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ON HYPERHOLOMORPHIC $F^{\alpha}_{\omega,G}(p,q,s)$ SPACES OF QUATERNION VALUED FUNCTIONS

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Abstract. The purpose of this paper is to define a new class of hyperholomorphic functions spaces, which will be called $F^{\alpha}_{\omega,G}(p,q,s)$ type spaces. For this class, we characterize hyperholomorphic weighted α-Bloch functions by functions belonging to $F^{\alpha}_{\omega,G}(p,q,s)$ spaces under some mild conditions. Moreover, we give some essential properties for the extended weighted little α-Bloch spaces. Also, we give the characterization for the hyperholomorphic weighted Bloch space by the integral norms of $F^{\alpha}_{\omega,G}(p,q,s)$ spaces of hyperholomorphic functions. Finally, we will give the relation between the hyperholomorphic $\mathcal{B}^{\alpha}_{\omega,0}$ type spaces and the hyperholomorphic valued-functions space $F^{\alpha}_{\omega,G}(p,q,s)$.

1. Introduction

Quaternions were introduced for the first time by William Rowan Hamilton in 1843. Quaternion analysis is the generalizations of the theory of holomorphic functions in one complex variable to Euclidean space. The concept of the hyperholomorphic functions based on the consideration of functions in the kernel of the generalized Cauchy-Riemann operator. Quaternions are also recognized as a powerful tool for modeling and solving problems in theoretical as well as applied mathematics (see [14]). The emergence of a large of software packages to perform

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computations in the algebra of the real quaternions (see [13]), or more generally, in Clifford Algebra has been enhanced by the increasing interest in using quaternions and their applications in almost all applied sciences (see [1, 2]).

2. Preliminaries

2.1. Analytic function spaces. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the complex unit disk. The well known Bloch space is defined by:

$$\mathcal{B} = \{f : f \text{ analytic in } \mathbb{D} \text{ and } \mathcal{B}(f) = \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty\}.$$

Composing the Möbius transform $\varphi_a(z)$, which maps the unit disk $\mathbb{D}$ onto itself, and the fundamental solution of the two-dimensional real Laplacian on $\mathbb{D}$, we have the Green's function $g(z, a) = \ln|\frac{1-\bar{a}z}{a-z}|$ with logarithmic singularity at $a \in \mathbb{D}$. Here, φ_a always stands for the Möbius transformation $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. Then, in [17] Zhao gave the following definition:

Definition 2.1. Let f be an analytic function in $\mathbb{D}$ and let $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. If

$$\|f\|^p_{F(p,q,s)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p(1 - |z|^2)^q g^s(z, a)dA(z) < \infty,$$

then $f \in F(p, q, s)$. Moreover, if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p(1 - |z|^2)^q g^s(z, a)dA(z) = 0,$$

then $f \in F_0(p, q, s)$.

Definition 2.2. (see [4]) Let a right-continuous and nondecreasing function $\omega : (0, 1] \to (0, \infty)$, the weighted Bloch space $\mathcal{B}_\omega$ is defined as the set of all analytic functions f on $\mathbb{D}$ satisfying

$$(1 - |z|)|f'(z)| \le C\omega(1 - |z|), \quad z \in \mathbb{D},$$

for some fixed $C = C_f > 0$. In the special case where $\omega \equiv 1$, $\mathcal{B}_\omega$ reduces to the classical Bloch space $\mathcal{B}$.

Definition 2.3. (see [15]) Let $0 < \alpha < \infty$ and $\omega : (0,1] \to (0,\infty)$. For an analytic function f in $\mathbb{D}$, we define the weighted α-Bloch space $\mathcal{B}^{\alpha}_{\omega}$, as follows:

$$\mathcal{B}^{\alpha}_{\omega} = \left\{ f : f \text{ analytic in } \mathbb{D} \text{ and } \|f\|_{\mathcal{B}^{\alpha}_{\omega}} = \sup_{z\in\mathbb{D}} \frac{(1-|z|)^{\alpha}\big|f'(z)\big|}{\omega(1-|z|)} < \infty \right\}.$$

Also, the little weighted α-Bloch space $\mathcal{B}^{\alpha}_{\omega,0}$ is a subspace of $\mathcal{B}^{\alpha}_{\omega}$ consisting of all $f \in \mathcal{B}^{\alpha}_{\omega}$, such that [7, 9]

$$\lim_{|z|\to 1^-} \frac{(1-|z|)^{\alpha}\big|f'(z)\big|}{\omega(1-|z|)} = 0.$$

2.2. Quaternion function spaces. To introduce the meaning of hyperholomorphic functions, let $\mathbb{H}$ be the skew field of quaternions. This means we can write each element $w \in \mathbb{H}$ in the form

$$w = w_0 + w_1 i + w_2 j + w_3 k, \quad w_0, w_1, w_2, w_3 \in \mathbb{R},$$

where $1, i, j, k$ are the basis elements of $\mathbb{H}$. For these elements we have the multiplication rules

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, kj = -jk = i, ki = -ik = j.$$

The conjugate element $\bar{w}$ is given by $\bar{w} = w_0 - w_1 i - w_2 j - w_3 k$. Then, we have the property

$$w\bar{w} = \bar{w}w = \|w\|^2 = w_0^2 + w_1^2 + w_2^2 + w_3^2.$$

Moreover, we can identify each vector $\vec{x} = (x_0, x_1, x_2) \in \mathbb{R}^3$ with a quaternion x of the form

$$x = x_0 + x_1 i + x_2 j.$$

In follows we will work in $\mathbb{B}_1(0) \subset \mathbb{R}^3$, the unit ball in the real three-dimensional space. We will consider functions f defined on $\mathbb{B}_1(0)$ with values in $\mathbb{H}$. We define a generalized Cauchy-Riemann operator D by

$$Df = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2},$$

and it's conjugate operator by

$$\overline{D}f = \frac{\partial f}{\partial x_0} - i\frac{\partial f}{\partial x_1} - j\frac{\partial f}{\partial x_2}.$$

For these operators, we have that

$$D\overline{D} = \overline{D}D = \Delta_3,$$

where Δ_3 is the Laplacian for functions defined over domains in $\mathbb{R}^3$. For $|a| < 1$, we will denote by

$$\varphi_a(x) = (a - x)(1 - \bar{a}x)^{-1}$$

the Möbius transform, which maps the unit ball onto itself. Furthermore, let

$$g(x, a) = \frac{1}{4\pi}\left(\frac{1}{|\varphi_a(x)|} - 1\right),$$

be the modified fundamental solution of the Laplacian in $\mathbb{R}^3$ composed with the Möbius transform $\varphi_a(x)$. Especially, we denote for all $p \geq 0$

$$g^p(x, a) = \frac{1}{4^p\pi^p}\left(\frac{1}{|\varphi_a(x)|} - 1\right)^p.$$

Let $f : \mathbb{B} \mapsto \mathbb{H}$ be a hyperholomorphic function. Then from [10], we have the seminorms

- $\mathcal{B}(f) = \sup\limits_{x\in\mathbb{B}}(1 - |x|^2)^{3/2}|\overline{D}f(x)|$,
- $Q_p(f) = \sup\limits_{a\in\mathbb{B}} \int_{\mathbb{B}} |\overline{D}f(x)|^2 g^p(x, a) d\mathbb{B}_x$.

Definition 2.4. Let $0 < \alpha < \infty$. Recall that the hyperholomorphic α-Bloch space (see [5]) is defined as follows:

$$\mathcal{B}^\alpha = \{f \in \ker D : \sup_{x\in\mathbb{B}}(1 - |x|^2)^{\frac{3\alpha}{2}}|\overline{D}f(x)| < \infty\},$$

the little α-Bloch type space $\mathcal{B}_0^\alpha$ is a subspace of $\mathcal{B}$ consisting of all $f \in \mathcal{B}^\alpha$ such that

$$\lim_{|x|\to 1^-}(1 - |x|^2)^{\frac{3\alpha}{2}}|\overline{D}f(x)| = 0.$$

Quite recently, El-Sayed Ahmed and Omran in [6], gave the following definition:

Definition 2.5. Let f be quaternion-valued function in $\mathbb{B}$. For $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. If

$$\|f\|^p_{F(p,q,s)} = \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3q}{2}} \left(1 - |\varphi_a(x)|^2\right)^s d\mathbb{B}_x < \infty,$$

then $f \in F(p, q, s)$. Moreover, if

$$\lim_{|a|\to 1} \int_{\mathbb{B}} |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3q}{2}} \left(1 - |\varphi_a(x)|^2\right)^s d\mathbb{B}_x = 0,$$

then $f \in F_0(p, q, s)$.

Ahmed and asiri in [8], gave the following definition:

Definition 2.6. For given right-continuous and nondecreasing function $\omega : (0,1] \to (0,\infty)$, and $0 < \alpha < \infty$. A quaternion-valued function f on $\mathbb{B}_1(0)$ is belong to the weighted α- Bloch space $\mathcal{B}^{\alpha}_{\omega}$, if

$$\|f\|_{\mathcal{B}^{\alpha}_{\omega}} = \sup_{x\in\mathbb{B}_1(0)} \frac{(1-|x|^2)^{\frac{3\alpha}{2}}}{\omega(1-|x|)} |\overline{D}f(x)| < \infty.$$

Moreover, A quaternion-valued function $f \in \mathcal{B}^{\alpha}_{\omega,0}$, if

$$\|f\|_{\mathcal{B}^{\alpha}_{\omega,0}} = \lim_{|x|\to 1^-} \frac{(1-|x|^2)^{\frac{3\alpha}{2}}}{\omega(1-|x|)} |\overline{D}f(x)| < \infty.$$

Now, we use the definition of green function in $\mathbb{R}^3$ (see [3])

$$G(x,a) = \frac{1-|\varphi_a(x)|^2}{|1-\overline{a}x|}.$$

Then, we introduce the following new definition of the so called the hyperholomorphic $F^{\alpha}_{\omega,G}(p,q,s)$ spaces.

Definition 2.7. Let $1 < \alpha,\ p < \infty$, $-2 < q < \infty$, $s > 0$, and $\omega : (0,1] \to (0,\infty)$. Assume that f be hyperholomorphic function in the unit ball $\mathbb{B}_1(0)$. Then, $f \in F^{\alpha}_{\omega,G}(p,q,s)$, if

$$\begin{aligned} &F^{\alpha}_{\omega,G}(p,q,s) \\ = &\left\{ f \in kerD : \sup_{a\in\mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p \frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)} \big(G(x,a)\big)^s d\mathbb{B}_x < \infty \right\}. \end{aligned}$$

The space $F^{\alpha}_{\omega,G,0}(p,q,s)$ is subspace of $F^{\alpha}_{\omega,G}(p,q,s)$ consisting of all functions $f \in F^{\alpha}_{\omega,G}(p,q,s)$, such that

$$\lim_{|a|\to 1^-} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p \frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)} \big(G(x,a)\big)^s d\mathbb{B}_x = 0.$$

Our objective in this article is twofold. First, we study the generalized quaternion spaces $F^{\alpha}_{\omega,G}(p,q,s)$. Second, we characterize their relations to the quaternion $\mathcal{B}^{\alpha}_{\omega,0}$ space. The following lemma, we will need in the sequel:

Lemma 2.1. (see [16]). *Let* $f : \mathbb{B}_1(0) \longrightarrow \mathbb{H}$ *be a hyperholomorphic function. Let* $0 < R < 1$, $1 < q$. *Then for every* $a \in \mathbb{B}_1(0)$

$$|\overline{D}f(a)|^q \le \frac{3\cdot 4^{2+q}}{\pi R^3(1-R^2)^{2q}(1-|a|^2)^3} \int_{\mathcal{M}(a,R)} \left|\overline{D}f(x)\right|^q d\mathbb{B}_x \ .$$

3. Characterization of $F^{\alpha}_{\omega,G}(p,q,s)$ spaces in Clifford Analysis

The relations between $F^{\alpha}_{\omega,G}(p,q,s)$ space and $\mathcal{B}^{\alpha}_{\omega}$ spaces are given in quaternion sense. The results in this section are extensions and generalize of the results (see [10, 11]).

Proposition 3.1. *Let α, $p \geq 1$, $-2 < q < \infty$, $s > 0$, $0 < R < 1$, and $\omega : (0,1] \to (0,\infty)$. Assume that f be hyperholomorphic function in the unit ball $\mathbb{B}_1(0)$. Then*

$$\frac{(1-|a|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|a|)}|\overline{D}f(a)|^p$$
$$\leq \frac{48(2)^{2p}(1+R)^s}{\pi R^3(1-R^2)^{2ps+s}(1-|a|^2)^3}\int_{\mathbb{B}_1(0)}|\overline{D}f(x)|^p\frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|a|)}\big(G(x,a)\big)^s d\mathbb{B}_x.$$

Proof. Let $\mathcal{M}(a,R) = \{x \in \mathbb{B}_1(0) : |\varphi_a(x)| = \frac{|a-x|}{|1-\overline{a}x|} < R\}$ be pseudo-hyperbolic ball with center a and radius R. Then

$$\int_{\mathbb{B}_1(0)}|\overline{D}f(x)|^p\frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)}\big(G(x,a)\big)^s d\mathbb{B}_x$$
$$\geq \int_{\mathcal{M}(a,R)}|\overline{D}f(x)|^p\frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)}\big(G(x,a)\big)^s d\mathbb{B}_x.$$

Since

$$|\varphi_a(x)| < R, \quad \forall x \in \mathcal{M}(a,R),$$

and

$$G(x,a) = \frac{1-|\varphi_a(x)|^2}{|1-\overline{a}x|}.$$

Then, we have

$$G(x,a) = \frac{1-R^2}{1+R}, \quad \textit{where } 1-R \leq |1-\overline{a}x| \leq 1+R.$$

Now, for fixed $R \in (0,1)$ and $a \in \mathbb{B}_1(0)$.
Let $\mathcal{E}(a,R) \subset \mathcal{M}(a,R)$, sutch that

$$\mathcal{E}(a,R) = \{x \in \mathbb{B}_1(0) : |x-a| < R|1-a|\}.$$

Then, we deduce that

$$\begin{aligned}
&\int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p \frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)} \big(G(x,a)\big)^s d\mathbb{B}_x \\
\geq\ & \left(\frac{1-R^2}{1+R}\right)^s \int_{\mathcal{M}(a,R)} |\overline{D}f(x)|^p \frac{(1-|x|^2)^{\frac{3q}{2}+2s}}{\omega^p(1-|x|)} d\mathbb{B}_x \\
\geq\ & \left(\frac{1-R^2}{1+R}\right)^s \int_{\mathcal{E}(a,R)} |\overline{D}f(x)|^p \frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)} d\mathbb{B}_x \\
\geq\ & \left(\frac{1-R^2}{1+R}\right)^s \frac{(1-|a|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|a|)} \int_{\mathcal{E}(a,R)} |\overline{D}f(x)|^p d\mathbb{B}_x.
\end{aligned}$$

Now, using Lemma 3.1, we obtain

$$\begin{aligned}
&\int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p \frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)} \big(G(x,a)\big)^s d\mathbb{B}_x \\
\geq\ & \left(\frac{1-R^2}{1+R}\right)^s \frac{(1-|a|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|a|)} \frac{\pi R^3(1-R^2)^{2p}(1-|a|^2)^3}{3(4)^{p+2}} |\overline{D}f(a)|^p \\
=\ & \frac{\pi R^3(1-R^2)^{2p+s}(1-|a|^2)^{\frac{3\alpha q}{2}+2s+3}}{3(4)^{p+2}(1+R)^s\omega^p(1-|a|)} |\overline{D}f(a)|^p,
\end{aligned}$$

which implies that,

$$\begin{aligned}
&\frac{(1-|a|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|a|)} |\overline{D}f(a)|^p \\
&\leq \frac{48(2)^{2p}(1+R)^s}{\pi R^3(1-R^2)^{2p+s}(1-|a|^2)^3} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p \frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)} \big(G(x,a)\big)^s d\mathbb{B}_x.
\end{aligned}$$

This completes the proof. □

Corollary 3.1. *From proposition 3.1, we get for* $\alpha,\ p \geq 1,\ -2 < q < \infty,\ s > 0,$ *and* $\omega : (0,1] \to (0,\infty)$ *that*

$$F^{\alpha}_{\omega,G}(p,q,s) \subset \mathcal{B}_{\omega}^{\frac{\alpha q+2s}{p}}.$$

Proposition 3.2. *Let* $\alpha,\ p \geq 1,\ -2 < q < \infty,\ s > 2,$ *and* $\omega : (0,1] \to (0,\infty)$. *Let* f *be a hyperholomorphic function in* $B_1(0)$, $\forall a \in B_1(0)$;

$|a|<1$ and $f \in \mathcal{B}_\omega^{\frac{\alpha q+2s}{p}}$. Then, we have that

$$\int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p \frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)} \big(G(x,a)\big)^s d\mathbb{B}_x \leq k(\mathcal{B}_\omega^\gamma(f)),$$

where $\gamma = \frac{\alpha q+2s}{p}$.

Proof. From (see [8, 10]), we have

$$\frac{(1-|x|^2)^{\frac{3\alpha}{2}}}{\omega^p(1-|x|)} |\overline{D}f(a)|^p \leq \mathcal{B}_\omega^\alpha(f).$$

Then,

$$\begin{aligned} &\int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p \frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)} \big(G(x,a)\big)^s d\mathbb{B}_x \\ \leq\ & (\mathcal{B}_\omega^\gamma(f)) \int_{\mathbb{B}_1(0)} \big(G(x,a)\big)^s d\mathbb{B}_x. \end{aligned}$$

Using the equality

$$G(x,a) = \frac{1-|\varphi_a(x)|^2}{|1-\overline{a}x|} = \frac{(1-|a|^2)(1-|x|^2)}{|1-\overline{a}x|^3}, \tag{1}$$

where

$$1-|x| \leq |1-\overline{a}x| \leq 1+|x|, \quad 1-|a| \leq |1-\overline{a}x| \leq 1+|a| \leq 2. \tag{2}$$

Then, we get

$$\begin{aligned} &\int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p \frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)} \big(G(x,a)\big)^s d\mathbb{B}_x \\ \leq\ & (\mathcal{B}_\omega^\gamma(f)) \int_{\mathbb{B}_1(0)} \frac{(1-|a|^2)^s(1-|x|^2)^s}{|1-\overline{a}x|^{3s}} d\mathbb{B}_x. \\ \leq\ & 2^{2s}(\mathcal{B}_\omega^\gamma(f)) \int_{\mathbb{B}_1(0)} \frac{(1-|a|)^s(1-|x|)^s}{(1-|x|^s)(1-|a|)^{2s}} d\mathbb{B}_x. \\ \leq\ & 2^{2s}(\mathcal{B}_\omega^\gamma(f)) \int_{\mathbb{B}_1(0)} \frac{1}{(1-|a|)^s} d\mathbb{B}_x. \\ \leq\ & k(\mathcal{B}_\omega^\gamma(f)). \end{aligned}$$

Therefore, the proof of proposition is complete. □

Corollary 3.2. *From proposition 3.2, we get for* $\alpha,\ p\geq 1,\ -2<q<\infty,\ s>2$, *and* $\omega:(0,1]\to(0,\infty)$ *that*

$$\mathcal{B}_{\omega}^{\frac{\alpha q+2s}{p}}\subset F^{\alpha}_{\omega,G}(p,q,s).$$

The results in Corollary 3.1 and Corollary 3.2 prove the following theorem, which give to us the characterization for the hyperholomorphic weighted Bloch space by the integral norms of $F^{\alpha}_{\omega,G}(p,q,s)$ spaces of hyperholomorphic functions.

Theorem 3.1. *Let f be a hyperholomorphic function in $B_1(0)$. Then for* $\alpha,\ p\geq 1,\ -2<q<\infty,\ s>2$, *and* $\omega:(0,1]\to(0,\infty)$, *we have*

$$\mathcal{B}_{\omega}^{\frac{\alpha q+2s}{p}}= F^{\alpha}_{\omega,G}(p,q,s).$$

For characterization the little hyperholomorphic weighted Bloch space, used the the same arguments in the previous theorem to prove the following theorem.

Theorem 3.2. *Let f be a hyperholomorphic function in $B_1(0)$. Then for* $\alpha,\ p\geq 1,\ -2<q<\infty,\ s>2$, *and* $\omega:(0,1]\to(0,\infty)$, *we have*

$$\mathcal{B}_{\omega,0}^{\frac{\alpha q+2s}{p}}= F^{\alpha}_{\omega,G,0}(p,q,s).$$

Theorem 3.3. *Let* $0<R<1$ *and* $\omega:(0,1]\to(0,\infty)$. *Then for the hyperholomorphic function f in $\mathbb{B}_1(0)$, the following are equivalent*

(a) $f\in\mathcal{B}_{\omega}^{\alpha q+2s}$.

(b) For each $-2<q<\infty,\ 1\leq\alpha<\infty$, *and* $0<p<\infty$

$$\sup_{a\in\mathbb{B}_1(0)}\int_{\mathbb{B}_1(0)}|\overline{D}f(x)|^p\frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)}\big(G(x,a)\big)^s d\mathbb{B}_x<+\infty.$$

(c) For each $-2<q<\infty,\ 1\leq\alpha<\infty$, *and* $0<p<\infty$

$$\sup_{a\in\mathbb{B}_1(0)}\int_{\mathcal{M}(a,R)}|\overline{D}f(x)|^p\frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)}d\mathbb{B}_x<+\infty.$$

(d) For each $-2<q<\infty,\ 1\leq\alpha<\infty$, *and* $0<p<\infty$

$$\sup_{a\in\mathbb{B}_1(0)}\frac{|\mathcal{M}(a,R)|^{\frac{\alpha q}{2}+\frac{2s}{3}}}{\omega^p(1-|a|)}\int_{\mathcal{M}(a,R)}|\overline{D}|f(x)|^p d\mathbb{B}_x<+\infty.$$

Proof. To prove (a) implies (b). Using (1) and (2), we have

$$\begin{aligned}
&\sup_{a\in\mathbb{B}_1(0)}\int_{\mathbb{B}_1(0)}|\overline{D}f(x)|^p\frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)}\big(G(x,a)\big)^s d\mathbb{B}_x\\
\leq\ &\sup_{a\in\mathbb{B}_1(0)}|\overline{D}f(x)|^p\frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)}\int_{\mathbb{B}_1(0)}\big(G(x,a)\big)^s d\mathbb{B}_x.
\end{aligned}$$

Using the same steps as in Proposition 3.2, we obtain

$$\begin{aligned}
&\sup_{a\in\mathbb{B}_1(0)}\int_{\mathbb{B}_1(0)}|\overline{D}f(x)|^p\frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)}\big(G(x,a)\big)^s d\mathbb{B}_x\\
\leq\ &2^{2s}\|f\|^p_{\mathcal{B}_\omega^{\frac{\alpha q+2s}{p}}}\int_{\mathbb{B}_1(0)}\frac{1}{(1-|a|)^s}d\mathbb{B}_x\\
\leq\ &K_1\|f\|^p_{\mathcal{B}_\omega^{\frac{\alpha q+2s}{p}}}\\
<\ &\infty.
\end{aligned}$$

(b) implies (c), using the same steps as in Proposition 3.1, we deduce that

$$\begin{aligned}
&\sup_{a\in\mathbb{B}_1(0)}\int_{\mathbb{B}_1(0)}|\overline{D}f(x)|^p\frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)}\big(G(x,a)\big)^s d\mathbb{B}_x\\
\geq\ &\left(\frac{1-R^2}{2}\right)^s\sup_{a\in\mathbb{B}_1(0)}\int_{\mathcal{M}(a,R)}|\overline{D}f(x)|^p\frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)}d\mathbb{B}_x.
\end{aligned}$$

For (c) implies (d), we use the fact $(1-|x|^2)^3\approx|\mathcal{M}(a,R)|$, $\forall\ x\in\mathcal{M}(a,R)$ (see [12]). Then

$$\begin{aligned}
&\sup_{a\in\mathbb{B}_1(0)}\int_{\mathcal{M}(a,R)}|\overline{D}f(x)|^p\frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)}d\mathbb{B}_x\\
\approx\ &\frac{|\mathcal{M}(a,R)|^{\frac{\alpha q}{2}+\frac{2s}{3}}}{\omega^p(1-|a|)}\sup_{a\in\mathbb{B}_1(0)}\int_{\mathcal{M}(a,R)}|\overline{D}f(x)|^p d\mathbb{B}_x.
\end{aligned}$$

For (d) implies (a). Form Lemma 2.1, we have

$$
\begin{aligned}
&|\overline{D}f(a)|^p\frac{(1-|a|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|a|)}\\
\leq\;& \frac{3\cdot 4^{2+p}(1-|a|^2)^{\frac{3\alpha q}{2}+2s}}{\pi R^3(1-R^2)^{2p}(1-|a|^2)^3\omega^p(1-|a|)}\int_{\mathcal{M}(a,R)}\left|\overline{D}f(x)\right|^p d\mathbb{B}_x\\
=\;& \frac{3\cdot 4^{2+p}(1-|a|^2)^{\frac{3\alpha q}{2}+2s}}{\pi R^3(1-R^2)^{2p}(1-|a|^2)^3\omega^p(1-|a|)}\\
&\times\frac{(1-R^2|a|^2)^{\frac{3\alpha q}{2}+2s}R^{\frac{3\alpha q}{2}+2s}}{(1-R^2|a|^2)^{\frac{3\alpha q}{2}+2s}R^{\frac{3\alpha q}{2}+2s}}\int_{\mathcal{M}(a,R)}\left|\overline{D}f(x)\right|^p d\mathbb{B}_x.
\end{aligned}
$$

Now, since

$$
|\mathcal{M}(a,R)|=\frac{(1-|a|^2)^3}{(1-R^2|a|^2)^3}R^3.
$$

Also, we used the following inequalities

$$
1-R^2\leq 1-R^2|a|^2\leq 1+R^2 \quad and \quad 1-|a|^2\leq 1-R^2|a|^2\leq 1+|a|^2.
$$

Then, we have

$$
\begin{aligned}
&|\overline{D}f(a)|^p\frac{(1-|a|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|a|)}\\
\leq\;& \frac{3\cdot 4^{2+p}|\mathcal{M}(a,R)|^{\frac{\alpha q}{2}+\frac{2s}{3}}}{\pi R^3(1-R^2)^{2p}(1-|a|^2)^3\omega^p(1-|a|)}\\
&\quad\times\frac{(1-R^2|a|^2)^{\frac{3\alpha q}{2}+2s}}{R^{\frac{3\alpha q}{2}+2s}}\int_{\mathcal{M}(a,R)}\left|\overline{D}f(x)\right|^p d\mathbb{B}_x\\
\leq\;& \frac{3\cdot 4^{2+p}|\mathcal{M}(a,R)|^{\frac{\alpha q}{2}+\frac{2s}{3}}}{\pi R^3(1-R^2)^{2p}(1-R^2)^3\omega^p(1-|a|)}\\
&\quad\times\frac{(1+R^2)^{\frac{3\alpha q}{2}+2s}}{R^{\frac{3\alpha q}{2}+2s}}\int_{\mathcal{M}(a,R)}\left|\overline{D}f(x)\right|^p d\mathbb{B}_x\\
=\;& \frac{3\cdot 4^{2+p}(1+R^2)^{3(\frac{\alpha q}{2}+\frac{2s}{3})}}{\pi R^{3(1+\frac{\alpha q}{2}+\frac{2s}{3})}(1-R^2)^{2p+3}}\times\frac{|\mathcal{M}(a,R)|^{\frac{\alpha q}{2}+\frac{2s}{3}}}{\omega^p(1-|a|)}\int_{\mathcal{M}(a,R)}\left|\overline{D}f(x)\right|^p d\mathbb{B}_x.
\end{aligned}
$$

Therefore, our theorem is proved. □

From Theorem 3.3, using the same arguments, we directly obtain the following theorem.

THEOREM 3.4. *Let $0 < R < 1$ and $\omega : (0,1] \to (0,\infty)$. Then for the hyperholomorphic function f in $\mathbb{B}_1(0)$, the following are equivalent*

(a) $f \in \mathcal{B}^{\alpha q+2s}_{\omega,0}$.

(b) For each $-2 < q < \infty$, $1 \le \alpha < \infty$, *and* $0 < p < \infty$

$$\lim_{|a|\to 1^-} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p \frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)} \big(G(x,a)\big)^s d\mathbb{B}_x = 0.$$

(c) For each $-2 < q < \infty$, $1 \le \alpha < \infty$, *and* $0 < p < \infty$

$$\lim_{|a|\to 1^-} \int_{\mathcal{M}(a,R)} |\overline{D}f(x)|^p \frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)} d\mathbb{B}_x = 0.$$

(d) For each $-2 < q < \infty$, $1 \le \alpha < \infty$, *and* $0 < p < \infty$

$$\lim_{|a|\to 1^-} \frac{|\mathcal{M}(a,R)|^{\frac{\alpha q}{2}+\frac{2s}{3}}}{\omega^p(1-|a|)} \int_{\mathcal{M}(a,R)} |\overline{D}|f(x)|^p d\mathbb{B}_x = 0.$$

The following theorem give another relation between the quaternion $\mathcal{B}^{\alpha}_{\omega}$ space and the quaternion valued-functions space $F^{\alpha}_{\omega,G}(p,q,s)$.

THEOREM 3.5. *Let f be a hyperholomorphic function in $B_1(0)$. Then for α, $p \ge 1$, $-2 < q < \infty$, $\beta, s > 0$, and $\omega : (0,1] \to (0,\infty)$, we have*

$$\|f\|^p_{\mathcal{B}_\omega^{\frac{\alpha q+2s}{p}}} \approx \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^p \frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)} (1-|\varphi_a(x)|^2)^{\beta} \big(G(x,a)\big)^s d\mathbb{B}_x.$$

Proof. From Theorem 3.3, we have

$$\|f\|^p_{\mathcal{B}_\omega^{\frac{\alpha q+2s}{p}}} \approx \int_{\mathcal{M}(a,R)} |\overline{D}f(x)|^p \frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)} d\mathbb{B}_x.$$

Let the constant

$$C(R) = (1-R^2)^{\beta} \left(\frac{1-R^2}{1+R}\right)^s,$$

since $C(R)$ depending on R is finite, then

$$\|f\|^p_{\mathcal{B}_\omega^{\frac{\alpha q+2s}{p}}} \tag{3}$$
$$\approx \sup_{a\in\mathbb{B}_1(0)}\int_{\mathcal{M}(a,R)}|\overline{D}f(x)|^p\frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)}(1-R^2)^{\beta}\left(\frac{1-R^2}{1+R}\right)^s d\mathbb{B}_x.$$

Since $x\in\mathcal{M}(a,R)$, then $|\varphi_a(x)|<R$, $|\varphi_a(x)|^2<R^2$, and $1-|\varphi_a(x)|^2>1-R^2$.

Then, we have

$$\|f\|^p_{\mathcal{B}_\omega^{\frac{\alpha q+2s}{p}}} \tag{4}$$
$$\approx \sup_{a\in\mathbb{B}_1(0)}\int_{\mathcal{M}(a,R)}|\overline{D}f(x)|^p\frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)}(1-R^2)^{\beta}\left(\frac{1-R^2}{1+R}\right)^s d\mathbb{B}_x$$
$$\leq \lambda\sup_{a\in\mathbb{B}_1(0)}\int_{\mathbb{B}_1(0)}|\overline{D}f(x)|^p\frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)}(1-|\varphi_a(x)|^2)^{\beta}\big(G(x,a)\big)^s d\mathbb{B}_x.$$

Conversely, we have

$$\sup_{a\in\mathbb{B}_1(0)}\int_{\mathbb{B}_1(0)}|\overline{D}f(x)|^p\frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)}(1-|\varphi_a(x)|^2)^{\beta}\big(G(x,a)\big)^s d\mathbb{B}_x$$
$$\leq \|f\|^p_{\mathcal{B}_\omega^{\frac{\alpha q+2s}{p}}}\sup_{a\in\mathbb{B}_1(0)}\int_{\mathbb{B}_1(0)}(1-|\varphi_a(x)|^2)^{\beta}\big(G(x,a)\big)^s d\mathbb{B}_x. \tag{5}$$

Using (1) and (2), we obtain

$$\sup_{a\in\mathbb{B}_1(0)}\int_{\mathbb{B}_1(0)}|\overline{D}f(x)|^p\frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)}(1-|\varphi_a(x)|^2)^{\beta}\big(G(x,a)\big)^s d\mathbb{B}_x$$
$$\leq \|f\|^p_{\mathcal{B}_\omega^{\frac{\alpha q+2s}{p}}}\sup_{a\in\mathbb{B}_1(0)}\int_{\mathbb{B}_1(0)}\frac{(1-|x|^2)^{\beta}(1-|a|^2)^{\beta}}{(1-|x|)^{\beta}}\frac{(1-|x|^2)^{s}(1-|a|^2)^{s}}{(1-|a|)^{2s}(1-|x|)^{s}}d\mathbb{B}_x.$$
$$\leq 2^{2(\beta+s)}\|f\|^p_{\mathcal{B}_\omega^{\frac{\alpha q+2s}{p}}}\sup_{a\in\mathbb{B}_1(0)}\int_{\mathbb{B}_1(0)}(1-|a|)^{\beta-s}d\mathbb{B}_x. \tag{6}$$

Then, we obtain

$$\|f\|^p_{\mathcal{B}_\omega^{\frac{\alpha q+2s}{p}}} \tag{7}$$
$$\geq \lambda\sup_{a\in\mathbb{B}_1(0)}\int_{\mathbb{B}_1(0)}\overline{D}|f(x)|^p\frac{(1-|x|^2)^{\frac{3\alpha q}{2}+2s}}{\omega^p(1-|x|)}(1-|\varphi_a(x)|^2)^{\beta}\big(G(x,a)\big)^s d\mathbb{B}_x.$$

From (4) and (7) the proof is complete.

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