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A FEW RESULTS ON JANOWSKI FUNCTIONS ASSOCIATED WITH k-SYMMETRIC POINTS

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Abstract. The purpose of the present paper is to introduce and study new subclasses of analytic functions which generalize the classes of Janowski functions with respect to k-symmetric points. We also study certain interesting properties like covering theorem, convolution condition, neighborhood results and argument theorem.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all function which are univalent in $\mathcal{U}$.

For f and g be analytic in $\mathcal{U}$, we say that the function f is subordinate to g in $\mathcal{U}$, if there exists an analytic function w in $\mathcal{U}$ such that $|w(z)| < 1$ with $w(0) = 0$, and $f(z) = g(w(z))$, and we denote this by $f(z) \prec g(z)$. If g is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = g(0)$

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and $f(\mathcal{U}) \subset g(\mathcal{U})$. The convolution or Hadamard product of two analytic functions $f, g \in \mathcal{A}$ where f is defined by (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, is

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

For any $f \in \mathcal{A}$, ρ-neighborhood of $f(z)$ can be defined as:

$$\mathcal{N}_\rho(f) = \left\{ g \in \mathcal{A} : g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \ \sum_{n=2}^{\infty} n|a_n - b_n| \leq \rho \right\}. \tag{1.2}$$

For $e(z) = z$, we can see that

$$\mathcal{N}_\rho(e) = \left\{ g \in \mathcal{A} : g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \ \sum_{n=2}^{\infty} n|b_n| \leq \rho \right\}. \tag{1.3}$$

The idea of neighborhoods was first introduced by Goodman [14] which was further generalized by Ruscheweyh [11]. He also proved that if $f \in \mathcal{A}$, $\rho > 0$ and η is a complex number with $|\eta| < \rho$, and

$$\frac{f(z) + \eta z}{1 + \eta} \in \mathcal{S}^*,$$

then $\mathcal{N}_\rho(f) \subset \mathcal{S}^*$. Where $\mathcal{S}^*$ is the class of starlike functions.

Using the principle of the subordination we define the class $\mathcal{P}$ of functions with positive real part.

Definition 1.1. [7] Let $\mathcal{P}$ denote the class of analytic functions of the form $p(z) = 1 + \sum\limits_{n=1}^{\infty} p_n z^n$ defined on $\mathcal{U}$ and satisfying $p(0) = 1$, $\operatorname{Re} p(z) > 0$, $z \in \mathcal{U}$.

Any function p in $\mathcal{P}$ has the representation $p(z) = \dfrac{1 + w(z)}{1 - w(z)}$, where $w \in \Omega$ and

$$\Omega = \{w \in \mathcal{A} : w(0) = 0, |w(z)| < 1\}. \tag{1.4}$$

The class $\mathcal{P}$ of functions with positive real part plays a crucial role in geometric function theory. Its significance can be seen from the fact that simple subclasses like class $\mathcal{S}^*$ of starlike, class $\mathcal{C}$ of convex functions, class of starlike functions with respect to symmetric points have been defined by using the concept of class of functions with positive real part.

Definition 1.2. [1] Let $\mathcal{P}[A,B]$, with $-1 \le B < A \le 1$, denote the class of analytic function p defined on $\mathcal{U}$ with the representation $p(z) = \dfrac{1+Aw(z)}{1+Bw(z)}$, $z \in \mathcal{U}$, where $w \in \Omega$.

Remark: $p \in \mathcal{P}[A,B]$ if and only if $p(z) \prec \dfrac{1+Az}{1+Bz}$.

In [6] the class $\mathcal{P}[A,B,\alpha]$ of generalized Janowski functions was introduced. For arbitrary numbers A, B, α, with $-1 \le B < A \le 1$, $0 \le \alpha < 1$, a function p analytic in $\mathcal{U}$ with $p(0) = 1$ is in the class $\mathcal{P}[A,B,\alpha]$ if and only if

$$p(z) \prec \frac{1+[(1-\alpha)A+\alpha B]z}{1+Bz} \Leftrightarrow p(z) = \frac{1+[(1-\alpha)A+\alpha B]w(z)}{1+Bw(z)},\ w \in \Omega.$$

The definition of starlike functions with respect to k-symmetric points is as follows.

Definition 1.3. For a positive integer k, let $\varepsilon = \exp\left(\frac{2\pi i}{k}\right)$ denote the k^{th} root of unity for $f \in \mathcal{A}$, let

$$M_{f,k}(z) = \sum_{v=1}^{k-1} \varepsilon^{-v} f(\varepsilon^v z).\frac{1}{\sum_{v=1}^{k-1}\varepsilon^{-v}}, \tag{1.5}$$

be its k-weighted mean function.

A function f in $\mathcal{A}$ is said to belong to the class $\mathcal{S}_k^*$ of functions starlike with respect to k-symmetric points if for every r close to 1 , $r < 1$, the angular velocity of f about the point $M_{f_k(z_0)}$ is positive at $z = z_0$ as z traverses the circle $|z| = r$ in the positive direction, that is $\Re\left\{\frac{zf'(z)}{f(z)-M_{f,k}(z_0)}\right\} > 0$ for $z = z_0$, $|z_0| = r$.

Definition 1.4. [12] A function f in $\mathcal{S}$ is starlike with respect to k-symmetric points, or briefly k-starlike if,

$$\Re\left\{\frac{zf'(z)}{f_k(z)}\right\} > 0,\ z \in \mathcal{U}, \tag{1.6}$$

where

$$f_k(z) = \frac{1}{k}\sum_{v=1}^{k-1} \varepsilon^{-v} f(\varepsilon^v z) \tag{1.7}$$

If $f(z)$ is defined by (1.1) then,

$$(1.8) \qquad f_k(z) = z + \sum_{n=2}^{\infty} \chi_n a_n z^n, \qquad (k = 2, 3, \dots).$$

$$(1.9) \qquad \chi_n = \begin{cases} 1, & n = lk + 1, \ l \in \mathbb{N}_0, \\ 0, & n \neq lk + 1. \end{cases}$$

Using the generalization of Janowski functions and the concept of k-symmetrical functions we define the following:

Definition 1.5. A function f in $\mathcal{A}$ is said to belong to the class $\mathcal{S}^k(A, B, \alpha)$,
$(-1 \leq B < A \leq 1), 0 \leq \alpha < 1$ if

$$\frac{zf'(z)}{f_k(z)} \prec \frac{1 + [(1-\alpha)A + \alpha B]z}{1 + Bz}, \ z \in \mathcal{U},$$

where $f_k(z)$ is defined by (1.8).

We note that for special values of k, α, A and B yield the following classes.
$\mathcal{S}^1(A, B, \alpha)$=$\mathcal{S}(A, B, \alpha)$ is the class introduced by Polatoglu, Bolcal, Sen and Yavuz, [6], $\mathcal{S}^k(A, B, 0)$=$\mathcal{S}^{(k)}(A, B)$ is the class studied by Kwon and Sim [3], $\mathcal{S}^k(1, -1, 0)$= $\mathcal{S}_k^* = \mathcal{S}_k^*(1, -1)$, the class is studied by Sakaguchi [12] and etc.
Fuad Alsarari and Latha in [5, 8, 13] studied some classes which related to Janowski type functions and symmetric points.

Definition 1.6. A function f in $\mathcal{A}$ is said to belong to the class $\mathcal{K}^k(A, B, \alpha)$,
$(-1 \leq B < A \leq 1), 0 \leq \alpha < 1$ if

$$\frac{(zf'(z))'}{f_k'(z)} \prec \frac{1 + [(1-\alpha)A + \alpha B]z}{1 + Bz}, \ z \in \mathcal{U}.$$

We need the following lemmas to prove our main results.

Lemma 1.7. *[6] Any function $f \in \mathcal{S}^*(A, B, \alpha)$ can be written in the form*

$$f(z) = \begin{cases} z\left(1 + Bw(z)\right)^{\frac{(1-\alpha)(A-B)}{B}}, & \text{if} \ \ B \neq 0, \\ z \exp\left[(1-\alpha)Aw(z)\right], & \text{if} \ \ B = 0, \end{cases}$$

where $w \in \Omega$.

LEMMA 1.8. *[6] Let $p \in \mathcal{P}[A, B, \alpha]$, then the set of the values of p is in the closed disc with center at $C(r)$ and having the radius $\rho(r)$, where*

$$\begin{cases} C(r) = \left(\frac{1-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}, 0\right), \rho(r) = \frac{(1-\alpha)(A-B)r}{1-B^2r^2} & \text{if } B \neq 0, \\ C(r) = (1, 0), \rho(r) = (1-\alpha)|A|r & \text{if } B = 0, \end{cases}$$

2. Main results

LEMMA 2.1. *Let $p \in \mathcal{P}[A, B, \alpha]$. Then*

$$p(r) \leq |p(z)| \leq q(r),$$

where

$$p(r) = \begin{cases} \dfrac{1-(1-\alpha)(A-B)r - B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}, & \text{if } B \neq 0, \\ 1-(1-\alpha)Ar, & \text{if } B = 0, \end{cases} \tag{2.1}$$

and

$$q(r) = \begin{cases} \dfrac{1+(1-\alpha)(A-B)r - B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}, & \text{if } B \neq 0, \\ 1+(1-\alpha)Ar, & \text{if } B = 0, \end{cases}$$

Proof. The set of the values of p is in the closed disc with center at $C(r) = \frac{1-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}$ and having the radius $\rho(r) = \frac{(1-\alpha)(A-B)r}{1-B^2r^2}$ using Lemma 1.8, that is

$$\left| p - \frac{1-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2} \right| \leq \frac{(1-\alpha)(A-B)r}{1-B^2r^2}. \tag{2.2}$$

Simplifying (2.2) we get the required result.

□

THEOREM 2.2. *If $f \in \mathcal{S}^k(A, B, \alpha)$, then*

$$f_k(z) = \begin{cases} z\left(1+Bw(z)\right)^{\frac{(1-\alpha)(A-B)}{B}}, & \text{if } B \neq 0, \\ z\exp[(1-\alpha)Aw(z)], & \text{if } B = 0, \end{cases} \tag{2.3}$$

for some $w \in \Omega$, where f_k are defined by (1.7).

Proof. Suppose that $f \in \mathcal{S}^k(A,B,\alpha)$, we can get

$$\frac{zf'(z)}{f_k(z)} \prec \frac{1+[(1-\alpha)A+\alpha B]z}{1+Bz}. \tag{2.4}$$

Replacing z by $\varepsilon^\nu z$ in (2.4), it follows that

$$\frac{\varepsilon^\nu z f'(\varepsilon^v z)}{f_k(\varepsilon^\nu z)} \prec \frac{1+[(1-\alpha)A+\alpha B]\varepsilon^\nu z}{1+B\varepsilon^\nu z} \prec \frac{1+[(1-\alpha)A+\alpha B]z}{1+Bz}.$$

Since $f_k(\varepsilon^\nu z) = \varepsilon^\nu f_k(z)$,

$$\frac{zf'(\varepsilon^\nu z)}{f_k(z)} \prec \frac{1+[(1-\alpha)A+\alpha B]z}{1+Bz}, \tag{2.5}$$

Letting $\nu = 0,1,2,\ldots,k-1$ in (2.5) and using the fact that $\mathcal{P}[A,B,\alpha]$ is a convex set, we deduce that

$$\frac{z\frac{1}{k}\sum_{\nu=0}^{k-1} f'(\varepsilon^\nu z)}{f_k(z)} \prec \frac{1+[(1-\alpha)A+\alpha B]z}{1+Bz},$$

or equivalently

$$\frac{zf_k'(z)}{f_k(z)} \prec \frac{1+[(1-\alpha)A+\alpha B]z}{1+Bz},$$

that is $f_k \in \mathcal{S}(A,B,\alpha)$, and by Lemma 1.7 we obtain our result. □

THEOREM 2.3. *If $f \in \mathcal{S}^k(A,B,\alpha)$, then*

$$|f(z)| \le \begin{cases} \displaystyle\int_0^r \frac{1+(1-\alpha)(A-B)\rho - B[(1-\alpha)A+\alpha B]\rho^2}{1-B^2\rho^2}(1+B\rho)^{\frac{(1-\alpha)(A-B)}{B}} d\rho, \\ \quad \text{if} \quad B \neq 0, \\ \displaystyle\int_0^r [1+(1-\alpha)A\rho]\exp[(1-\alpha)A\rho]\, d\rho, \\ \quad \text{if} \quad B = 0, \end{cases}$$

where $|z| \le r < 1$.

Proof. Integrating the function f' along the close segment connecting the origin with an arbitrary $z \in \mathcal{U}$, and observing that a point on this segment is of the form $\zeta = \rho e^{i\theta}$, with $\rho \in [0,r]$, where $\theta = \arg z$ and $r = |z|$, we get

$$f(z) = \int_0^z f'(\zeta)d\zeta, \; z = re^{i\theta},$$

hence

$$|f(z)| = \left|\int_0^r f'\left(\rho e^{i\theta}\right)e^{i\theta}d\rho\right| \le \int_0^r \left|f'\left(\rho e^{i\theta}\right)e^{i\theta}\right| d\rho.$$

For an arbitrary function $f \in \mathcal{S}^k(A, B, \alpha)$, we have

$$\frac{zf'(z)}{f_k(z)} = p(z), \qquad p \in \mathcal{P}[A, B, \alpha],$$

we need to study the following cases:

(i) If $B \neq 0$, then there exists a function $w \in \Omega$, such that $f_k(z) = z\,(1 + Bw(z))^{\frac{(1-\alpha)(A-B)}{B}}$, and therefore

$$|f'(z)| \leq \frac{1 + (1-\alpha)(A-B)r - B[(1-\alpha)A + \alpha B]r^2}{1 - B^2r^2} |1 + Bw(z)|^{\frac{(1-\alpha)(A-B)}{B}}, \quad |z| \leq r < 1. \tag{2.6}$$

Since $w \in \Omega$, we have

$$|1 + Bw(z)| \leq 1 + |B|r, \ |z| \leq r < 1.$$

Case 1. If $B > 0$, using the fact that $-1 \leq B < A \leq 1$ and $0 \leq \alpha < 1$, we have

$$|1 + Bw(z)|^{\frac{(1-\alpha)(A-B)}{B}} \leq (1 + |B|r)^{\frac{(1-\alpha)(A-B)}{B}}, \ |z| \leq r < 1,$$

and from (2.6) we obtain

$$|f'(z)| \leq \frac{1 + (1-\alpha)(A-B)r - B[(1-\alpha)A + \alpha B]r^2}{1 - B^2r^2} (1 + |B|r)^{\frac{(1-\alpha)(A-B)}{B}}, \quad |z| \leq r < 1. \tag{2.7}$$

Case 2. If $B < 0$, from the fact that $-1 \leq B < A \leq 1$ and $0 \leq \alpha < 1$, we have

$$(1 - |B|r)^{\frac{(1-\alpha)(A-B)}{B}} \geq |1 + Bw(z)|^{\frac{(1-\alpha)(A-B)}{B}}, \ |z| \leq r < 1,$$

and from (2.6) we obtain

$$\frac{1 + (1-\alpha)(A-B)r - B[(1-\alpha)A + \alpha B]r^2}{1 - B^2r^2} (1 - |B|r)^{\frac{(1-\alpha)(A-B)}{B}} \geq |f'(z)|, \quad |z| \leq r < 1. \tag{2.8}$$

Now, combining the inequalities (2.7) and (2.8), we finally conclude that

(2.9)
$$|f'(z)| \leq \frac{1+(1-\alpha)(A-B)r - B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}(1+Br)^{\frac{(1-\alpha)(A-B)}{B}},$$
$|z| \leq r < 1.$

then

$$\begin{aligned}|f(z)| &\leq \int_0^r \left|f'\left(\rho e^{i\theta}\right)e^{i\theta}\right| d\rho \\ &\leq \int_0^r \frac{1+(1-\alpha)(A-B)\rho - B[(1-\alpha)A+\alpha B]\rho^2}{1-B^2\rho^2}(1+B\rho)^{\frac{(1-\alpha)(A-B)}{B}} d\rho,\end{aligned}$$

that is

$$|f(z)| \leq \int_0^r \frac{1+(1-\alpha)(A-B)\rho - B[(1-\alpha)A+\alpha B]\rho^2}{1-B^2\rho^2}(1+B\rho)^{\frac{(1-\alpha)(A-B)}{B}} d\rho,$$
$|z| \leq r < 1.$

(ii) If $B = 0$, there exists a function $w \in \Omega$, such that $f_k(z) = z\exp\left[(1-\alpha)Aw(z)\right]$, and therefore

$$|f'(z)| \leq [1+(1-\alpha)Ar]\left|\exp\left[(1-\alpha)Aw(z)\right]\right|,\ |z| \leq r < 1. \tag{2.10}$$

Since $\left|\exp\left[(1-\alpha)Aw(z)\right]\right| = \exp\left[(1-\alpha)A\operatorname{Re} w(z)\right]$, $z \in \mathcal{U}$, using a similar computation as in the previous case, we deduce

$$\left|\exp\left[(1-\alpha)Aw(z)\right]\right| \leq \exp\left[(1-\alpha)Ar\right],\ |z| \leq r < 1.$$

Thus, (2.10) yields

$$|f'(z)| \leq [1+(1-\alpha)Ar]\exp\left[(1-\alpha)Ar\right],\ |z| \leq r < 1, \tag{2.11}$$

and hence

$$|f(z)| \leq \int_0^r \left|f'\left(\rho e^{i\theta}\right)e^{i\theta}\right| d\rho \leq \int_0^r [1+(1-\alpha)|A|\rho]\exp\left[(1-\alpha)A\rho\right] d\rho,$$

that is

$$|f(z)| \leq \int_0^r [1+(1-\alpha)A\rho]\exp\left[(1-\alpha)A\rho\right] d\rho,\ |z| \leq r < 1.$$

□

THEOREM 2.4. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, *be analytic in* $\mathcal{U}$, *for* $-1 \leq B < A \leq 1$, *and* $0 \leq \alpha < 1$, *if*

$$\sum_{n=2}^{\infty} \{(n-\chi_n) + |[(1-\alpha)A + \alpha B]\chi_n - Bn|\} |a_n| \leq (A-B)(1-\alpha).$$

Then $f(z) \in \mathcal{S}^k(A, B, \alpha)$.

Proof. For the proof of Theorem 2.4, it suffices to show that the values for $\frac{zf'(z)}{f_k(z)}$, satisfy

$$\left|\frac{zf'(z) - f_k(z)}{[(1-\alpha)A + \alpha B]f_k(z) - Bzf'(z)}\right| \leq 1.$$

Consider

$$\begin{aligned} &\left|\frac{zf'(z) - f_k(z)}{[(1-\alpha)A + \alpha B]f_k(z) - Bzf'(z)}\right| \\ = &\left|\frac{\sum_{n=2}^{\infty}(n-\chi_n)a_n z^{n-1}}{[(1-\alpha)A + \alpha B] - B + \sum_{n=2}^{\infty}\{[(1-\alpha)A + \alpha B]\chi_n - Bn\} a_n z^{n-1}}\right| \\ \leq &\frac{\sum_{n=2}^{\infty}(n-\chi_n)|a_n||z|^{n-1}}{(1-\alpha)(A-B) - \sum_{n=2}^{\infty}|[(1-\alpha)A + \alpha B]\chi_n - Bn| |a_n||z|^{n-1}} \\ \leq &\frac{\sum_{n=2}^{\infty}(n-\chi_n)|a_n|}{(1-\alpha)(A-B) - \sum_{n=2}^{\infty}|[(1-\alpha)A + \alpha B]\chi_n - Bn| |a_n|}. \end{aligned}$$

This last expression is bounded above by 1 if

$$\sum_{n=2}^{\infty} \{(n-\chi_n) + |[(1-\alpha)A + \alpha B]\chi_n - Bn|\} |a_n| \leq (1-\alpha)(A-B),$$

hence $\left|\frac{zf'(z) - f_k(z)}{[(1-\alpha)A + \alpha B]f_k(z) - Bzf'(z)}\right| \leq 1$, and Theorem 2.4 is proved. □

THEOREM 2.5. *A function* $f \in \mathcal{S}^k(A, B, \alpha)$ *if and only if*
(2.12)

$$\frac{1}{z}\left[f * \left\{\frac{z}{(1-z)^2}(1 + Be^{i\phi}) - q(z)(1 + [(1-\alpha)A + \alpha B]e^{i\phi})\right\}\right] \neq 0$$

where
$-1 \leq B < A \leq 1, 0 \leq \alpha < 10 \leq \phi < 2\pi$ *and* $q(z)$ *is given by* (2.17).

Proof. Suppose that $f \in \mathcal{S}^k(A, B, \alpha)$, then

$$\frac{zf'(z)}{f_k(z)} \prec \frac{1 + [(1-\alpha)A + \alpha B]z}{1 + Bz},$$

if and only if

$$\frac{zf'(z)}{f_k(z)} \neq \frac{1 + [(1-\alpha)A + \alpha B]e^{i\phi}}{1 + Be^{i\phi}}. \tag{2.13}$$

For all $z \in \mathcal{U}$ and $0 \leq \phi < 2\pi$. It is easy to know the condition (2.13) can be written as

$$\frac{1}{z}[zf'(z)(1 + Be^{i\phi}) - f_k(z)(1 + [(1-\alpha)A + \alpha B]e^{i\phi})] \neq 0, \tag{2.14}$$

on the other hand, it well known that

$$zf'(z) = f(z) * \frac{z}{(1-z)^2} \tag{2.15}$$

and

$$f_k(z) = f(z) * q(z), \tag{2.16}$$

where

$$q(z) = \frac{1}{k}\sum_{v=0}^{k-1} \frac{z}{1 - \varepsilon^v z}. \tag{2.17}$$

Substituting (2.15) and (2.16) into (2.14) we get (2.12). □

To find some neighborhood results for the class $f \in \mathcal{S}^k(A, B, \alpha)$ analogous to those obtained by Ruschewegh [11], we introduce the following concept of neighborhood

Definition 2.6. For $-1 \leq B < A \leq 1, 0 \leq \alpha < 1, 0 \leq \phi < 2\pi$ and $\rho \geq 0$ we define $\mathcal{N}^k(A, B, \alpha; f, \rho)$ the neighborhood of a function $f \in \mathcal{A}$ as

$$\mathcal{N}^k(A, B, \alpha; f, \rho) = \left\{ g \in \mathcal{A} : g(z) = z + \sum_{n=2}^{\infty} b_n z^n, d(f, g) = \sum_{n=2}^{\infty} \frac{(n - \chi_n) + |[(1-\alpha)A + \alpha B]\chi_n - Bn|}{(1-\alpha)(A - B)} |b_n - a_n| \leq \rho \right\}, \tag{2.18}$$

where $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and χ_n is defined by (1.9).

REMARK 2.7. For parametric values $k = A = -B = 1$, and $\alpha = 0$ (2.18) reduces to (1.2).

THEOREM 2.8. *Let $f \in \mathcal{A}$, and for all complex number η, with $|\mu| < \rho$, if*

$$\frac{f(z) + \eta z}{1 + \eta} \in \mathcal{S}^k(A, B, \alpha). \tag{2.19}$$

Then

$$\mathcal{N}^k(A, B, \alpha; f, \rho) \subset \mathcal{S}^k(A, B, \alpha).$$

Proof. We assume that a function g defined by $g(z) = \sum_{n=2}^{\infty} b_n z^n$ is in the class $\mathcal{N}^k(A, B, \alpha; f, \rho)$. In order to prove the theorem, we only need to prove that $g \in \mathcal{S}^k(A, B, \alpha)$. We would prove this claim in next three steps.
We first note that Theorem 2.5 is equivalent to

$$f \in \mathcal{S}^k(A, B, \alpha) \Leftrightarrow \frac{1}{z}[(f * t_\phi)(z)] \neq 0, \quad z \in \mathcal{U}, \tag{2.20}$$

where

$$t_\phi(z) = \frac{\frac{z}{(1-z)^2}(1 + Be^{i\phi}) - q(z)(1 + [(1-\alpha)A + \alpha B]e^{i\phi})}{(1-\alpha)(B-A)e^{i\phi}},$$

where $0 \leq \phi < 2\pi, -1 \leq B < A \leq 1, 0 \leq \alpha < 1$ and q is given by (2.17).
We can write $t_\phi(z) = z + \sum_{n=2}^{\infty} t_n z^n$,
where

$$t_n = \frac{(n - \chi_n) + |[(1-\alpha)A + \alpha B]\chi_n - Bn|}{(1-\alpha)(B-A)e^{i\phi}}, \tag{2.21}$$

and where χ_n is defined by (1.9). Secondly we obtain that (2.19) is equivalent to

$$\left| \frac{f(z) * t_\phi(z)}{z} \right| \geq \rho, \tag{2.22}$$

because, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ and satisfy (2.19), then (2.20) is equivalent to

$$t_\phi \in \mathcal{S}^k(A, B, \alpha, \sigma) \Leftrightarrow \frac{1}{z}\left[\frac{f(z) * t_\phi(z)}{1 + \eta}\right] \neq 0, \qquad |\eta| < \rho.$$

Thirdly letting $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ we notice that

$$\begin{aligned}\left|\frac{g(z) * t_\phi(z)}{z}\right| &= \left|\frac{f(z) * t_\phi(z)}{z} + \frac{(g(z) - f(z)) * t_\phi(z)}{z}\right| \\ &\geq \rho - \left|\frac{(g(z) - f(z)) * t_\phi(z)}{z}\right|, \quad \text{(by using (2.22))} \\ &= \rho - \left|\sum_{n=2}^{\infty} (b_n - a_n) t_n z^n\right|, \\ &\geq |z| \sum_{n=2}^{\infty} \left[\frac{(n - \chi_n) + |[(1-\alpha)A + \alpha B]\chi_n - Bn|}{(1-\alpha)(B-A)e^{i\phi}}\right] |b_n - a_n| \\ &\geq \rho - \rho = 0, \qquad \text{by applying (2.21).}\end{aligned}$$

This prove that

$$\frac{g(z) * t_\phi(z)}{z} \neq 0, \qquad z \in \mathcal{U}.$$

In view of our observations (2.20), it follows that $g \in \mathcal{S}^k(A, B, \alpha)$. This completes the proof of the theorem. □

When $k = A = -B = 1$ and $\alpha = 0$ in the above theorem we get (1.3) proved by Ruschewyh in [11].

COROLLARY 2.9. *Let $\mathcal{S}^*$ be the class of starlike functions. Let $f \in \mathcal{A}$ and for all complex number η, with $|\mu| < \rho$, if*

$$\frac{f(z) + \eta z}{1 + \eta} \in \mathcal{S}^*, \tag{2.23}$$

then $\mathcal{N}_\sigma(f) \subset \mathcal{S}^*$.

THEOREM 2.10. *Let $f \in \mathcal{S}^k(A, B, \alpha)$. Then*

$$|\arg f'(z)| \leq \begin{cases} \dfrac{(A-B)(1-\alpha)}{B} \arcsin(Br) \\ \qquad\qquad + \arcsin\left(\frac{(A-B)(1-\alpha)}{1 - B[(1-\alpha)A + \alpha B]r^2}\right), & \text{if } B \neq 0, \\ (1-\alpha)Ar + \arcsin((1-\alpha)Ar), & \text{if } B = 0, \end{cases}$$

where

Proof. Suppose that $f \in \mathcal{S}^k(A, B, \alpha)$, then

$$|\arg f'(z)| \leq \left|\arg \frac{f_k(z)}{z}\right| + |\arg p(z)|, \tag{2.24}$$

where $p \in P[A, B, \alpha]$, using Theorem 2.2 for $B \neq 0$, we have

$$\frac{f_k(z)}{z} = (1 + Bw(z))^{\frac{(1-\alpha)(A-B)}{B}},$$

we discuss the following cases Case (1), $B > 0$.

$$\begin{aligned}
&\left|(1 + Bw(z))^{\frac{[(1-\alpha)A+\alpha B]-B}{B}}\right| \\
&= \left|\exp\left\{\frac{[(1-\alpha)A + \alpha B] - B}{B} \log(1 + Bw(z))\right\}\right| \\
&= \exp\left\{\frac{[(1-\alpha)A + \alpha B] - B}{B} \log|(1 + Bw(z))|\right\} \\
&= |(1 + Bw(z))|^{\frac{[(1-\alpha)A+\alpha B]-B}{B}} \\
&\leq (1 + Br)^{\frac{[(1-\alpha)A+\alpha B]-B}{B}}.
\end{aligned}$$

Case (2) $B < 0$.
Let $B = -C, C > 0$. Then

$$\begin{aligned}
\left|(1 + Bw(z))^{\frac{[(1-\alpha)A+\alpha B]-B}{B}}\right| &= \left|\left\{(1 - Cw(z))^{-1}\right\}^{\frac{[(1-\alpha)A-\alpha C]+C}{C}}\right| \\
&= \left|(1 - Cw(z))^{-1}\right|^{\frac{[(1-\alpha)A-\alpha C]+C}{C}} \\
&\leq \left(\frac{1}{1 - Cr}\right)^{\frac{[(1-\alpha)A-\alpha C]+C}{C}} \\
&= (1 + Br)^{\frac{[(1-\alpha)A+\alpha B]-B}{B}}.
\end{aligned}$$

Combining the cases (1) and (2), we get

$$\begin{aligned}
&\left|\arg\left(\frac{f_k(z)}{z}\right)\right| \\
&\leq \frac{[(1-\alpha)A + \alpha B] - B}{B} |\arg(1 + Br)| \\
&\leq \frac{[(1-\alpha)A + \alpha B] - B}{B} \arcsin(Br).
\end{aligned} \tag{2.25}$$

For $B = 0$ it is clear

$$\left|\arg\left(\frac{f_k(z)}{z}\right)\right| \leq (1 - \alpha)Ar. \tag{2.26}$$

Now using (2.2) in Lemma 2.1 for $p \in P[A, B, \alpha]$, we have

$$\left| p - \frac{1 - B[(1-\alpha)A + \alpha B]r^2}{1 - B^2 r^2} \right| \leq \frac{(1-\alpha)(A-B)r}{1 - B^2 r^2}, \tag{2.27}$$

from which it follows that

$$|\arg p(z)| \leq \arcsin\left(\frac{(1-\alpha)(A-B)r}{1 - B[(1-\alpha)A + \alpha B]r^2} \right). \tag{2.28}$$

For $B = 0$, directly we get

$$|\arg p(z)| \leq \arcsin((1-\alpha)Ar). \tag{2.29}$$

From (2.25),(2.26),(2.28) and (2.29) we get the proof. □

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