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MODAL, NECESSITY, SUFFICIENCY AND CO-SUFFICIENCY OPERATORS

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Abstract. We investigate the properties of modal, necessity, sufficiency and co-sufficiency operators. We show that their operations induce various relations, respectively.

1. Introduction

Pawlak [5] introduced rough set theory to generalize the classical set theory. Rough approximations are defined by a partition of the universe which is corresponding to the equivalence relation about information. An information consists of (X, A) where X is a set of objects and A is a set of attributes, a map $a : X \to P(A_a)$ where A_a is the value set of the attribute a. Recently, intensional modal-like logics with the propositional operators induced by relations are important mathematical tools for data analysis and knowledge processing [1-3, 6-9].

In this paper, we investigate the properties of modal, necessity, sufficiency and co-sufficiency operators. We show that their operations induce various relations, respectively.

2. Preliminaries

Definition 2.1. [2] Let $P(X), P(Y)$ be the families of subsets on X and Y, respectively. Then a map $F : P(X) \to P(Y)$ is called
(1) *modal operator* if $F(\bigcup_{i\in\Gamma} A_i) = \bigcup_{i\in\Gamma} F(A_i),\ F(\emptyset) = \emptyset$,
(2) *necessity operator* if $F(\bigcap_{i\in\Gamma} A_i) = \bigcap_{i\in\Gamma} F(A_i),\ F(X) = Y$,
(3) *sufficiency operator* if $F(\bigcup_{i\in\Gamma} A_i) = \bigcap_{i\in\Gamma} F(A_i),\ F(\emptyset) = Y$,
(4) *co-sufficiency operator* if $F(\bigcap_{i\in\Gamma} A_i) = \bigcup_{i\in\Gamma} F(A_i),\ F(X) = \emptyset$.

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(5) a dual operator F^{∂} is defined by $F^{\partial}(A) = F(A^c)^c$. Moreover, we define $F^c(A) = (F(A))^c$ and $F^*(A) = F(A^c)$.

Definition 2.2. [2,4] Let $R \subset P(X \times Y)$ be a relation. For each $A \in P(X)$, we define operations $(y, x) \in R^{-1}$ iff $(x, y) \in R$ and $[R], [[R]], \langle R\rangle, \langle\langle R\rangle\rangle, [R]^*, \langle R\rangle^* : P(X) \to P(Y)$ as follows:

$$[R](A) = \{y \in Y \mid (\forall x)((x, y) \in R \to x \in A)\},$$

$$[[R]](A) = \{y \in Y \mid (\forall x \in X)(x \in A \to (x, y) \in R)\}$$

$$\langle R\rangle(A) = \{y \in Y \mid (\exists x \in X)((x, y) \in R,\ x \in A)\}$$

$$\langle\langle R\rangle\rangle(A) = \{y \in Y \mid (\exists x \in X)((x, y) \in R^c,\ x \in A^c)\},$$

$$[R]^*(A) = \{y \in Y \mid (\forall x \in X)((x, y) \in R \to x \in A^c)\}$$

$$\langle R\rangle^*(A) = \{y \in Y \mid (\exists x \in X)((x, y) \in R,\ x \in A^c)\}.$$

Theorem 2.3. [2] *Let $R \subset P(X \times Y)$ be a relation.*
(1) $\langle R\rangle$ is a modal operator and $[R]$ is a necessity operator with $\langle R\rangle(A) = ([R](A^c))^c = [R]^{\partial}(A)$, for each $A \in P(X)$.
(2) If $F : P(X) \to P(Y)$ is a modal operator on $P(X)$, there exists a unique relation $R_F \subset P(X \times Y)$ such that $\langle R_F\rangle = F$ and $[R_F] = F^{\partial}$ where $(x, y) \in R_F$ iff $y \in F(\{x\})$.
(3) $R_{\langle R\rangle} = R$.

3. Modal, necessity, sufficiency and co-sufficiency operators

Lemma 3.1. *Let $F, G : P(X) \to P(Y)$ be operators. Then the following properties hold:*
(1) $(F^{\partial})^{\partial} = F$, $(F^c)^c = F$ and $(F^*)^* = F$.
(2) $(F^{\partial})^* = (F^*)^{\partial}$, $(F^{\partial})^c = (F^c)^{\partial}$ and $(F^*)^c = (F^c)^* = F^{\partial}$.
(3) $(F \cup G)^{\partial} = F^{\partial} \cap G^{\partial}$, $(F \cup G)^* = F^* \cup G^*$ and $(F \cup G)^c = F^c \cap G^c$.
(4) $F, G : P(X) \to P(Y)$ are modal operators, then $F \cup G$ is a model operator and it's dual operator $F^{\partial} \cap G^{\partial}$ is a necessity operator.
(5) $F, G : P(X) \to P(Y)$ are necessity operators, then $F \cap G$ is a necessity operator and it's dual operator $F^{\partial} \cap G^{\partial}$ is a model operator.

Proof. (1) $(F^\partial)^\partial(A) = (F^\partial(A^c))^c = F(A)$.

(2) $(F^\partial)^*(A) = F^\partial(A^c) = F^c(A) = (F^*(A^c))^c = (F^*)^\partial(A)$.

$(F^\partial)^c(A) = (F^\partial(A))^c = F(A^c) = (F^c(A^c))^c = (F^c)^\partial(A)$.

(3) $(F \cup G)^\partial(A) = ((F \cup G)(A^c))^c = (F(A^c))^c \cap (F(A^c))^c = F^\partial(A) \cap G^\partial(A)$. Other cases are similarly proved.

(4) and (5) are easily proved from (3).

□

Lemma 3.2. *(1) A map $F : P(X) \to P(Y)$ is a modal operator iff $F^\partial : P(X) \to P(Y)$ is a necessity operator.*

(2) A map $F : P(X) \to P(Y)$ is a sufficiency operator iff $F^\partial : P(X) \to P(Y)$ is a co-sufficiency operator operator.

(3) A map $F : P(X) \to P(Y)$ is a modal operator iff $F^c : P(X) \to P(Y)$ is a sufficient operator.

(4) A map $F : P(X) \to P(Y)$ is a co-sufficiency operator iff $F^c : P(X) \to P(Y)$ is a necessity operator operator.

(5) A map $F : P(X) \to P(Y)$ is a sufficiency operator iff $F^* : P(X) \to P(Y)$ is a necessity operator operator.

(6) A map $F : P(X) \to P(Y)$ is a modal operator iff $F^* : P(X) \to P(Y)$ is a co-sufficiency operator.

Proof. (1) Let $F : P(X) \to P(Y)$ be a modal operator.

$$\begin{aligned} F^\partial(\bigcap_{i\in\Gamma} A_i) &= \Big(F(\bigcup_{i\in\Gamma} A_i^c)\Big)^c = \Big(\bigcup_{i\in\Gamma} F(A_i^c)\Big)^c \\ &= \bigcap_{i\in\Gamma}(F(A_i^c))^c = \bigcap_{i\in\Gamma} F^\partial(A_i). \\ F^\partial(X) &= \Big(F(X^c)\Big)^c = \Big(F(\emptyset)\Big)^c = Y. \end{aligned}$$

Conversely, $(F^\partial)^\partial(A) = (F^\partial(A^c))^c = F(A)$.

$$\begin{aligned} F(\bigcup_{i\in\Gamma} A_i) &= \Big(F^\partial(\bigcap_{i\in\Gamma} A_i^c)\Big)^c = \Big(\bigcap_{i\in\Gamma} F^\partial(A_i^c)\Big)^c \\ &= \bigcup_{i\in\Gamma}(F^\partial(A_i^c))^c = \bigcup_{i\in\Gamma} F(A_i). \\ F(\emptyset) &= \Big(F^\partial((\emptyset)^c)\Big)^c = F(X)^c = \emptyset. \end{aligned}$$

(2), (3) and (4) are similarly proved as same in (1).

□

THEOREM 3.3. *Let $R \subset P(X \times Y)$ be a relation.*

(1) $\langle\langle R\rangle\rangle^*$ is a modal operator and $[[R]]^*$ is a necessity operator with $\langle\langle R\rangle\rangle^*(A) = ([[R]]^*(A^c))^c = ([[R]]^*)^\partial(A)$ for each $A \in P(X)$.

(2) If $F : P(X) \to P(Y)$ is a modal operator on $P(X)$, there exists a unique relation $R_F \subset P(X \times Y)$ such that $\langle\langle R_F\rangle\rangle^* = F$ and $[[R_F]]^* = F^\partial$ where $(x, y) \in R_F$ iff $y \in F(\{x\})^c$.

(3) $R_{\langle\langle R\rangle\rangle^*} = R$.

Proof. (1) We have $\langle\langle R\rangle\rangle^*(A) = ([[R]]^*(A^c))^c = ([[R]]^*)^\partial(A)$ from:

$$\begin{aligned} y \in ([[R]]^*(A^c))^c \text{ iff } & \Big((\forall x \in X)(X \in A \to (x, y) \in R)\Big)^c \\ \text{iff } & \Big((\forall x \in X)((x, y) \in R^c, x \in A)^c\Big)^c \\ \text{iff } & (\exists x \in X)((x, y) \in R^c, x \in A) \\ \text{iff } & y \in \langle\langle R\rangle\rangle^*(A). \end{aligned}$$

(2) Since $A = \bigcup_{x \in A}\{x\}$ and $F(A) = \bigcup_{x \in A} F(\{x\})$, we have

$$\begin{aligned} y \in \langle\langle R_F\rangle\rangle^*(A) \text{ iff } & (\exists x \in X)((x, y) \in R_F^c \;\&\; x \in A) \\ \text{iff } & (\exists x \in X)(y \in F(\{x\}) \;\&\; x \in A) \\ \text{iff } & y \in \bigcup_{x \in A} F(\{x\}) = F(\bigcup_{x \in A}\{x\}) = F(A). \end{aligned}$$

$$\begin{aligned} y \in [[R_F]]^*(A) \text{ iff } & (\forall x \in X)(x \in A^c \to (x, y) \in R_F)) \\ \text{iff } & (\forall x \in X)((x, y) \in R_F^c \to x \in A)) \\ \text{iff } & (\forall x \in X)(y \in F(\{x\}) \to x \in A)) \\ \text{iff } & \Big((\exists x \in X)(y \in F(\{x\}) \;\&\; x \in A^c)\Big)^c \\ \text{iff } & y \in \Big(\bigcup_{x \in A^c} F(\{x\}))\Big)^c \\ \text{iff } & y \in \Big(F(\bigcup_{x \in A^c}\{x\})\Big)^c = (F(A^c))^c = F^\partial(A). \end{aligned}$$

(3)

$$
\begin{aligned}
(x,y) \in R_{\langle\langle R\rangle\rangle^*} &\text{ iff } y \in \langle\langle R\rangle\rangle^*(\{x\})^c \\
&\text{ iff } \Big((\exists z \in X)((z,y) \in R^c \ \&\ z \in \{x\}\Big)^c \\
&\text{ iff } (x,y) \in (R^c)^c = R.
\end{aligned}
$$

□

THEOREM 3.4. *Let $R \in P(X \times Y)$ be a relation.*

(1) $[[R]]$ *is a sufficiency operator and* $\langle\langle R\rangle\rangle$ *is a co-sufficiency operator with* $\langle\langle R\rangle\rangle(A) = ([[R]](A^c))^c = [[R]]^\partial(A)$ *for each* $A \in P(X)$.

(2) *If* $F : P(X) \to P(Y)$ *is a sufficiency operator on* $P(X)$, *there exists a unique relation* $R_F \in P(X \times Y)$ *such that* $[[R_F]] = F$ *and* $\langle\langle R_F\rangle\rangle = F^\partial$ *where* $(x,y) \in R_F$ *iff* $y \in F(\{x\})$.

(3) $R_{[[R]]} = R$.

Proof. (1) We have $\langle\langle R\rangle\rangle(A) = ([[R]](A^c))^c = [[R]]^\partial(A)$ from:

$$
\begin{aligned}
x \in ([[R]](A^c))^c &\text{ iff } \Big((\forall y \in X)(y \in A^c \to (x,y) \in R)\Big)^c \\
&\text{ iff } \Big((\forall y \in X)(((x,y) \in R^c \ \&\ y \in A^c)^c\Big)^c \\
&\text{ iff } (\exists y \in X)((x,y) \in R^c \ \&\ y \in A^c) \\
&\text{ iff } x \in \langle\langle R\rangle\rangle(A).
\end{aligned}
$$

(2) Since $F(\bigcup_{x\in A}\{x\}) = \bigcap_{x\in A} F(\{x\})$, we have

$$
\begin{aligned}
y \in [[R_F]](A) &\text{ iff } (\forall x \in X)(x \in A \to (x,y) \in R_F) \\
&\text{ iff } (\forall x \in X)(x \in A \to y \in F(\{x\})) \\
&\text{ iff } y \in \bigcap_{x\in A} F(\{x\}) = F(\bigcup_{x\in A}\{x\}) = F(A).
\end{aligned}
$$

$$
\begin{aligned}
y \in \langle\langle R_F\rangle\rangle(A) &\text{ iff } (\exists x \in X)((x,y) \in R_F^c \ \&\ x \in A^c) \\
&\text{ iff } (\exists x \in X)(y \in F(\{x\})^c \ \&\ x \in A^c) \\
&\text{ iff } y \in \bigcup_{x\in A^c} F(\{x\})^c = (\bigcap_{x\in A^c} F(\{x\}))^c \\
&\text{ iff } y \in (F(\bigcup_{x\in A^c}\{x\}))^c = (F(A^c))^c = F^\partial(A).
\end{aligned}
$$

(3)

$$\begin{aligned}(x,y)\in R_{[[R]]} &\text{ iff } (\forall z\in X)(z\in\{x\}\to(z,y)\in R)\\ &\text{ iff } (x,y)\in R.\end{aligned}$$

□

THEOREM 3.5. *Let $R\in P(X\times Y)$ be a relation.*

(1) *$[R]^*$ is a sufficiency operator and $\langle R\rangle^*$ is a co-sufficiency operator with $[R]^*(A)=(\langle R\rangle^*(A^c))^c$.*

(2) *If $F:P(X)\to P(Y)$ is a sufficiency operator on $P(X)$, there exists a unique relation $R_F\in P(X\times Y)$ such that $[R_F]^*=F$ and $\langle R_F\rangle^*=F^\partial$ where $(x,y)\in R_F$ iff $y\in F(\{x\})^c$.*

(3) *$R_{[R]^*}=R$.*

Proof. (1)

$$\begin{aligned}y\in(\langle R\rangle^*(A^c))^c &\text{ iff } \Big((\exists x\in X)(x\in A\ \&\ (x,y)\in R)\Big)^c\\ &\text{ iff } (\forall x\in X)((x,y)\in R\to x\in A^c)\\ &\text{ iff } y\in[R]^*(A).\end{aligned}$$

(2)

$$\begin{aligned}y\in[R_F]^*(A) &\text{ iff } (\forall x\in X)((x,y)\in R_F\to x\in A^c)\\ &\text{ iff } (\forall x\in X)(x\in A\to y\in F(\{x\}))\\ &\text{ iff } y\in\bigcap_{x\in A}F(\{x\})=F(\bigcup_{x\in A}\{x\})=F(A).\end{aligned}$$

$$\begin{aligned}y\in\langle R_F\rangle^*(A) &\text{ iff } (\exists x\in X)((x,y)\in R_F\ \&\ x\in A^c)\\ &\text{ iff } (\exists x\in X)(y\in F(\{x\})^c\ \&\ x\in A^c)\\ &\text{ iff } \Big((\forall x\in X)(x\in A^c\to y\in F(\{x\}))\Big)^c\\ &\text{ iff } y\in\Big(\bigcap_{x\in A^c}(F(\{x\}))\Big)^c\\ &\text{ iff } y\in\Big(F(\bigcup_{x\in A^c}\{x\})\Big)^c=(F(A^c))^c\\ &\text{ iff } y\in F^\partial(A).\end{aligned}$$

(3)

$$\begin{aligned}(x,y)\in R_{[R]^*} &\text{ iff } y\in [R]^*(\{x\}^c)^c\\ &\text{ iff } \Big((\forall z\in X)((z,y)\in R\to z\in\{x\}^c)\Big)^c\\ &\text{ iff } (x,y)\in R.\end{aligned}$$

□

THEOREM 3.6. *Let $R\subset P(X\times Y)$ be a relation.*

(1) *If $F : P(X)\to P(Y)$ is a necessity operator on $P(X)$, there exists a unique relation $R_F\in P(X\times Y)$ such that $[R_F]=F$ and $\langle R_F\rangle = F^\partial$ where $(x,y)\in R_F$ iff $y\in F(\{x\}^c)^c$.*

(2) $R_{[R]}=R$.

Proof. (1)

$$\begin{aligned}y\in [R_F](A) &\text{ iff } (\forall x\in X)((x,y)\in R_F\to x\in A)\\ &\text{ iff } (\forall x\in X)(y\in F(\{x\}^c)^c\to x\in A)\\ &\text{ iff } (\forall x\in X)(x\in A^c\to y\in F(\{x\}^c))\\ &\text{ iff } y\in \bigcap_{x\in A^c} F(\{x\}^c)=F(\bigcap_{x\in A^c}\{x\}^c)=F(A).\end{aligned}$$

$$\begin{aligned}y\in \langle R_F\rangle(A) &\text{ iff } (\exists x\in X)((x,y)\in R_F \ \&\ x\in A)\\ &\text{ iff } (\exists x\in X)(y\in F(\{x\}^c)^c \ \&\ x\in A)\\ &\text{ iff } \Big((\forall x\in X)(x\in A\to y\in F(\{x\}^c))\Big)^c\\ &\text{ iff } y\in \Big(\bigcap_{x\in A} F(\{x\}^c)\Big)^c=\Big(F(\bigcap_{x\in A}\{x\}^c)\Big)^c\\ &\text{ iff } y\in F(A^c)^c \text{ iff } y\in F^\partial(A).\end{aligned}$$

(2)

$$\begin{aligned}(x,y)\in R_{[R]} &\text{ iff } y\in [R](\{x\}^c)^c\\ &\text{ iff } \Big((\forall z\in X)((z,y)\in R\to z\in\{x\}^c)\Big)^c\\ &\text{ iff } (x,y)\in R.\end{aligned}$$

□

Theorem 3.7. *Let $R \in P(X \times Y)$ be a relation.*

(1) *If $F : P(X) \to P(Y)$ is a co-sufficiency operator on $P(X)$, there exists a unique relation $R_F \in P(X \times Y)$ such that $\langle\langle R_F \rangle\rangle = F$ and $[[R_F]] = F^{\partial}$ where $(x, y) \in R_F$ iff $y \in F(\{x\}^c)^c$.*

(2) $R_{\langle\langle R_F \rangle\rangle} = R$.

Proof. (1)

$$\begin{aligned} y \in \langle\langle R_F \rangle\rangle(A) &\text{ iff } (\exists x \in X)((x, y) \in R_F^c \ \&\ x \in A^c) \\ &\text{ iff } (\exists x \in X)(y \in F(\{x\}^c) \ \&\ x \in A^c) \\ &\text{ iff } y \in \bigcup_{x \in A^c} F(\{x\}^c) = F(\bigcap_{x \in A^c} \{x\}^c) = F(A). \end{aligned}$$

$$\begin{aligned} y \in [[R_F]](A) &\text{ iff } (\forall x \in X)(x \in A \to (x, y) \in R_F) \\ &\text{ iff } (\forall x \in X)(x \in A \to y \in F(\{x\}^c)^c) \\ &\text{ iff } \Big((\exists x \in X)(x \in A \ \&\ y \in F(\{x\}^c))\Big)^c \\ &\text{ iff } y \in \Big(\bigcup_{x \in A} F(\{x\}^c)\Big)^c \\ &\text{ iff } y \in \Big(F(\bigcap_{x \in A} \{x\}^c)\Big)^c \\ &\text{ iff } y \in F(A^c)^c = F^{\partial}(A). \end{aligned}$$

(2)

$$\begin{aligned} (x, y) \in R_{\langle\langle R \rangle\rangle} &\text{ iff } y \in \langle\langle R \rangle\rangle(\{x\}^c)^c \\ &\text{ iff } \Big((\exists z \in X)((z, y) \in R^c \ \&\ z \in \{x\}^c)\Big)^c \\ &\text{ iff } (x, y) \in R. \end{aligned}$$

□

Theorem 3.8. *Let $R \in P(X \times Y)$ be a relation.*

(1) *If $F : P(X) \to P(Y)$ is a necessity operator on $P(X)$, there exists a unique relation $R_F \in P(X \times Y)$ such that $[[R_F]]^* = F$ and $\langle\langle R_F \rangle\rangle^* = F^{\partial}$ where $(x, y) \in R_F$ iff $y \in F(\{x\}^c)$.*

(2) $R_{[[R_F]]^*} = R$.

Proof. (1)

$$\begin{aligned} y \in [[R_F]]^*(A) &\text{ iff } (\forall x \in X)(x \in A^c \to (x,y) \in R_F) \\ &\text{ iff } (\forall x \in X)(x \in A^c \to y \in F(\{x\}^c)) \\ &\text{ iff } y \in \bigcap_{x \in A^c} F(\{x\}^c) = F(\bigcap_{x \in A^c} \{x\}^c) = F(A). \end{aligned}$$

$$\begin{aligned} x \in \langle\langle R_F \rangle\rangle^*(A) &\text{ iff } (\exists x \in X)((x,y) \in R_F^c \;\&\; x \in A) \\ &\text{ iff } (\exists x \in X)(y \in F(\{x\}^c)^c \;\&\; x \in A) \\ &\text{ iff } \Big((\forall x \in X)(x \in A \to y \in F(\{x\}^c))\Big)^c \\ &\text{ iff } y \in \Big(\bigcap_{x \in A} F(\{x\}^c)\Big)^c = \Big(F(\bigcap_{x \in A} \{x\}^c)\Big)^c \\ &\text{ iff } y \in F(A^c)^c \text{ iff } y \in F^\partial(A). \end{aligned}$$

(2)

$$\begin{aligned} (x,y) \in R_{[[R_F]]^*} &\text{ iff } y \in [[R_F]]^*(\{x\}^c) \\ &\text{ iff } (\forall z \in X)(z \in \{x\} \to (z,y) \in R) \\ &\text{ iff } (x,y) \in R. \end{aligned}$$

□

THEOREM 3.9. *Let $R \in P(X \times Y)$ be a relation.*
(1) *If $F : P(X) \to P(Y)$ is a co-sufficiency operator on $P(X)$, there exists a unique relation $R_F \in P(X \times Y)$ such that $\langle R_F \rangle^* = F$ and $[R_F]^* = F^\partial$ where $(x,y) \in R_F$ iff $y \in F(\{x\}^c)$.*
(2) *$R_{\langle R_F \rangle^*} = R$.*

Proof. (1)

$$\begin{aligned} y \in \langle R_F \rangle^* &\text{ iff } (\exists x \in X)((x,y) \in R_F \;\&\; x \in A^c) \\ &\text{ iff } (\exists x \in X)(y \in F(\{x\}^c) \;\&\; x \in A^c) \\ &\text{ iff } y \in \bigcup_{x \in A^c} F(\{x\}^c) = F(\bigcap_{x \in A^c} \{x\}^c) = F(A). \end{aligned}$$

$$\begin{aligned} y \in [R_F]^*(A) &\text{ iff } (\forall x \in X)(x \in (x,y) \in R_F \to x \in A^c) \\ &\text{ iff } \Big((\exists x \in X)(x \in A \;\&\; (x,y) \in R_F)\Big)^c \\ &\text{ iff } \Big((\exists x \in X)(x \in A \;\&\; y \in F(\{x\}^c))\Big)^c \\ &\text{ iff } y \in \Big(\bigcup_{x \in A} F(\{x\}^c)\Big)^c \\ &\text{ iff } y \in \Big(F(\bigcap_{x \in A} \{x\}^c)\Big)^c \\ &\text{ iff } y \in F(A^c)^c = F^{\partial}(A). \end{aligned}$$

(2)

$$\begin{aligned} (x,y) \in R_{\langle R_F \rangle^*} &\text{ iff } y \in \langle R_F \rangle^*(\{x\}^c) \\ &\text{ iff } \Big((\exists z \in X)((z,y) \in R^c \;\&\; z \in \{x\})\Big)^c \\ &\text{ iff } (x,y) \in R. \end{aligned}$$

□

Example 3.10. Let $X = \{a,b,c\}$ and $Y = \{x,y,z\}$ be a set. Define $F, G : P(X) \to P(Y)$ as

$$F(\{a\}) = \emptyset, F(\{b\}) = \{x\}, F(\{c\}) = \{y,z\},$$

$$G(\{a\}) = X, G(\{b\}) = \{x,y\}, G(\{c\}) = \{y,z\},$$

$$H(\{b,c\}) = \{x\}, H(\{c,a\}) = \{x,y\}, H(\{a,b\}) = \{z\}.$$

(1) If F is a modal operator, then, by Theorem 2.3,

$$F(A) = \begin{cases} \emptyset, & \text{if } A \in \{\emptyset, \{a\}\}, \\ \{x\}, & \text{if } A \in \{\{b\}, \{a,b\}\}, \\ \{y,z\}, & \text{if } A \in \{\{c\}, \{a,c\}\}, \\ Y, & \text{if } A \in \{\{b,c\}, X\}. \end{cases}$$

Since $(x, y) \in R_F$ iff $y \in F(\{x\})$, we obtain:

$$R_F = \{(b, x), (c, y), (c, z)\}, \langle R_F \rangle = F,\ [R_F] = F^{\partial}.$$

(2) If F is a modal operator, then, by Theorem 3.3, we obtain F as same in (1). Since $(x, y) \in R_F$ iff $y \in F(\{x\})^c$, we obtain:

$$R_F = \{(a, x), (a, y), (a, z), (b, y), (b, z), (c, x)\},$$

$$\langle\langle R_F \rangle\rangle^* = F, [[R_F]]^* = F^{\partial}.$$

(3) If G is a sufficiency operator, then, by Theorem 3.4,

$$G(A) = \begin{cases} Y, & \text{if } A \in \{\emptyset, \{a\}\}, \\ \{x, y\}, & \text{if } A \in \{\{b\}, \{a, b\}\}, \\ \{y, z\}, & \text{if } A \in \{\{c\}, \{a, c\}\}, \\ \{y\}, & \text{if } A \in \{\{b, c\}, X\}. \end{cases}$$

Since $(x, y) \in R_G$ iff $y \in G(\{x\})$, we obtain:

$$R_G = \{(a, x), (a, y), (a, z), (b, x), (b, y), (c, y), (c, z)\},$$

$$[[R_G]] = G, \langle\langle R_G \rangle\rangle = G^{\partial}.$$

(4) If G is a sufficiency operator, then, by Theorem 3.5, we obtain G as same in (3). Since $(x, y) \in R_G$ iff $y \in G(\{x\})^c$, we obtain:

$$R_G = \{(b, z), (c, x)\}, [R_G]^* = G, \langle R_G \rangle^* = G^{\partial}.$$

(5) If H is a necessity operator, then, by Theorem 3.6,

$$H(A) = \begin{cases} \emptyset, & \text{if } A \in \{\emptyset, \{a\}, \{b\}\}, \\ \{x\}, & \text{if } A \in \{\{c\}, \{b, c\}\}, \\ \{x, y\}, & \text{if } A = \{a, c\}, \\ \{z\}, & \text{if } A = \{a, b\}, \\ Y, & \text{if } A = X. \end{cases}$$

Since $(x,y)\in R_H$ iff $y\in H(\{x\}^c)^c$, we obtain:

$$R_H=\{(a,y),(a,z),(b,z),(c,x),(c,y)\},[R_H]=H,\langle R_H\rangle=H^\partial.$$

(6) If H is a necessity operator, then, by Theorem 3.8, we obtain H as same in (5). Since $(x,y)\in R_H$ iff $y\in H(\{x\}^c)$, we obtain:

$$R_H=\{(a,x),(b,x),(b,y),(c,z)\},[[R_H]]^*=H,\langle\langle R_H\rangle\rangle^*=H^\partial.$$

(7) If H is a co-sufficiency operator, then, by Theorem 3.7, we have:

$$H(A)=\begin{cases}\emptyset, & \text{if } A=X,\\ \{x,y\}, & \text{if } A\in\{\{c\},\{a,c\}\},\\ \{x,z\}, & \text{if } A=\{b\},\\ \{z\}, & \text{if } A=\{a,b\},\\ \{x\}, & \text{if } A=\{b,c\},\\ Y, & \text{if } A\in\{\emptyset,\{a\}\}.\end{cases}$$

Since $(x,y)\in R_H$ iff $y\in H(\{x\}^c)^c$, we obtain:

$$R_H=\{(a,y),(a,z),(b,z),(c,x),(c,y)\},\langle\langle R_H\rangle\rangle=H,[[R_H]]=H^\partial.$$

(8) If H is a co-sufficiency operator, then, by Theorem 3.9, we obtain H as same in (7). Since $(x,y)\in R_H$ iff $y\in H(\{x\}^c)$, we obtain:

$$R_H=\{(a,x),(b,x),(b,y),(c,z)\},\langle R_H\rangle^*=H,[R_H]^*=H^\partial.$$

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