(x_1,x_2)$, but of $F(a_1x_1,a_2x_2)$, in which a_1 and a_2 can be thought as scaling factors.

THEOREM 3.1 (scaling). *Let* $F\in\mathcal{F}_{A_1,A_2}$ *be given by* (1.8) *and let* a_1 *and* a_2 *be nonzero real numbers. Then we have*

$$T^{(p)}_{\vec{q}}[F(a_1\cdot,a_2\cdot)](y_1,y_2)=T^{(p)}_{(q_1/a_1^2,q_2/a_2^2)}[F](a_1y_1,a_2y_2)\tag{3.1}$$

for s-a.e. $(y_1,y_2)\in B^2$.

Proof. For all $\lambda_1>0,\lambda_2>0$ and s-a.e. $(y_1,y_2)\in B^2$,

$$T_{\vec{\lambda}}[F(a_1\cdot,a_2\cdot)](y_1,y_2)=\int_{B^2}F(a_1(\lambda_1^{-1/2}x_1+y_1),a_2(\lambda_2^{-1/2}x_2+y_2))$$
$$d(\nu\times\nu)(x_1,x_2)$$

Using the expression (1.8), Fubini theorem and (1.2), we have

$$\begin{aligned}&T_{\vec\lambda}[F(a_1\cdot,a_2\cdot)](y_1,y_2)\\&=\int_{B^2}\int_H \exp\Big\{i\sum_{j=1}^2 (A_j^{1/2}h+a_j(\lambda_j^{-1/2}x_j+y_j))^\sim\Big\}\,d\sigma(h)\,d(\nu\times\nu(x_1,x_2)\\&=\int_H \exp\Big\{\sum_{j=1}^2\Big[i(A_j^{1/2}h,a_jy_j)^\sim-\frac{a_j^2}{2\lambda_j}|A_j^{1/2}h|^2\Big]\Big\}\,d\sigma(h).\end{aligned}$$

Extending analytically and using the dominated convergence theorem, we obtain

$$\begin{aligned}&T_{\vec q}^{(p)}[F(a_1\cdot,a_2\cdot)](y_1,y_2)\\&=\int_H \exp\Big\{i\sum_{j=1}^2\Big[(A_j^{1/2}h,a_jy_j)^\sim-\frac{a_j^2}{2q_j}|A_j^{1/2}h|^2\Big]\Big\}\,d\sigma(h).\end{aligned}$$

Finally by (2.7) we see that the last expression is equal to the right hand side of (3.1) which completes the proof. □

Next corollary follows immediately from the scaling theorem above by putting $a_1=a_2=-1$. This result is called time reversal because we replace (x_1,x_2) by $(-x_1,-x_2)$ in $F(x_1,x_2)$ to get $F(-x_1,-x_2)$. The transform of this new functional is obtained by simply replacing (y_1,y_2) by $(-y_1,-y_2)$ in the transform of $F(x_1,x_2)$.

Corollary 3.2 (time reversal). *Let $F\in\mathcal{F}_{A_1,A_2}$ be given by* (1.8). *Then we have*

$$T_{\vec q}^{(p)}[F(-\cdot,-\cdot)](y_1,y_2)=T_{\vec q}^{(p)}[F](-y_1,-y_2) \tag{3.2}$$

for s-a.e. $(y_1,y_2)\in B^2$.

Our next theorem is useful to obtain Fourier-Feynman transforms of new functionals from the Fourier-Feynman transforms of old functionals which we know their Fourier-Feynman transforms.

Theorem 3.3 (modulation). *Let $F\in\mathcal{F}_{A_1,A_2}$ be given by* (1.8) *and let $(w_1,w_2)\in H^2$. Then we have*

(3.3)

$$T_{\vec q}^{(p)}\Big[F(\cdot,\cdot)\cos\Big(\sum_{j=1}^2 q_j(w_j,\cdot)^\sim\Big)\Big](y_1,y_2)=\frac12(K(w_1,w_2)+K(-w_1,-w_2))$$

and

(3.4)

$$T_{\vec{q}}^{(p)}\Big[F(\cdot,\cdot)\sin\Big(\sum_{j=1}^{2}q_j(w_j,\cdot)^{\sim}\Big)\Big](y_1,y_2)=\frac{1}{2i}(K(w_1,w_2)-K(-w_1,-w_2)),$$

where

$$K(w_1,w_2)=\exp\Big\{i\sum_{j=1}^{2}\Big[q_j(w_j,y_j)^{\sim}-\frac{q_j}{2}|w_j|^2\Big]\Big\} \qquad (3.5)$$
$$T_{\vec{q}}^{(p)}[F(\cdot-w_1,\cdot-w_2)](y_1,y_2)$$

for s-a.e. $(y_1,y_2)\in B^2$.

Proof. Using the identity $\cos\theta=\frac{1}{2}(e^{i\theta}+e^{-i\theta})$ and the linearity (2.4) of the Fourier-Feynman transform $T_{\vec{q}}^{(p)}$, we get

$$\begin{aligned}&T_{\vec{q}}^{(p)}\Big[F(\cdot,\cdot)\cos\Big(\sum_{j=1}^{2}q_j(w_j,\cdot)^{\sim}\Big)\Big](y_1,y_2)\\&=\frac{1}{2}\Big(T_{\vec{q}}^{(p)}\Big[F(\cdot,\cdot)\exp\Big\{i\sum_{j=1}^{2}q_j(w_j,\cdot)^{\sim}\Big\}\Big](y_1,y_2)\\&\quad+T_{\vec{q}}^{(p)}\Big[F(\cdot,\cdot)\exp\Big\{-i\sum_{j=1}^{2}q_j(w_j,\cdot)^{\sim}\Big\}\Big](y_1,y_2)\Big).\end{aligned}$$

Now by the time shifting theorem or frequency shifting theorem we obtain (3.3). Using the identity $\sin\theta=\frac{1}{2i}(e^{i\theta}-e^{-i\theta})$ the second conclusion is proved similarly. □

Since the Dirac measure concentrated at $h=0$ in H is a complex Borel measure, the constant function $F\equiv 1$ belongs to $\mathcal{F}_{A_1,A_2}$. Hence we have the following corollary.

Corollary 3.4. *Let* $(w_1,w_2)\in B^2$. *Then we have*

$$\begin{aligned}&T_{\vec{q}}^{(p)}\Big[\cos\Big(\sum_{j=1}^{2}q_j(w_j,\cdot)^{\sim}\Big)\Big](y_1,y_2)\\&=\cos\Big(\sum_{j=1}^{2}q_j(w_j,y_j)^{\sim}\Big)\exp\Big\{-\frac{i}{2}\sum_{j=1}^{2}q_j|w_j|^2\Big\}\end{aligned} \qquad (3.6)$$

and

$$T_{\vec{q}}^{(p)}\Big[\sin\Big(\sum_{j=1}^{2} q_j(w_j,\cdot)^{\sim}\Big)\Big](y_1,y_2) \tag{3.7}$$
$$= \sin\Big(\sum_{j=1}^{2} q_j(w_j,y_j)^{\sim}\Big)\exp\Big\{-\frac{i}{2}\sum_{j=1}^{2} q_j|w_j|^2\Big\}$$

for s-a.e. $(y_1,y_2)\in B^2$.

Proof. Since

$$T_{\vec{q}}^{(p)}[F(\cdot - w_1,\cdot - w_2)](y_1,y_2) = T_{\vec{q}}^{(p)}[F](y_1-w_1,y_2-w_2)=1$$

for $F\equiv 1$, by the modulation property Theorem 3.3 and Euler's formula, the results follow immediately. □

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