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SENSITIVITY ANALYSIS OF A SHAPE CONTROL PROBLEM FOR THE NAVIER–STOKES EQUATIONS

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Abstract. We deal with a sensitivity analysis of an optimal shape control problem for the stationary Navier–Stokes system. A two–dimensional channel flow of an incompressible, viscous fluid is examined to determine the shape of a bump on a part of the boundary that minimizes the viscous drag. By using the material derivative method and adjoint variables for a shape sensitivity analysis, we derive the shape gradient of the design functional for the model problem.

1. Introduction

We examine a sensitivity analysis of a shape optimization problem for the stationary Navier–Stokes equations of incompressible viscous flow. Specifically, we want to determine the shape of a bump along the wall of a two-dimensional channel that leads to a minimum value for the viscous drag. Existence results for this problem were given in [15] where one may also find a derivation of the model problem.

Sensitivity analyses in a shape control problem are concerned with the relationship between available control parameters and responses of the

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state variables and design functional to changes in those parameters. This relationship is embodied in the sensitivities and shape gradient, i.e., roughly speaking the derivatives of the state variables and design functional, respectively, with respect to parameters that determine the shape of the boundary. Thus, the computation and analysis of sensitivities and of the shape gradient play a central role in a shape sensitivity analysis. Simplified situations have been studied by using the normal variation method for the shape sensitivity; see, e.g., [12] and [21]. In our setting, the domain and state variables are not smooth enough to accommodate the normal variation method. Instead, we will use the material derivative method to describe the domain perturbation and to compute the shape gradient. The application of the material derivative method to shape sensitivity analyses was systematized by Zolésio in [31]. In [8] and [9], Delfour and Zolésio developed a shape calculus that may be used to determine the shape gradient and the shape Hessian. We will also use the adjoint equation technique to simplify the computation of the shape gradient. Our aim here is to provide a systematic shape sensitivity analysis for a problem in which the viscous drag is minimized through the use of shape modifications and to derive a useful formula for the shape gradient of the design functional.

The plan of the rest of the paper is as follows. In the remainder of this section, we describe the model problem and introduce some notation. Then, in section 2, we state some results of [15] concerning the existence of optimal solutions. In section 3, we discuss some notions concerning the material derivative method. In section 4, a shape sensitivity analysis and the adjoint equation method are used to derive the shape gradient for our model problem. Then, the regularity of the state and adjoint systems to justify the sensitivity is discussed in section 5.

1.1. The model problem. We consider the two–dimensional incompressible flow of a viscous fluid passing through a channel having a finite depth; see Figure 1. Let $\mathbf{g}_1$ and $\mathbf{g}_2$ be the preset velocities at the inflow Γ_1 and outflow Γ_2 of the channel, respectively. Along the bottom and top sides of the channel the velocity vanishes. The arc $\Gamma_b(\alpha)$, which is part of the bottom boundary, represents the bump, which is to be determined.

Let the boundary shape corresponding to the bump be represented by the graph of the curve $\alpha : [M_1, M_2] \to \mathbb{R}$. The domain Ω_α is composed of two fixed rectangles and a domain with an unknown boundary. Thus,

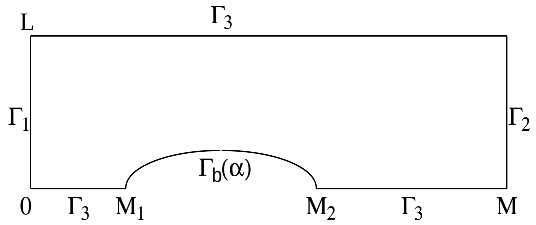

FIGURE 1. Domain Ω_α for flow through a channel with a bump.

the domain Ω_α is determined by the shape of the unknown boundary $\Gamma_b(\alpha)$ which we assume is given by

$$\Gamma_b(\alpha) = \{(x_1, x_2) \in [M_1, M_2] \times [0, L] \mid x_2 = \alpha(x_1)\},$$

where $\alpha(x_1)$ is a function to be determined by the optimization process. Let $\Gamma_\alpha = \partial\Omega_\alpha = \cup_{i=1}^3 \Gamma_i \cup \Gamma_b(\alpha)$ so that $\Gamma_3 = \Gamma_\alpha - \Gamma_1 \cup \Gamma_2 \cup \Gamma_b(\alpha)$. Assume that both end points of $\Gamma_b(\alpha)$ are fixed (at $x_1 = M_1, x_2 = 0$ and $x_1 = M_2, x_2 = 0$) for all admissible domains. Since the domain Ω_α is determined by the shape of $\Gamma_b(\alpha)$, one may define the admissible family of curves defining $\Gamma_b(\alpha)$ as follows:

$$\begin{aligned}\mathcal{U}_{ad} \;&= \{\, \alpha \in C^{0,1}([M_1, M_2]) \mid 0 \le \alpha(x_1) \le L\,,\\ &\qquad |\alpha(x_1) - \alpha(\overline{x}_1)| \le \beta |x_1 - \overline{x}_1| \quad \forall x_1, \overline{x}_1 \in [M_1, M_2]\}\,,\end{aligned}$$

where the positive constant β is chosen in such a way that $\mathcal{U}_{ad} \neq \emptyset$. We have denoted the set of Lipschitz continuous functions in $[M_1, M_2]$ by the symbol $C^{0,1}([M_1, M_2])$.

The condition $|\alpha(x_1) - \alpha(\overline{x}_1)| \le \beta|x_1 - \overline{x}_1|$ is invoked to prevent the "blow-up" of the boundary, i.e., to suppress excessive oscillations of $\Gamma_b(\alpha)$. In [22], an example is provided illustrating the observation that when the boundary is allowed to oscillate, the limit of a sequence that minimizes the objective functional may have nothing to do with the given optimization problem.

We consider, for each $\alpha \in \mathcal{U}_{ad}$, the viscous, incompressible, stationary Navier–Stokes equations in nondimensional form in Ω_α. Let $\mathbf{u} = (u_1, u_2)^T$ denote the velocity and p the pressure. Then, we have

$$-\nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega_\alpha \tag{1.1}$$

and

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega_\alpha \tag{1.2}$$

along with the Dirichlet boundary conditions

$$\mathbf{u} = \mathbf{g} = \begin{cases} \mathbf{g}_1 & \text{on } \Gamma_1 \\ \mathbf{g}_2 & \text{on } \Gamma_2 \\ \mathbf{0} & \text{on } \Gamma_3 \cup \Gamma_b(\alpha)\,, \end{cases} \tag{1.3}$$

where $\mathbf{f}$ and $\mathbf{g}_i$, $i = 1, 2$, are given functions. Here, Δ and ∇ denote the Laplacian and gradient operators in $\mathbb{R}^2$, respectively, $\mathbf{f}$ denotes the given external force, and, in the nondimensional form of the Navier–Stokes equations, ν denotes the reciprocal of the Reynolds number *Re*. Note that the constant density has been absorbed into the pressure and the body force.

One can examine several objectives for determining the shape of the bump, e.g., the reduction of the drag due to viscosity or the identification of the velocity at a fixed vertical slit downstream of the bump. To fix ideas, we focus on the minimization of the design functional

$$\begin{aligned} \mathcal{J}(\alpha) = \mathcal{J}\Big(\Omega_\alpha, \mathbf{u}(\alpha)\Big) \quad &= 2\nu \int_{\Omega_\alpha} D(\mathbf{u}) : D(\mathbf{u})\, d\Omega \\ &= \frac{\nu}{2} \sum_{i,j=1}^{2} \int_{\Omega_\alpha} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 d\Omega\,, \end{aligned} \tag{1.4}$$

where $\mathbf{u}(\alpha)$ is a solution of (1.1)-(1.3) in Ω_α and $D(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$ is the deformation tensor for the flow $\mathbf{u}$. This functional represents the rate of energy dissipation due to deformation. Physically, except for an unimportant additive constant whose value depends on the data $\mathbf{f}$, $\mathbf{g}_1$, and $\mathbf{g}_2$, this functional represents the viscous drag of the flow. In (1.4), the colon denotes the scalar product operator between two tensors. (Again, our results remain valid if we consider other functionals such as the identification of the velocity at a location downstream of the bump.)

The extremal problem we consider is then given as follows:

$$\begin{aligned} &\min_{\alpha \in \mathcal{U}_{ad}} \mathcal{J}(\Omega(\alpha), \mathbf{u}(\alpha)) \quad \text{such that, for some } p(\alpha), \\ &(\mathbf{u}(\alpha), p(\alpha)) \text{ is a solution of (1.1)-(1.3) in } \Omega(\alpha)\,. \end{aligned} \tag{1.5}$$

1.2. Notation. Throughout, depending on the context, $\mathcal{I}$ will denote the identity mapping or the identity matrix; C denotes a generic constant whose value also depends on context. We denote by $H^s(\Omega)$, $s \in \mathbb{R}$, the standard Sobolev space of order s with respect to the set Ω, which is either the flow domain Ω_α, or its boundary Γ_α, or part of its boundary.

Whenever m is a nonnegative integer, the inner product over $H^m(\Omega)$ is given by

$$(f,g)_{m,\Omega} = (f,g)_{0,\Omega} + \sum_{0<|\lambda|\le m} (D^\lambda f, D^\lambda g)_{0,\Omega}\,,$$

where $(f,g)_{0,\Omega} = \int_\Omega fg\, d\Omega$ denotes the inner product over $H^0(\Omega) = L^2(\Omega)$ and λ denotes a multi-index. Hence, we naturally associate the norm on $H^m(\Omega)$ with $\|f\|_{m,\Omega} = \sqrt{(f,f)_{m,\Omega}}$. Whenever there is no chance for confusion, we will, for the flow domain Ω_α, let $(\cdot,\cdot)_{m,\Omega_\alpha} = (\cdot,\cdot)_m$ and $\|\cdot\|_{m,\Omega_\alpha} = \|\cdot\|_m$.

For vector–valued functions and spaces, we use boldface notation. For example, $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^n$ denotes the space of $\mathbb{R}^n$–valued functions such that each component belongs to $H^s(\Omega)$. Of special interest to us is the space

$$\mathbf{H}^1(\Omega) = \left\{ v_j \in L^2(\Omega) \;\Big|\; \frac{\partial v_j}{\partial x_k} \in L^2(\Omega) \quad \text{for } j,k = 1,2 \right\}$$

equipped with the norm $\|\mathbf{v}\|_1 = (\sum_{i=1}^{2} \|v_i\|_1^2)^{1/2}$. For $\Gamma_s \subset \Gamma = \partial\Omega$ with nonzero measure, we also consider the subspace

$$\mathbf{H}^1_{\Gamma_s}(\Omega) = \{\, \mathbf{v} \in \mathbf{H}^1(\Omega) \,|\, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_s \,\}\,;$$

we let $\mathbf{H}^1_0(\Omega) = \mathbf{H}^1_\Gamma(\Omega)$. For any $\mathbf{v} \in \mathbf{H}^1(\Omega)$, we let

$$|\!|\!|\mathbf{v}|\!|\!| = 2\left(\int_\Omega D(\mathbf{v}) \,:\, D(\mathbf{v})\, d\Omega \right)^{1/2} = \frac{1}{2}\Big(\sum_{i,j=1}^{2} \|\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\|_0^2 \Big)^{1/2}.$$

By applying Korn's inequality and a compactness argument, we obtain that whenever Ω is a Lipschitz continuous bounded domain and Γ_s is a subset of Γ with a positive measure, then, there exists a positive constant C such that

$$|\!|\!|\mathbf{v}|\!|\!| \ge C\|\mathbf{v}\|_1 \quad \text{for all } \mathbf{v} \in \mathbf{H}^1_{\Gamma_s}(\Omega)\,. \tag{1.6}$$

Note that the constant C in (1.6) is independent of the choice of $\mathbf{v}$. Thus, we have that $|\!|\!|\cdot|\!|\!|$ is a norm which is equivalent to the norm $\|\cdot\|_{1,\Omega}$ on $\mathbf{H}^1_{\Gamma_s}(\Omega)$. Hence, if we take the inner product on $\mathbf{H}^1_{\Gamma_s}(\Omega)$ to be $((\mathbf{u},\mathbf{v}))_1 = 2(D(\mathbf{u}), D(\mathbf{v}))_{0,\Omega}$, then $|\!|\!|\mathbf{u}|\!|\!| = ((\mathbf{u},\mathbf{u}))_1^{1/2}$.

For each $\alpha \in C^{0,1}([M_1, M_2])$, let $\Gamma_0(\alpha) = \Gamma_3 \cup \Gamma_b(\alpha)$ and $\Gamma_g = \Gamma_1 \cup \Gamma_2$ so that $\Gamma_\alpha = \Gamma_0(\alpha) \cup \Gamma_g$. Since $\mathbf{u} = \mathbf{0}$ on $\Gamma_0(\alpha)$, we may define a generalized velocity space as

$$\mathbf{V}_\alpha = \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega_\alpha) = \{\, \mathbf{u} \in \mathbf{H}^1(\Omega_\alpha) \;\mid\; \mathbf{u} = \mathbf{0} \;\text{ on } \Gamma_0(\alpha) \,\}\,;$$

$\mathbf{V}_\alpha$ is the space of $\mathbf{H}^1(\Omega_\alpha)$–functions that vanish on $\Gamma_0(\alpha)$, i.e., $\mathbf{V}_\alpha$ is the space on which the homogeneous essential boundary condition is imposed. Let $\mathbf{V}^*_\alpha$ be the dual space of $\mathbf{V}_\alpha$. Note that $\mathbf{V}^*_\alpha$ is a subspace of $\mathbf{H}^{-1}(\Omega_\alpha)$, where the latter is the dual space of $\mathbf{H}^1_0(\Omega_\alpha)$. The duality pairing between $\mathbf{V}^*_\alpha$ and $\mathbf{V}_\alpha$ is denoted by $< \cdot, \cdot >_{-1}$.

Let

$$\mathbf{W}_\alpha = \mathbf{H}^{1/2}(\Gamma_g) = \{\, \mathbf{s} \in \mathbf{H}^{1/2}(\Gamma) \mid \mathbf{s} = 0 \;\text{ on } \Gamma_0(\alpha) \,\}\,;$$

$\mathbf{W}_\alpha$ is the subspace of $\mathbf{H}^{1/2}(\Gamma)$ consisting of functions that vanish on $\Gamma_0(\alpha)$. Let $\mathbf{W}^*_\alpha$ denote its dual space and let $< \cdot, \cdot >_{-1/2,\Gamma_g}$ denote the duality pairing between $\mathbf{W}^*_\alpha$ and $\mathbf{W}_\alpha$.

Since Γ_g is smooth, the trace mapping $\gamma_{\Gamma_g} : \mathbf{H}^1(\Omega_\alpha) \to \mathbf{W}_\alpha = \mathbf{H}^{1/2}(\Gamma_g)$ is well–defined and $\mathbf{W}_\alpha = \gamma_{\Gamma_g}(\mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega_\alpha)) = \gamma_{\Gamma_g}\mathbf{V}_\alpha$ for each $\alpha \in \mathcal{U}_{ad}$. Now, let $\mathbf{g}$ be an element of $\mathbf{W}_\alpha = \mathbf{H}^{1/2}(\Gamma_g)$. It is well–known that $\mathbf{W}_\alpha$ is a Hilbert space with the norm

$$\|\mathbf{g}\|_{1/2,\Gamma_g} = \inf_{\mathbf{v}\in\mathbf{V}_\alpha,\, \gamma_{\Gamma_g}\mathbf{v}=\mathbf{g}} \|\mathbf{v}\|_{1,\Omega_\alpha} \qquad \forall\, \mathbf{g} \in \mathbf{W}_\alpha\,.$$

Let $\mathbf{s}$ belong to $\mathbf{W}^*_\alpha$. By the definition of the dual norm, we note that

$$\|\mathbf{s}\|_{-1/2,\Gamma_g} = \sup_{\mathbf{g}\in\mathbf{W}_\alpha,\, \mathbf{g}\neq\mathbf{0}} \frac{< \mathbf{s}, \mathbf{g} >_{-1/2,\Gamma_g}}{\|\mathbf{g}\|_{1/2,\Gamma_g}} \quad \forall\, \mathbf{s} \in \mathbf{W}^*_\alpha\,.$$

It is shown in [15] that

$$\|\mathbf{s}\|_{-1/2,\Gamma_g} = \sup_{\mathbf{v}\in\mathbf{V}_\alpha,\, \mathbf{v}\neq\mathbf{0}} \frac{< \mathbf{s}, \gamma_{\Gamma_g}\mathbf{v} >_{-1/2,\Gamma_g}}{\|\mathbf{v}\|_{1,\Omega_\alpha}} \quad \forall\, \mathbf{s} \in \mathbf{W}^*_\alpha \tag{1.7}$$

provides an alternate and equivalent definition for the dual norm $\|\cdot\|_{-1/2,\Gamma_g}$. In the sequel we will simply write $< \mathbf{s}, \mathbf{v} >_{-1/2,\Gamma_g}$ instead of $< \mathbf{s}, \gamma_{\Gamma_g}\mathbf{v} >_{-1/2,\Gamma_g}$ whenever $\mathbf{s} \in \mathbf{W}^*_\alpha$ and $\mathbf{v} \in \mathbf{V}_\alpha$.

Since the pressure is determined only up to a constant in the mathematical formulation of the Navier–Stokes equations with velocity boundary conditions, we define the space of generalized pressures to be

$$S_\alpha = L^2_0(\Omega_\alpha) = \left\{ p \in L^2(\Omega_\alpha) \;\middle|\; \int_{\Omega_\alpha} p \, d\Omega = 0 \right\}.$$

Thus, S_α consists of square integrable functions having zero mean over Ω_α.

2. Existence of optimal solutions

We now state some results of [15] concerning the existence of optimal solutions satisfying (1.5). We first recast this problem into a precise function space setting.

2.1. Weak variational formulation of the state equations. For the weak variational formulation, we will use the forms

$$\begin{aligned} a_\alpha(\mathbf{u},\mathbf{v}) &= 2\int_{\Omega_\alpha} D(\mathbf{u}) : D(\mathbf{v})\, d\Omega \\ &= \frac{1}{2}\sum_{i,j=1}^{2}\int_{\Omega_\alpha}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right) d\Omega\,, \end{aligned}$$

$$b_\alpha(\mathbf{v},q) = -\int_{\Omega_\alpha} q\,\nabla\cdot\mathbf{v}\, d\Omega = -\sum_{i=1}^{2}\int_{\Omega_\alpha} q\,\frac{\partial v_i}{\partial x_i}\, d\Omega\,,$$

and

$$c_\alpha(\mathbf{w},\mathbf{u},\mathbf{v}) = \int_{\Omega_\alpha}(\mathbf{w}\cdot\nabla)\mathbf{u}\,\cdot\,\mathbf{v}\, d\Omega = \sum_{i,j=1}^{2}\int_{\Omega_\alpha} w_j\frac{\partial u_i}{\partial x_j} v_i\, d\Omega\,.$$

Obviously, $a_\alpha(\cdot,\cdot)$ is a continuous bilinear form on $\mathbf{H}^1(\Omega_\alpha)\times\mathbf{H}^1(\Omega_\alpha)$ and $b_\alpha(\cdot,\cdot)$ is a continuous bilinear form on $\mathbf{H}^1(\Omega_\alpha)\times L^2(\Omega_\alpha)$; also, $c_\alpha(\cdot,\cdot,\cdot)$ is a continuous trilinear form on $\mathbf{H}^1(\Omega_\alpha)\times\mathbf{H}^1(\Omega_\alpha)\times\mathbf{H}^1(\Omega_\alpha)$ which can be verified by the Sobolev embedding of $\mathbf{H}^1(\Omega_\alpha)\subset\mathbf{L}^4(\Omega_\alpha)$ and Hölder's inequality. As a consequence of (1.6), we have the coercivity property

$$a_\alpha(\mathbf{v},\mathbf{v}) \ge C\|\mathbf{v}\|_1^2 \quad \forall\,\mathbf{v}\in\mathbf{V}_\alpha\,; \tag{2.1}$$

we also have the inf–sup condition (or LBB–condition)

$$\inf_{q\in S_\alpha}\ \sup_{\mathbf{v}\in\mathbf{H}_0^1(\Omega_\alpha)} \frac{b_\alpha(\mathbf{v},q)}{\|\mathbf{v}\|_1\|q\|_0} \ge C\,. \tag{2.2}$$

For details concerning these forms and their properties, one may consult [11], [19], or [30].

One can show that (1.1)–(1.3) have the following weak formulation: for each $\alpha\in\mathcal{U}_{ad}$, find $\mathbf{u}\in\mathbf{V}_\alpha$, $p\in S_\alpha$, and $\mathbf{t}\in\mathbf{W}_\alpha^*$ satisfying

$$\nu a_\alpha(\mathbf{u},\mathbf{v}) + c_\alpha(\mathbf{u},\mathbf{u},\mathbf{v}) + b_\alpha(\mathbf{v},p) - <\mathbf{t},\mathbf{v}>_{-1/2,\Gamma_g} = <\mathbf{f},\mathbf{v}>_{-1} \tag{2.3}$$

for all $\mathbf{v}\in\mathbf{V}_\alpha$, and

$$b_\alpha(\mathbf{u},q) = 0 \quad \forall q\in S_\alpha\,, \tag{2.4}$$

(2.5) $$< \mathbf{s}, \mathbf{u} >_{-1/2,\Gamma_g} = < \mathbf{s}, \mathbf{g} >_{-1/2,\Gamma_g} \quad \forall \mathbf{s} \in \mathbf{W}_\alpha^* .$$

In showing that (2.3) is a weak formulation of (1.1), it is convenient to replace the viscous term in the latter with $2\nu\nabla \cdot (D(\mathbf{u}))$; the equivalence of the two forms of the viscous terms follows from the incompressibility constraint (1.2). Also, since Γ_g is smooth, the trace mapping $\gamma_{\Gamma_g} : \mathbf{V}_\alpha \to \mathbf{W}_\alpha$ is well–defined and $\mathbf{W}_\alpha = \gamma_{\Gamma_g}\mathbf{V}_\alpha$ for each $\alpha \in \mathcal{U}_{ad}$; hence, (2.5) is well–justified. Note that the inhomogeneous boundary condition on the velocity is enforced weakly through the use of Lagrange multipliers; see [1], [13], [14], and [15].

It can be shown that, in the sense of distributions, $\mathbf{t}$ is the stress vector on Γ_g, i.e.,

$$\mathbf{t} = -p\mathbf{n} + 2\nu D(\mathbf{u}) \cdot \mathbf{n} = -p\mathbf{n} + \nu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) \cdot \mathbf{n} \quad \text{on } \Gamma_g .$$

Existence and uniqueness results for solutions of the system (2.3)–(2.5) are contained in the following theorem; for a proof, one may consult [11], [13], [19], [24], or [30].

THEOREM 2.1. *Let $\alpha \in \mathcal{U}_{ad}$ be fixed and let the data satisfy $\mathbf{f} \in \mathbf{V}_\alpha^*$, $\mathbf{g} \in \mathbf{W}_\alpha$, and the compatibility condition $\int_{\Gamma(\alpha)} \mathbf{g} \cdot \mathbf{n}\, d\Gamma = 0$. Then,*

(I) *there exists at least one solution $(\mathbf{u}, p, \mathbf{t}) \in \mathbf{V}_\alpha \times S_\alpha \times \mathbf{W}_\alpha^*$ of* (2.3)–(2.5)*;*

(II) *the set of velocity fields that are solutions of* (2.3)–(2.5) *is closed in $\mathbf{H}^1(\Omega_\alpha)$ and is compact in $\mathbf{L}^2(\Omega_\alpha)$; and*

(III) *if $\nu > \nu_0(\Omega_\alpha; \mathbf{f}, \mathbf{g})$ for some positive constant ν_0 whose value is determined by the given data, then the set of solutions of* (2.3)–(2.5) *is composed of a single element.*

Note that the solutions of (2.3)–(2.5) exist for any Reynolds number; however, (III) implies that uniqueness can be guaranteed only for "large enough" values of ν or for "small enough" values of the data $(\mathbf{f}, \mathbf{g}) \in \mathbf{V}_\alpha^* \times \mathbf{W}_\alpha$.

2.2. The extremal problem. In the notation introduced in subsections 1.2 and 2.1, the design functional $\mathcal{J}$ defined in (1.4) can be expressed in the form

(2.6) $$\mathcal{J}(\alpha) = \mathcal{J}\Big(\Omega_\alpha, \mathbf{u}(\alpha)\Big) = 2\nu \int_{\Omega_\alpha} D(\mathbf{u}) : D(\mathbf{u})\, d\Omega$$

Due to the regularity of $\mathbf{u}(\alpha)$, it is obviously followed that $\mathcal{J}(\alpha) < \infty$ for every $\alpha \in \mathcal{U}_{ad}$. We introduce the admissibility set of controls and velocities

$$\mathcal{V}_{ad} = \Big\{ (\alpha, \mathbf{u}(\alpha)) \in \mathcal{U}_{ad} \times \mathbf{V}_\alpha \;\Big|\; \exists\, p(\alpha) \in S_\alpha \text{ and } \mathbf{t}(\alpha) \in \mathbf{W}^*_\alpha \text{ such that} (\mathbf{u}(\alpha), p(\alpha), \mathbf{t}(\alpha)) \text{ is a solution of (2.3)–(2.5)} \Big\}.$$

Then, the extremal problem (1.5) can be restated in the following precise form:

$$\min_{(\alpha,\mathbf{u}(\alpha))\in\mathcal{V}_{ad}} \mathcal{J}(\alpha, \mathbf{u}(\alpha))\,. \tag{2.7}$$

We state the following existence result of [15] and [17].

THEOREM 2.2. *There exists at least one optimal solution* $(\alpha^*, \mathbf{u}(\alpha^*)) \in \mathcal{V}_{ad}$ *for the problem (2.7).*

3. The material derivative method

In this section, we give a brief overview of the material derivative method, following closely the presentations of [8], [29], [31], and [16].

To describe continuous variations of a shape, the perturbation process from a fixed domain Ω to a domain Ω_t parametrized by a "time" t can be formalized through a (smooth) vector field $\mathbf{V}(t, \cdot)$ defined in a neighborhood of Ω and a locally one-to-one transformation $\mathcal{F}_t$. For a given point $\mathbf{p} \in \Omega$, let us consider the system of differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{V}(t, \mathbf{x}(t)) \quad \text{and} \quad \mathbf{x}(0) = \mathbf{p}\,. \tag{3.1}$$

For some $\sigma > 0$ this naturally induces a locally one-to-one transformation $\mathcal{F}_t$ such that

$$\mathcal{F}_t(\mathbf{p}) = \mathbf{x}(t) = \mathbf{p} + \int_0^t \mathbf{V}(s, \mathbf{x}(s))\,ds\,,$$

for $0 \le t < \sigma$ and $\mathbf{p} \in \Omega \subset \mathbb{R}^n$. As a result, the reference domain Ω is locally transported to $\Omega_t = \mathcal{F}_t(\Omega)$ in the $\mathbf{V}$-direction, where $\Omega = \Omega_0 = \mathcal{F}_0(\Omega)$. Naturally, we assume the boundary is preserved under the transformation $\mathcal{F}_t$, i.e., $\partial\Omega_t = \mathcal{F}_t(\partial\Omega)$ and $\partial\Omega = \mathcal{F}_0(\partial\Omega)$. In case that the dependency of $\mathcal{F}_t$ on $\mathbf{V}$ should be emphasized, we will write $\mathcal{F}_t = \mathcal{F}_{(t,\mathbf{V})}$.

By the local existence and uniqueness theorem for the system of ordinary differential equations and the integral representation of the trajectory $\mathcal{F}_t$, it is not difficult to show that

$$\mathcal{F}_{(t_1+t_2,\mathbf{V})}(\mathbf{q}) = \mathcal{F}_{(t_2,\mathbf{V}_{t_1})}(\mathcal{F}_{(t_1,\mathbf{V})}(\mathbf{q}))$$

for all $t_1,\ t_2 \geq 0$ such that $0 \leq t_1,\ t_2,\ t_1+t_2 < \sigma$ and all points $\mathbf{q}$ in a neighborhood $\mathcal{O}_{\mathbf{p}}$ of $\mathbf{p}$, where $\mathbf{V}_{t_1}(s,\cdot) = \mathbf{V}(t_1+s,\cdot)$.

For practical applications, we consider the case wherein all the perturbations of a domain are contained in a fixed domain $\widehat{\Omega}$. To be more precise, let us assume that $\widehat{\Omega}$ is a bounded open set containing Ω and that, for some $\widehat{t} > 0$, $\mathbf{V} : [\,0,\widehat{t}\,] \times \widehat{\Omega} \longrightarrow \mathbb{R}^n$ denotes a continuous vector field. Suppose that $t \mapsto \mathbf{V}(t,\mathbf{x})$ is continuous for each $\mathbf{x} \in \widehat{\Omega}$ and $\mathbf{V}(t,\cdot)$ is Lipschitz continuous, i.e., there exists a positive constant $C > 0$ such that

$$|\mathbf{V}(t,\mathbf{x}_1) - \mathbf{V}(t,\mathbf{x}_2)| \leq C\,|\mathbf{x}_1 - \mathbf{x}_2| \tag{3.2}$$

for all $t \in [\,0,\widehat{t}\,]$ and every $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\widehat{\Omega}$. Then, for any $\mathbf{p} \in \Omega$, there exists a $\widehat{\sigma} \in (0,\widehat{t}\,)$, an open neighborhood $\mathcal{O}_{\mathbf{p}}$ of $\mathbf{p}$ in $\widehat{\Omega}$, and a one–to–one transformation

$$\mathcal{F}_t : \mathcal{O}_{\mathbf{p}} \to \mathcal{F}_t(\mathcal{O}_{\mathbf{p}}) \subset \mathbb{R}^n \qquad \text{for } \ 0 \leq t < \widehat{\sigma}\,,$$

such that $t \mapsto \mathcal{F}_t(\mathbf{p})$ is a unique solution of (3.1) for $0 \leq t < \widehat{\sigma}$. We assume that

$$\bigcup_{\mathbf{p}\in\Omega,\ 0\leq t<\widehat{\sigma}} \mathcal{F}_t(\mathcal{O}_{\mathbf{p}}) \subset \widehat{\Omega}\,. \tag{3.3}$$

This condition is needed to guarantee the existence of the inverse $\mathcal{F}_t^{-1}$ of $\mathcal{F}_t$ for $0 \leq t < \widehat{\sigma}$ whenever $\mathbf{V}(t,\cdot)$ is defined on $\widehat{\Omega}$.

Since $\overline{\Omega}$ is compact, there exists a finite open covering $\{\mathcal{O}_i\}_{i=1}^m$ of $\overline{\Omega}$ in $\widehat{\Omega}$ with correponding positive numbers $\sigma_1,\ldots,\sigma_m$ and transformations $\{\mathcal{F}_t^{(i)}\}$ such that $\mathcal{F}_t^{(i)} : \mathcal{O}_i \to \mathcal{F}_t^{(i)}(\mathcal{O}_i) \subset \widehat{\Omega}$ is one-to-one for $0 \leq t < \sigma_i$. Let us take $\mathcal{O} = \bigcup_{i=1}^m \mathcal{O}_i$ and $\sigma = \min\{\sigma_1,\cdots,\sigma_m\}$. Notice that by the uniqueness of the solution of the differential equations, $\mathcal{F}_t^{(i)}(\mathbf{q}) = \mathcal{F}_t^{(j)}(\mathbf{q})$ for $\mathbf{q} \in \mathcal{O}_i \cap \mathcal{O}_j$. So if we unambiguously patch the transformations together by defining

$$\mathcal{F}_t(\mathbf{q}) = \mathcal{F}_t^{(i)}(\mathbf{q}) \qquad \text{if } \ \mathbf{q} \in \mathcal{O}_i\,,$$

then clearly $\mathcal{F}_t : \Omega \to \Omega_t \subset \widehat{\Omega}$ is a one–to–one transformation for $0 \leq t < \sigma$. The continuity of $\mathcal{F}_t(\cdot)$ for all $0 \leq t < \sigma$ easily follows from the integral expression for $\mathcal{F}_t$ and the Lipschitz continuity of $\mathbf{V}(t,\cdot)$. Furthermore, if $\mathbf{V}(t,\cdot)$ is of class C^k over $\widehat{\Omega}$, from classical regularity results, it follows that $\mathcal{F}_t(\cdot)$ is also of class C^k over $\widehat{\Omega}$. Clearly, $(0,\sigma) \ni t \mapsto \mathcal{F}_t(\mathbf{x})$ is continuously differentiable for each $\mathbf{x} \in \Omega_t$.

Next, we consider the inverse $\mathcal{F}_t^{-1}$ of $\mathcal{F}_t$. Note that if $t \mapsto \mathbf{V}(t,\cdot)$ were defined in a neighborhood $(-\sigma,\sigma)$ of 0 and $\dfrac{\partial \mathbf{V}}{\partial t} = \mathbf{0}$, then $\mathcal{F}_{t_1+t_2}(\mathbf{q}) = \mathcal{F}_{t_1}(\mathcal{F}_{t_2}(\mathbf{q}))$ for all t_1 and t_2 such that $-\sigma < t_1,\ t_2,\ t_1+t_2 < \sigma$ and all points $\mathbf{q}$ in a neighborhood $\mathcal{O}_\mathbf{p}$ of $\mathbf{p}$. In this case, $\{\mathcal{F}_t\}_{-\sigma<t<\sigma}$ is a local one parameter group of transformations whose inverse is given by $\mathcal{F}_t^{-1} = \mathcal{F}_{-t}$ for $-\sigma < t < \sigma$.

Since this is not the case, to discuss the inverse $\mathcal{F}_t^{-1}$, we consider the following system of differential equations:

$$
\begin{aligned}
\dot{\mathbf{p}}(s) &= -\mathbf{V}(t-s,\mathbf{p}(s)) \quad \text{for } 0 \le s \le t\,,\\
\mathbf{p}(0) &= \mathbf{x} = \mathcal{F}_t(\mathbf{p}(t)) \quad \text{for } \ \mathbf{x} \in \Omega_t \subset \widehat{\Omega}\,.
\end{aligned}
\tag{3.4}
$$

This introduces a unique Lipschitzian solution $\mathcal{E}_t(\mathbf{x}) = \mathbf{p}(t)$ which, according to the following lemma, is an inverse of $\mathcal{F}_t$.

Lemma 3.1. *Under the assumptions* (3.1) *and* (3.2), *the transformation $\mathcal{E}_t$ induced from* (3.4) *is an inverse of $\mathcal{F}_t$. Moreover, if $\mathbf{V}(t,\cdot)$ is $C^k(\widehat{\Omega})$, so is $\mathcal{F}_t^{-1} = \mathcal{E}_t$.*

Proof. For $0 \le s \le t$, consider the map $s \mapsto \mathcal{F}_{t-s}(\mathbf{p})$. Since $\mathcal{F}_{t-s}(\mathbf{p}) = \mathbf{x}(t-s)$, $\ s \mapsto \mathcal{F}_{t-s}(\mathbf{p})$ is a solution of (3.4), i.e., $\mathcal{F}_{t-s} = \mathcal{E}_s$. Hence $\mathcal{E}_t(\mathcal{F}_t(\mathbf{p})) = \mathbf{p}(t) = \mathcal{F}_{t-t}(\mathbf{p}) = \mathbf{p}$ so that $\mathcal{E}_t$ is a left inverse of $\mathcal{F}_t$. To show that $\mathcal{E}_t$ is also a right inverse of $\mathcal{F}_t$, we consider the function $\mathbf{y}(\xi) = \mathbf{p}(t-\xi)$. Since from (3.4)

$$\dot{\mathbf{y}}(\xi) = \frac{d}{d\xi}\mathbf{p}(t-\xi) = -\,\dot{\mathbf{p}}(t-\xi) = \mathbf{V}(t-(t-\xi),\mathbf{p}(t-\xi)) = \mathbf{V}(\xi,\mathbf{y}(\xi))\,,$$

$\mathbf{y}(\xi)$ is a solution of

$$\dot{\mathbf{y}}(\xi) = \mathbf{V}(\xi,\mathbf{y}(\xi)) \quad \text{and} \quad \mathbf{y}(0) = \mathbf{p}(t)\,.$$

Then, it follows that

$$\mathbf{x} = \mathbf{p}(0) = \mathbf{y}(t) = \mathcal{F}_t(\mathbf{p}(t)) = \mathcal{F}_t(\mathcal{E}_t(\mathbf{x}))$$

and thus $\mathcal{E}_t = \mathcal{F}_t^{-1}$. Note that we can simply write $(\mathcal{F}_{(t,\mathbf{V})})^{-1} = \mathcal{F}_{(t,-\mathbf{V}_t)}$, where $\mathbf{V}_t(s,\cdot) = \mathbf{V}(t-s,\cdot)$.

The regularity result for $\mathcal{E}_t$ can be proved in the same way as for $\mathcal{F}_t$ using the regularity of $\mathbf{V}_t(s,\cdot)$ and (3.4). $\square$

Note that $\mathcal{F}_{(t+h,\mathbf{V})} - \mathcal{F}_{(t,\mathbf{V})} = (\mathcal{F}_{(h,\mathbf{V}_t)} - \mathcal{I}) \circ \mathcal{F}_{(t,\mathbf{V})}$ for $t,\, h > 0$ from which it follows that $\dfrac{\partial}{\partial t}\mathcal{F}_{(t,\mathbf{V})}(\mathbf{p}) = \mathbf{V}(t, \mathcal{F}_{(t,\mathbf{V})}(\mathbf{p}))$. Hence,

$$\mathbf{V}(t,\mathbf{x}) = \frac{\partial \mathcal{F}_t}{\partial t}(\mathcal{F}_t^{-1}(\mathbf{x})) \quad \text{for every } \ \mathbf{x} \in \Omega_t \quad \text{and} \quad 0 \le t < \sigma\,. \tag{3.5}$$

The arguments given so far can be simply stated as follows. If $\mathbf{V}(t,\cdot)$ is of class C^k with $k \ge 0$ over $\widehat{\Omega}$, there exists a C^k–diffeomorphism $\mathcal{F}_t$ from Ω onto Ω_t and, vice versa, if $\{\mathcal{F}_t\}_{0\le t<\sigma}$ is a family of C^k–diffeomorphisms, $\mathbf{V}$ can be recovered from (3.5) and $\mathbf{V}(t,\cdot)$ is also of class C^k. Note that if $\mathbf{V}(t,\cdot)$ is Lipschitz continous, so is $\mathcal{F}_t$, and vice versa.

Remark 3.2. Delfour and Zolésio([6]) showed that for any domain Ω in $\mathbb{R}^n$ and a (smooth) velocity $\mathbf{V} : [\,0,\widehat{t}\,] \times \overline{\Omega} \to \mathbb{R}^n$ satisfying

$$\begin{aligned} \mathbf{V}(t,\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \ &= 0 \quad \text{if the outward normal } \mathbf{n}(\mathbf{x}) \text{ is defined for a.e. } \mathbf{x} \in \partial\Omega \\ \mathbf{V}(t,\mathbf{x}) \ &= 0 \quad \text{otherwise}\,, \end{aligned}$$

the solution $\mathcal{F}_t$ of (3.1) maps $\overline{\Omega}$ into $\overline{\Omega}$ for all $0 \le t \le \widehat{t}$. The above conditions guarantee the nontransversality along $\partial\Omega$, i.e., $\mathcal{F}_{(t,\mathbf{V})}(\Omega) = \Omega$.

Let us briefly describe the variation of a function due to the domain perturbation. Throughout this discussion, we assume that

$$\bigcup_{t\in[\,0,\widehat{t}\,]} \{t\} \times \Omega_t \subset [\,0,\widehat{t}\,] \times \widehat{\Omega}\,.$$

Let $z = z_t(\mathbf{x}) = z(t,\mathbf{x})$ be a regular function defined on $\Omega_t = \mathcal{F}_t(\Omega)$. Then, the composite function $z_t \circ \mathcal{F}_t$ is defined on a fixed reference domain Ω. The material derivative (or Lagrangian derivative) of z_t at $\mathbf{p} \in \Omega$ in the $\mathbf{V}$–direction is defined by the following semi–derivative :

$$\dot{z}(\mathbf{p};\mathbf{V}) = \left\{ \frac{d}{dt}\, z_t(\mathcal{F}_{(t,\mathbf{V})}(\mathbf{p})) \right\}\Bigg|_{t=0^+}\,.$$

If $\{\Omega_t\}_{0\le t\le \widehat{t}}$ is a class of domains with the uniform extension property, we can consider z_t as a restriction of $\widehat{z}$ to $\{t\} \times \Omega_t$, where $\widehat{z}$ is defined globally in $[\,0,\widehat{t}\,] \times \widehat{\Omega}$, i.e.,

$$\widehat{z}(t,\mathbf{x}) = \widehat{P}(z_t(\mathcal{F}_t))(\mathcal{F}_t^{-1}(\mathbf{x}))\,,$$

where $\widehat{P}$ denotes the Calderon extension operator to $\widehat{\Omega}$. Then, using the chain rule, the material derivative can be written as

$$\text{(3.6)}\quad \dot{z}(\mathbf{p};\mathbf{V}) = \lim_{t\to 0^+} \frac{\widehat{z}(t,\mathcal{F}_t(\mathbf{p})) - \widehat{z}(0,\mathbf{p})}{t} = \frac{\partial \widehat{z}}{\partial t}\Big|_{(0,\mathbf{p})} + (\mathbf{V}\cdot\nabla z)\Big|_{(0,\mathbf{p})},$$

where $\nabla z = (\frac{\partial z}{\partial x_1}, \cdots, \frac{\partial z}{\partial x_n})^T$. Similarly, if $\mathbf{v} = \mathbf{v}_t(\mathbf{x}) = \mathbf{v}(t,\mathbf{x})$ is a vector–valued function defined in Ω_t and $\widehat{\mathbf{v}}(t,\cdot)$ is its extension to $\widehat{\Omega}$, the material derivative of $\mathbf{v}_t$ can be written as

$$\dot{\mathbf{v}}(\mathbf{p};\mathbf{V}) = \frac{\partial \widehat{\mathbf{v}}}{\partial t}\Big|_{(0,\mathbf{p})} + (\mathbf{V}\cdot\nabla\mathbf{v})\Big|_{(0,\mathbf{p})}.$$

Next, we consider functionals. Let $\mathcal{K}(\Omega)$ be any domain functional on $\widehat{\Omega}$. The rate of variation of $\mathcal{K}(\Omega)$ at the reference domain Ω with respect to the domain perturbation may be measured by the directional semi–derivative

$$d\mathcal{K}(\Omega;\mathbf{V}) = \lim_{t\to 0^+} \frac{\mathcal{K}(\Omega_t) - \mathcal{K}(\Omega)}{t} = \left\{\frac{d}{dt}\mathcal{K}(\mathcal{F}_{(t,\mathbf{V})}(\Omega))\right\}\Big|_{t=0^+}.$$

A functional $\mathcal{K}(\Omega)$ is said to be shape differentiable if

(a) $d\mathcal{K}(\Omega;\mathbf{V})$ exists for all directions $\mathbf{V}$ and
(b) $\mathbf{V} \mapsto d\mathcal{K}(\Omega;\mathbf{V})$ is linear and continuous over appropriate admissible vector fields.

If $\mathcal{K}(\Omega)$ is shape differentiable, one can interpret $\mathbf{V} \mapsto d\mathcal{K}(\Omega;\mathbf{V})$ in the distributional sense, i.e.,

$$d\mathcal{K}(\Omega;\mathbf{V}) =< \mathcal{G}(\Omega), \mathbf{V} >,$$

where $\mathcal{G}(\Omega)$ is a vector-valued distribution of a finite order acting on the appropriate trial function space which itself is determined by the regularity of the feasible domains. In this case, $\mathcal{G}(\Omega)$ is called the shape gradient of the design functional $\mathcal{K}(\Omega)$ and is usually written as

$$\mathcal{G}(\Omega) = \operatorname{grad}\mathcal{K}(\Omega).$$

Then, the shape optimization problem is rendered into the problem of finding $\mathcal{G}(\Omega)$.

For the structure of the shape gradient of a design functional, we now list some fundamental properties. This structure will be applied to the shape sensitivity analysis in the next section to obtain the shape gradient for the problem (2.7). Moreover, by using this structure, one can relax the regularity requirements for the feasible domains which are commonly invoked in shape optimization problems.

THEOREM 3.3. *Assume that* $\mathbf{V}$ *belongs to a class of vector fields satisfying* (3.2) *and* (3.3). *Suppose* $\mathcal{K}(\Omega)$ *is shape differentiable at* $\Omega \subset \widehat{\Omega}$. *Then,*

(I) $d\mathcal{K}(\Omega;\mathbf{V}) = d\mathcal{K}(\Omega;\mathbf{V}(0)) \quad \forall\, \mathbf{V} \in C^0([\,0,\widehat{t}\,]\,;\, C^k(\widehat{\Omega};\mathbb{R}^n))\,;$
(II) *the support of* $\mathcal{G}(\Omega) \subset \partial\Omega$; *and*
(III) (Hadamard's Structure Theorem) *there exists a scalar distribution* $g(\partial\Omega)$ *of a finite order such that*

$$d\mathcal{K}(\Omega;\mathbf{V}) = < \mathcal{G}(\Omega), \mathbf{V}(0) >_{\Omega} = < g(\partial\Omega)\,, \mathbf{V}(0)\cdot\mathbf{n} >_{\partial\Omega}\,, \tag{3.7}$$

where $\mathbf{V}(0)\cdot\mathbf{n}$ *is the normal component of* $\mathbf{V}(0) = \mathbf{V}(0,\cdot)$ *on* $\partial\Omega$.

Proof. See, e.g., [8], [29] or [16]. □

Once this $g(\partial\Omega)$ is obtained, a typical optimization algorithm such as gradient method may be employed to determine an optimal shape.

REMARK 3.4. From the first equality in (3.7), $d\mathcal{K}(\Omega;\mathbf{V})$ can be written as

$$d\mathcal{K}(\Omega;\mathbf{V}) = < \mathcal{G}(\Omega), (\mathbf{V}(0)\cdot\mathcal{N})\mathcal{N} >\,,$$

where $\mathcal{N}$ is a unitary extension of $\mathbf{n}$ to $\overline{\widehat{\Omega}}$. Such an extension always exists if $\partial\Omega$ is of class C^k, $(k \geq 1)$, which can be verified by using local atlases along the boundary of the domain and patching them together using cutoff functions; see [29] or [31] for details. Hence, $d\mathcal{K}(\Omega;\mathbf{V})$ can be written in integral form as

$$d\mathcal{K}(\Omega;\mathbf{V}) = \int_{\widehat{\Omega}} (\mathcal{G}(\Omega)\cdot\mathcal{N})(\mathbf{V}(0)\cdot\mathcal{N})\, d\Omega\,.$$

Consequently, in the representations of (3.7) for the shape gradient, $g(\partial\Omega)$ can be related to $\mathcal{G}(\Omega)$ via

$$g(\partial\Omega) = \gamma_{\partial\Omega}(\mathcal{G}(\Omega)\cdot\mathcal{N})$$

and, conversely,

$$\mathcal{G}(\Omega) = \ {}^{T}\gamma_{\partial\Omega}(g(\partial\Omega)\cdot\mathbf{n})\,.$$

For the specific problem considered in the next section, two standard examples of functionals are useful; these are:

$$\mathcal{K}_1(\Omega_t) = \int_{\Omega_t} y_t\, d\Omega_t \qquad \text{and} \qquad \mathcal{K}_2(\Omega_t) = \int_{\partial\Omega_t} y_t\, d\partial\Omega_t\,,$$

where $y_t(\mathbf{x}) = y(t, \mathbf{x})$ is a function defined on $\Omega_t \subset \widehat{\Omega}$ or $\partial\Omega_t$, respectively. Let $\widehat{y}$ be a uniform extension of y_t in $\widehat{\Omega}$. Then, under some reasonable assumptions on the regularity for the feasible domains and the class of functions, one can obtain

$$d\mathcal{K}_1(\Omega; \mathbf{V}) = \int_\Omega \frac{\partial \widehat{y}}{\partial t}\, d\Omega + \int_{\partial\Omega} (\mathbf{V}(0,\cdot) \cdot \mathbf{n}) y|_{t=0}\, d\partial\Omega \tag{3.8}$$

and

$$d\mathcal{K}_2(\Omega; \mathbf{V}) = \int_{\partial\Omega} \Big[\frac{\partial \widehat{y}}{\partial t} + (\mathbf{V}(0,\cdot) \cdot \mathbf{n}) \left(\frac{\partial y}{\partial \mathbf{n}} + \kappa y \right) \Big|_{t=0} \Big]\, d\partial\Omega\,. \tag{3.9}$$

Here, κ denotes the curvature of the boundary curve $\partial\Omega$ when the spatial dimension of the domain is 2 and the mean curvature of the boundary surface $\partial\Omega$ when the spatial dimension is 3. These formulations were introduced by many authors. For derivations, one may refer to [29] and [31].

In these two standard examples for functionals, $d\mathcal{K}_i(\Omega; \mathbf{V})$ consists of two main components: a linear term $\mathbf{V} \cdot \mathbf{n}$ on the boundary and a shape derivative term $\widehat{y}\,' = \widehat{y}\,'(\Omega; \mathbf{V}) = \dfrac{\partial \widehat{y}}{\partial t}$. In order to obtain the shape gradients for $\mathcal{K}_1$ and $\mathcal{K}_2$, it should be justified that $\mathbf{V} \mapsto \dfrac{\partial \widehat{y}}{\partial t}(\Omega; \mathbf{V})$ is linear and continuous over appropriate admissible vector fields. This implies that $\widehat{y}\,'(\Omega; \mathbf{V})$ should be represented as a linear function of $\mathbf{V}$. A major step toward the shape sensitivity analysis is to give a sense to $\widehat{y}\,'(\Omega; \mathbf{V})$ and to find an appropriate linear model for $\widehat{y}\,'(\Omega; \mathbf{V})$.

4. Shape sensitivity analysis

In this section we determine the shape gradient of the design functional (1.4) or (2.6) using the material derivative method. Specifically, we wish to compute the shape gradient of $\mathcal{J}$ at Ω_α in the direction of a specified deformation vector field $\mathbf{V}$. The major difficulty to be overcome is the lack of regularity for the shape derivative of the state variables. For elliptic equations, the shape derivative of the state variables is usually expressed as a solution of a boundary value problem which depends on the state variables and the normal component of the design velocity along the boundary of the reference domain. Once the existence of the solution for this boundary value problem is settled, this equation can be adopted to suppress the regularity requirement and to facilitate the computation of the shape gradient. However, for nonlinear problems, it is rarely expected that one is able to resolve the equations containing the shape derivative.

To get around these difficulties, we will employ adjoint variables that play a central role in eliminating the cumbersome shape derivative of the state variables. Though direct computation is still widely used, the adjoint variable technique seems to provide a more sound mathematical justification for the shape sensitivity analysis. Similar techniques have been systematically studied in the series of papers [5]– [7] using the theory for the differentiability of the parametrized minmax function of [4] and [10]. However, their applications are mainly based on the uniqueness of the saddle points for a Lagrangian formulation, which can hardly be expected in nonlinear problems such as ours. One can also find a rough framework using a Lagrange multiplier technique in [3]. For a historical background and simple applications in structural engineering, one may refer to [18] and some of the references cited therein. In this section we will mainly concentrate on computing the shape gradient straightforwardly and somewhat formally, leaving the justification to the next section.

4.1. Characterization of domain perturbations. We first note that the main contribution of the deformation to the shape gradient comes from the first order perturbation of the identity operator. In a sufficiently small neighborhood of $t = 0$, one can estimate the deformation at $\mathbf{p} \in \Omega$ as follows. Using (3.1), one has that

$$\mathbf{x}(t) = \mathbf{x}(0) + t\,\dot{\mathbf{x}}(0) + \mathbf{o}(t, \mathbf{x}(0)) = \mathbf{p} + t\,\mathbf{V}(0, \mathbf{p}) + \mathbf{o}(t, \mathbf{p})\,,$$

where $\mathbf{o}(t, \mathbf{p})$ denotes the remainder function such that $\lim_{t\to 0^+} \frac{1}{t}\,|\mathbf{o}(t, \mathbf{p})|_{\mathbb{R}^n} = 0$. Hence, $\mathbf{p} + t\,\mathbf{V}(0, \mathbf{p})$ can be considered as a linear approximation to $\mathcal{F}_t(\mathbf{p})$. Observe that $\mathcal{F}_t(\mathbf{p})$ and $\mathbf{p} + t\,\mathbf{V}(0, \mathbf{p})$ yield the same design velocity at $t = 0^+$. Hence, by Theorem 3.3, one can easily infer that they yield the same shape gradient and shape derivative. Thus, we can take

$$\mathcal{F}_t(\mathbf{p}) = \mathbf{p} + t\,\mathbf{V}(0, \mathbf{p}) = (\mathcal{I} + t\,\mathbf{V})(0, \mathbf{p}) \qquad \text{for } 0 \le t < \sigma\,. \tag{4.1}$$

From the second expression for the deformation, we can regard $\mathcal{F}_t(\mathbf{p})$ as a first order perturbation of the identity operator over the reference domain.

Remark 4.1. If $\sup\limits_{\mathbf{p}\in\mathbb{R}^n} \sum\limits_{|\lambda|\le 1} |D^\lambda \mathbf{V}| < 1$, we easily see that $\mathcal{I}+\mathbf{V}$ and $D(\mathcal{I}+\mathbf{V}) = \mathcal{I}+D\mathbf{V}$ are invertible, where here $D(\mathcal{I}+\mathbf{V})$ and $D\mathbf{V}$ denote the Jacobian matrices for the deformations. This is used in [23] and [26] to show that if $\mathbf{V}$ is of class C^2 and $\sup\limits_{\mathbf{p}\in\mathbb{R}^n} \sum\limits_{|\lambda|\le 1} |D^\lambda \mathbf{V}| < 1$, then $\mathcal{I}+\mathbf{V}$ is C^2–diffeomorphism. Using this device, they derived a shape calculus to accommodate the Hadamard formula for the shape gradient.

The choice of $\mathbf{V}$ is crucial in the shape sensitivity analysis. In our problem, we want to keep the variation of $\Gamma_b(\alpha)$ within the rectangular region Ω_0 depicted by the shaded region in Figure 2, i.e., we want that $\Gamma_b(\alpha) \subset \overline{\Omega}_0$ for every $\alpha \in \mathcal{U}_{ad}$. Note that then $\Omega_\alpha \subset \widehat{\Omega}$, where the latter is the rectangular domain also depicted in Figure 2. An appropriate choice for the velocity is then given by $\mathbf{V} = (0, V_2)^T$.

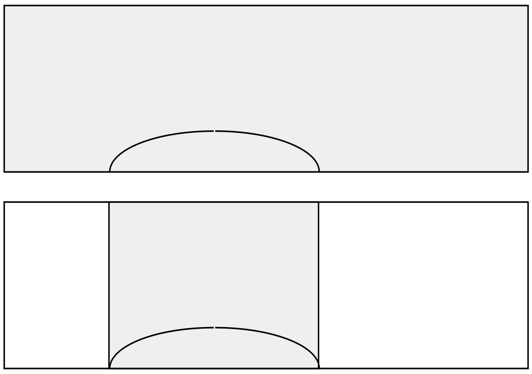

Figure 2. Domains Ω_0 (top) and $\widehat{\Omega}$ (bottom).

Utilizing the mapping technique, V_2 can be characterized as follows. For a fixed $\alpha \in \mathcal{U}_{ad}$, we associate a bijection

$$F_\alpha : \widehat{\Omega} \longrightarrow \Omega_\alpha \ ((\widehat{x}_1, \widehat{x}_2) \mapsto (p_1, p_2))$$

via

$$p_1 = \widehat{x}_1 \quad \text{and} \quad p_2 = \begin{cases} L + \dfrac{(\widehat{x}_2 - L)(L - \alpha(\widehat{x}_1))}{L} & \text{if } M_1 \le \widehat{x}_1 \le M_2 \\ \widehat{x}_2 & \text{otherwise}. \end{cases}$$

Let $\vartheta \in C^{0,1}([M_1, M_2])$ such that $\vartheta(M_1) = \vartheta(M_2) = 0$ and there exists $\sigma > 0$ such that the graph of $\alpha + t\vartheta$ lies in $\overline{\widehat{\Omega}}$ for $0 \le t < \sigma$. We may extend ϑ to $[0, M]$ by defining $\vartheta = 0$ over $[0, M_1] \cup [M_2, M]$. If we consider a bijection

$$F_{\alpha+t\vartheta} : \Omega_\alpha \longrightarrow \Omega(\alpha + t\vartheta) \quad ((\widehat{x}_1, \widehat{x}_2) \mapsto (x_1, x_2)) \,,$$

the composite $F_{\alpha+t\vartheta} \circ F_\alpha^{-1} : \Omega_\alpha \longrightarrow \Omega(\alpha + t\vartheta)$ $((p_1, p_2) \mapsto (x_1, x_2))$ is given by

$$\begin{aligned} x_1 &= p_1 \,, \\ x_2 &= \begin{cases} p_2 + t\dfrac{(p_2 - L)\vartheta(p_1)}{(\alpha(p_1) - L)} & \text{if } M_1 \le p_1 \le M_2 \\ p_2 & \text{otherwise} \,. \end{cases} \end{aligned} \tag{4.2}$$

Since $0 \le \alpha(p_1) < L$ for all $p_1 \in [M_1, M_2]$, the mapping (4.2) is well–defined and $(x_1, x_2) = (p_1, p_2) + t\,(0, V_2(p_1, p_2))$, where

$$V_2(p_1, p_2) = \begin{cases} \dfrac{(p_2 - L)\vartheta(p_1)}{(\alpha(p_1) - L)} & \text{if } M_1 \le p_1 \le M_2 \\ 0 & \text{otherwise} \,. \end{cases} \tag{4.3}$$

Hence, for the perturbation of the domain, it is reasonable to consider the transformation

$$\mathcal{F}_t(p_1, p_2) = (p_1, p_2) + t\,\mathbf{V}(p_1, p_2) = F_{\alpha+t\vartheta} \circ F_\alpha^{-1}(p_1, p_2) \,,$$

where $\mathbf{V} = (0, V_2)^T$ is an autonomous vector field. Clearly, $\mathcal{F}_t$ is a one–to–one transformation from Ω_α onto $\Omega(\alpha + t\vartheta)$ whose inverse is given by $\mathcal{F}_t^{-1}(x_1, x_2) = (p_1, p_2)$, where

$$p_1 = x_1 \quad \text{and} \quad p_2 = \begin{cases} x_2 + t\dfrac{(L - x_2)\vartheta(x_1)}{(\alpha(x_1) - L + t\vartheta(x_1))} & \text{if } M_1 \le x_1 \le M_2 \\ x_2 & \text{otherwise} \,. \end{cases}$$

Note that $\mathbf{V}(p_1, \alpha(p_1)) = (0, \vartheta(p_1))^T$ for all $p_1 \in [M_1, M_2]$. Thus, $\mathbf{V} = (0, \vartheta)^T$ along $\Gamma_b(\alpha)$ and $\mathbf{V} = \mathbf{0}$ along $\Gamma(\alpha) - \Gamma_b(\alpha)$.

4.2. The shape derivative and sensitivities. We now consider the variation of the functional $\mathcal{J}(\alpha)$ given in (2.6) with respect to domain perturbations, i.e., with respect to design functions $\alpha \in \mathcal{U}_{ad}$. Note that since $\mathcal{J}(\alpha) = \mathcal{J}(\alpha, \mathbf{u}(\alpha))$, the computation of the shape gradient may involve the sensitivities of the state variables. Recall that $\mathbf{V} = (0, \vartheta)^T$ along $\Gamma_b(\alpha)$ for any $\vartheta \in C^{0,1}([M_1, M_2])$ such that $\vartheta(M_1) = \vartheta(M_2) = 0$. Since the perturbation of a domain is determined by the variation of the boundary part $\Gamma_b(\alpha)$, for the computation of $\inf\limits_{\alpha\in\mathcal{U}_{ad}} \mathcal{J}(\alpha, \mathbf{u}(\alpha))$, we try to find a semi–derivative

$$d\mathcal{J}(\alpha;\vartheta) \equiv \left\{\frac{d}{dt}\,\mathcal{J}(\alpha + t\,\vartheta)\right\}\bigg|_{t=0^+} = \lim_{t\to 0^+} \frac{\mathcal{J}(\alpha_t) - \mathcal{J}(\alpha)}{t}\,,$$

where $\alpha_t = \alpha + t\,\vartheta$ for $\vartheta \in C^{0,1}([M_1, M_2])$. Then, this will yield the information for the gradient of the design functional $\mathcal{J}(\alpha)$. For this purpose, we assume that $\mathbf{f} \in \mathbf{L}^2(\widehat{\Omega})$. This assumption is needed to guarantee the existence of weak shape derivative of $\mathbf{f}$ in the space of $\mathbf{H}^{-1}(\widehat{\Omega})$ and the regularity of the function space; see section 5.

Let $\mathbf{u}(\alpha_t) \in \mathbf{H}^1(\Omega_t)$ be a solution of the incompressible Navier–Stokes equations (2.3)–(2.5) over $\Omega_t = \Omega(\alpha_t)$. Then, by (2.6), we have that

$$\mathcal{J}(\alpha_t) = 2\nu \int_{\Omega_t} D(\mathbf{u}(\alpha_t)) : D(\mathbf{u}(\alpha_t))\, d\Omega_t\,.$$

The function space $\mathbf{H}^1(\Omega_t)$ depends on the "time" t. To remove this dependence, two methods are widely used (see [9]):

(i) using the homeomorphism $\mathcal{F}_t$, one can transform back onto the reference domain $\Omega(\alpha)$; or
(ii) using the uniform extension property (assuming an adequate regularity of the boundary of the domain), the situation can be considered to be a mere restriction of a function space which is defined on $\widehat{\Omega}$.

To be consistent with the arguments used in [15] to prove the the existence theorem (Theorem 2.2), we choose the second method.

Let $\widehat{\mathbf{u}}(t, \mathbf{x}) = P_{\widehat{\Omega}}(\mathbf{u}(\alpha_t) \circ \mathcal{F}_t) \circ \mathcal{F}_t^{-1}(\mathbf{x})$. Then, $\widehat{\mathbf{u}}(t, \cdot)$ is a uniform extension of $\mathbf{u}(\alpha_t)$ to $\widehat{\Omega}$ such that $\mathbf{u}(\alpha_t) = \widehat{\mathbf{u}}\big|_{\{t\}\times\Omega_t}$. From (3.8), we have that

$$\begin{aligned} d\mathcal{J}(\alpha;\vartheta) &= \left\{\frac{d}{dt} 2\nu \int_{\Omega(\alpha_t)} D(\mathbf{u}(\alpha_t)) : D(\mathbf{u}(\alpha_t))\, d\Omega_t\right\}\bigg|_{t=0^+} \\ &= 4\nu \int_{\Omega_\alpha} D(\mathbf{u}(\alpha)) : D(\widehat{\mathbf{u}}')\, d\Omega \\ &\qquad + 2\nu \int_{\Gamma_\alpha} D(\mathbf{u}(\alpha)) : D(\mathbf{u}(\alpha))\, \mathbf{V}(0,\cdot)\cdot\mathbf{n}\, d\Gamma\,, \end{aligned} \tag{4.4}$$

where $\widehat{\mathbf{u}}'$ denotes the shape derivative of the extension $\widehat{\mathbf{u}}$. This expression gives the change in $\mathcal{J}$ in the direction of ϑ evaluated at α.

The state system (2.3)–(2.5) with respect to $\Omega_t = \Omega(\alpha_t)$ may be expressed in the form

$$\int_{\Omega_t} \Big(2\nu D(\mathbf{u}(\alpha_t)) : D(\mathbf{v}) + \mathbf{u}(\alpha_t) \cdot \nabla \mathbf{u}(\alpha_t) \cdot \mathbf{v} \\ - p(\alpha_t) \nabla \cdot \mathbf{v} - \mathbf{f} \cdot \mathbf{v} \Big) \, d\Omega_t - \int_{\Gamma_t} \mathbf{t}(\alpha_t) \cdot \mathbf{v} \, d\Gamma_t = 0 \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega_t) \, , \tag{4.5}$$

$$\int_{\Omega_t} q \, \nabla \cdot \mathbf{u}(\alpha_t) \, d\Omega_t = 0 \quad \forall \, q \in L^2_0(\Omega_t) \, , \tag{4.6}$$

and

$$< \mathbf{s}, \mathbf{u}(\alpha_t) - \mathbf{g} >_{-1/2, \Gamma_t} = 0 \quad \forall \, \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_t) \, , \tag{4.7}$$

where $\Gamma_t = \partial\Omega_t = \Gamma(\alpha_t)$. Note that we are now enforcing the boundary condition (1.3) on all of the boundary Γ_t through the use of a Lagrange multiplier technique, while in (2.3)–(2.5) we only did so for the part of the boundary Γ_g along which the boundary condition was of the inhomogeneous type. By making this change it becomes clear that the test functions $\mathbf{v}$, q, and $\mathbf{s}$ in (4.5)–(4.7) are independent of t. Indeed, one may choose $\mathbf{v}$ to be the restriction to Ω_t on an arbitrary function belonging to $\mathbf{H}^1(\widehat{\Omega})$. Also, the fact that $\int_{\Gamma_t} \mathbf{g} \cdot \mathbf{n} \, d\Gamma_t = 0$ implies that (4.6) holds for $q =$ constant so that that equation holds for all $q \in L^2(\Omega_t)$. Thus, one may choose q to be the restriction to Ω_t on an arbitrary function belonging to $L^2(\widehat{\Omega})$. Furthermore, in (4.7), we may choose $\mathbf{s}$ to be the restriction to Γ_t of an arbitrary function in $\mathbf{H}^1(\widehat{\Omega})$. Then, using (3.8) and (3.9) and the facts that $\mathbf{v}$, q, $\mathbf{s}$, $\mathbf{f}$, and $\mathbf{g}$ are independent of t, we have by differentiating (4.5)–(4.7), that for all $\mathbf{v} \in \mathbf{H}^1(\Omega_\alpha)$, $q \in L^2_0(\Omega_\alpha)$ and $\mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_\alpha)$

$$\begin{aligned} &\int_{\Omega_\alpha} \Big(2\nu D(\widehat{\mathbf{u}}') : D(\mathbf{v}) + \mathbf{u}(\alpha) \cdot \nabla \widehat{\mathbf{u}}' \cdot \mathbf{v} \\ &\qquad + \widehat{\mathbf{u}}' \cdot \nabla \mathbf{u}(\alpha) \cdot \mathbf{v} - \widehat{p}' \nabla \cdot \mathbf{v} \Big) \, d\Omega - \int_{\Gamma_\alpha} \widehat{\mathbf{t}}\,' \cdot \mathbf{v} \, d\Gamma \\ &= - \int_{\Gamma_\alpha} \Big(2\nu D(\mathbf{u}(\alpha)) : D(\mathbf{v}) + \mathbf{u}(\alpha) \cdot \nabla \mathbf{u}(\alpha) \cdot \mathbf{v} \\ &\qquad - p(\alpha) \nabla \cdot \mathbf{v} - \mathbf{f} \cdot \mathbf{v} \Big) \mathbf{V}(0, \cdot) \cdot \mathbf{n} \, d\Gamma \\ &\quad + \int_{\Gamma_\alpha} \Big(\big(\mathbf{n} \cdot \nabla (\mathbf{t}(\alpha) \cdot \mathbf{v}) + \kappa (\mathbf{t}(\alpha) \cdot \mathbf{v}) \big) \mathbf{V}(0, \cdot) \cdot \mathbf{n} \Big) \, d\Gamma \ , \end{aligned} \tag{4.8}$$

$$\int_{\Omega_\alpha} q\nabla\cdot\widehat{\mathbf{u}}'\,d\Omega = -\int_{\Gamma_\alpha}\big(q\nabla\cdot\mathbf{u}(\alpha)\big)\mathbf{V}(0,\cdot)\cdot\mathbf{n}\,d\Gamma\,, \tag{4.9}$$

and

$$\int_{\Gamma_\alpha}\mathbf{s}\cdot\widehat{\mathbf{u}}'\,d\Gamma = -\int_{\Gamma_\alpha}\big(\mathbf{n}\cdot\nabla(\mathbf{s}\cdot\mathbf{u}(\alpha)) + \kappa\mathbf{s}\cdot\mathbf{u}(\alpha)\big)\mathbf{V}(0,\cdot)\cdot\mathbf{n}\,d\Gamma \tag{4.10}$$

It should be emphasized that these equations should be viewed formally; in particular, the right-hand sides are not necessarily well-defined for the indicated function spaces.

4.3. Adjoint equations. We now want to define adjoint variables which will enable one to compute the shape gradient without having to directly consider the equations dealing with the shape sensitivities. Formally, one may derive equations for the adjoint variables by introducing the Lagrangian $\mathcal{L}:\mathcal{U}_{ad}\times\mathbf{V}_\alpha\times S_\alpha\times\mathbf{W}^*_\alpha\times\mathbf{V}_\alpha\times S_\alpha\times\mathbf{W}^*_\alpha\to\mathbb{R}$ defined by

$$\begin{aligned}\mathcal{L}(\alpha,\mathbf{u},p,\mathbf{t},\boldsymbol{\mu},\xi,\boldsymbol{\tau}) = \mathcal{J}(\alpha,\mathbf{u}) - \{\,&\nu a_\alpha(\mathbf{u},\boldsymbol{\mu}) + c_\alpha(\mathbf{u},\mathbf{u},\boldsymbol{\mu}) + b_\alpha(\boldsymbol{\mu},p)\\ &- <\mathbf{t},\boldsymbol{\mu}>_{-1/2,\Gamma_g} - <\mathbf{f},\boldsymbol{\mu}>_{-1} + b_\alpha(\mathbf{u},\xi) - <\boldsymbol{\tau},\mathbf{u}-\mathbf{g}>_{-1/2,\Gamma_g}\,\}\,,\end{aligned}$$

where $(\boldsymbol{\mu},\xi,\boldsymbol{\tau})\in\mathbf{V}_\alpha\times S_\alpha\times\mathbf{W}^*_\alpha$ are the adjoint variables. Formally, the adjoint equations are defined from the Euler–Lagrange equations for the Lagrangian.

Clearly, variations in the Lagrange multipliers $\boldsymbol{\mu}$, ξ, and $\boldsymbol{\tau}$ recover the constraints (2.3)–(2.5). From the variations in the state variables $\mathbf{u}$, p, and $\mathbf{t}$ one can derive the adjoint state equations

$$\begin{aligned}\nu a_\alpha(\mathbf{w},\boldsymbol{\mu}) + c_\alpha(\mathbf{w},\mathbf{u},\boldsymbol{\mu}) + c_\alpha(\mathbf{u},\mathbf{w},\boldsymbol{\mu}) + b_\alpha(\mathbf{w},\xi)&\\ - <\boldsymbol{\tau},\mathbf{w}>_{-1/2,\Gamma_g} = 2\nu a_\alpha(\mathbf{u},\mathbf{w})\quad\forall\mathbf{w}\in\mathbf{V}_\alpha\,,&\end{aligned} \tag{4.11}$$

$$b_\alpha(\boldsymbol{\mu},r) = 0\quad\forall r\in S_\alpha\,, \tag{4.12}$$

and

$$<\mathbf{z},\boldsymbol{\mu}>_{-1/2,\Gamma_g} = 0\quad\forall\mathbf{z}\in\mathbf{W}^*_\alpha \tag{4.13}$$

for the adjoint variables $\boldsymbol{\mu}(\alpha)$, $\xi(\alpha)$, and $\boldsymbol{\tau}(\alpha)$. One easily sees that, formally, (4.11)–(4.13) are equivalent to the system

$$-\nu\Delta\boldsymbol{\mu} - \mathbf{u}\cdot\nabla\boldsymbol{\mu} + \boldsymbol{\mu}\cdot(\nabla\mathbf{u})^T + \nabla\xi = -2\nu\Delta\mathbf{u}\quad\text{in }\Omega_\alpha \tag{4.14}$$

$$\nabla\cdot\boldsymbol{\mu} = 0\quad\text{in }\Omega_\alpha\,, \tag{4.15}$$

$$\boldsymbol{\mu} = \mathbf{0}\quad\text{on }\Gamma(\alpha)\,, \tag{4.16}$$

and

$$\boldsymbol{\tau} = -\xi \mathbf{n} + \nu D(\boldsymbol{\mu}) \cdot \mathbf{n} - 2\nu D(\mathbf{u}) \cdot \mathbf{n} \quad \text{on } \Gamma_g \,. \tag{4.17}$$

Equations (4.14)–(4.17) are the adjoint of the Navier–Stokes equations linearized about the state $\mathbf{u}$. It should be noted that the velocity, pressure, and stress triplet $(\mathbf{u}, p, \mathbf{t})$ that is a solution of the state equations correspond to the triplet $(\boldsymbol{\mu}, \xi, \boldsymbol{\tau})$ for the adjoint equations. It is interesting to note that the adjoint equations have the similar form to those of [18] for a simplified minimum drag profile problem.

Remark 4.2. One should note that, unlike the sensitivity equations (4.8)–(4.10), the adjoint equations (4.11)–(4.13) are well-defined in the indicated function spaces.

The fact that the adjoint equations (4.11)–(4.13) may be derived as Euler Lagrange equations for the Lagrangian $\mathcal{L}$ is merely used for motivational purposes. One could simply directly introduce these equations without any reference to the Lagrangian functional. The existence and regularity of solutions of the adjoint equations will be discussed in section 5.

4.4. Adjoint variables and the shape gradient. We now use the sensitivity equations (4.8)–(4.10) and the adjoint equations (4.11)–(4.13) to eliminate the sensitivities from the expression (4.4) for the shape derivative of the design functional $\mathcal{J}$.

First, in (4.8), set $\mathbf{v} = \boldsymbol{\mu}(\alpha)$, where $(\boldsymbol{\mu}(\alpha), \xi(\alpha), \boldsymbol{\tau}(\alpha))$ is a solution of the adjoint equations (4.11)–(4.13) over Ω_α; then,

$$\begin{aligned}
&\int_{\Omega_\alpha} \Big(2\nu D(\widehat{\mathbf{u}}') : D(\boldsymbol{\mu}(\alpha)) + \mathbf{u}(\alpha) \cdot \nabla \widehat{\mathbf{u}}' \cdot \boldsymbol{\mu}(\alpha) + \widehat{\mathbf{u}}' \cdot \nabla \mathbf{u}(\alpha) \cdot \boldsymbol{\mu}(\alpha) \Big)\, d\Omega \\
&\qquad + \int_{\Gamma_b(\alpha)} \Big(2\nu D(\mathbf{u}(\alpha)) : D(\boldsymbol{\mu}(\alpha)) - p(\alpha) \nabla \cdot \boldsymbol{\mu}(\alpha) \\
&\qquad\qquad - \mathbf{n} \cdot \nabla \boldsymbol{\mu}(\alpha) \cdot \mathbf{t}(\alpha) \Big) \mathbf{V}(0, \cdot) \cdot \mathbf{n} \, d\Gamma = 0 \,,
\end{aligned} \tag{4.18}$$

where we have used $\mathbf{V}(0, \cdot) = \mathbf{0}$ on $\Gamma_\alpha - \Gamma_b(\alpha)$, $\boldsymbol{\mu} = \mathbf{0}$ on Γ_α, which follows from (4.11) and (4.13), and

$$\int_{\Omega_\alpha} \widehat{p}' \, \nabla \cdot \boldsymbol{\mu}(\alpha) \, d\Omega = 0$$

which follows from (4.12) with $r = \widehat{p}'$. Note that by virtue of the fact that $\boldsymbol{\mu}(\alpha) = \mathbf{0}$ on Γ_α, (4.12) holds not only for $r \in L^2_0(\Omega_\alpha)$, but also for $r \in L^2(\Omega_\alpha)$ so that we are justified in setting $r = \widehat{p}'$ in (4.12).

Now, let us consider the first adjoint equation (4.11) over Ω_α. If we again use a Lagrange multiplier technique to enforce the boundary condition $\boldsymbol{\mu}(\alpha) = \mathbf{0}$ on all of Γ_α and not just on $\Gamma_\alpha - \Gamma_g$, we then have that the equation takes the form

$$\int_{\Omega_\alpha} \Big(2\nu D(\boldsymbol{\mu}(\alpha)) : D(\mathbf{w}) + \mathbf{u}(\alpha) \cdot \nabla \mathbf{w} \cdot \boldsymbol{\mu}(\alpha) + \mathbf{w} \cdot \nabla \mathbf{u}(\alpha) \cdot \boldsymbol{\mu}(\alpha) \\ - \xi(\alpha) \nabla \cdot \mathbf{w} - \; 4\nu D(\mathbf{u}(\alpha)) : D(\mathbf{w}) \Big) \, d\Omega - \int_{\Gamma_\alpha} \boldsymbol{\tau}(\alpha) \cdot \mathbf{w} \, d\Gamma = 0$$

for all $\mathbf{w} \in \mathbf{H}^1(\Omega)$. Setting $\mathbf{w} = \widehat{\mathbf{u}}'$, which we are now justified to do, results in

$$\int_{\Omega_\alpha} \Big(2\nu D(\boldsymbol{\mu}(\alpha)) : D(\widehat{\mathbf{u}}') + \mathbf{u}(\alpha) \cdot \nabla \widehat{\mathbf{u}}' \cdot \boldsymbol{\mu}(\alpha) + \widehat{\mathbf{u}}' \cdot \nabla \mathbf{u}(\alpha) \cdot \boldsymbol{\mu}(\alpha) \\ - \xi(\alpha) \nabla \cdot \widehat{\mathbf{u}}' - 4\nu D(\mathbf{u}(\alpha)) : D(\widehat{\mathbf{u}}') \Big) \, d\Omega - \int_{\Gamma_\alpha} \boldsymbol{\tau}(\alpha) \cdot \widehat{\mathbf{u}}' \, d\Gamma = 0 \,. \tag{4.19}$$

Next, in (4.9), we set $q = \xi(\alpha)$, where $\xi(\alpha)$ is the adjoint pressure, to yield

$$- \int_{\Omega_\alpha} \xi(\alpha) \nabla \cdot \widehat{\mathbf{u}}' \, d\Omega = \int_{\Gamma_b(\alpha)} \big(\xi(\alpha) \nabla \cdot \mathbf{u}(\alpha) \big) \mathbf{V}(0,\cdot) \cdot \mathbf{n} \, d\Gamma \,, \tag{4.20}$$

where we have used the fact that $\mathbf{V}(0,\cdot) = \mathbf{0}$ on $\Gamma_\alpha - \Gamma_b(\alpha)$. Setting, in (4.10), $\mathbf{s} = \boldsymbol{\tau}$, where $\boldsymbol{\tau}(\alpha)$ is the adjoint stress, we have that

$$\int_{\Gamma_\alpha} \boldsymbol{\tau}(\alpha) \cdot \widehat{\mathbf{u}}' \, d\Gamma = - \int_{\Gamma_b(\alpha)} \big(\mathbf{n} \cdot \nabla \mathbf{u}(\alpha) \cdot \boldsymbol{\tau}(\alpha) \big) \mathbf{V}(0,\cdot) \cdot \mathbf{n} \, d\Gamma \,, \tag{4.21}$$

where we have used the fact that $\mathbf{V}(0,\cdot) = \mathbf{0}$ on $\Gamma_\alpha - \Gamma_b(\alpha)$ and that $\mathbf{u}(\alpha) = \mathbf{0}$ on $\Gamma_b(\alpha)$. The substitution of (4.20) and (4.21) into (4.19) yields

$$\int_{\Omega_\alpha} \Big(2\nu D(\boldsymbol{\mu}(\alpha)) : D(\widehat{\mathbf{u}}') + \mathbf{u}(\alpha) \cdot \nabla \widehat{\mathbf{u}}' \cdot \boldsymbol{\mu}(\alpha) + \widehat{\mathbf{u}}' \cdot \nabla \mathbf{u}(\alpha) \cdot \boldsymbol{\mu}(\alpha) \\ - \; 4\nu D(\mathbf{u}(\alpha)) : D(\widehat{\mathbf{u}}') \Big) \, d\Omega \\ + \int_{\Gamma_b(\alpha)} \Big(\xi(\alpha) \nabla \cdot \mathbf{u}(\alpha) + \mathbf{n} \cdot \nabla \mathbf{u}(\alpha) \cdot \boldsymbol{\tau}(\alpha) \Big) \mathbf{V}(0,\cdot) \cdot \mathbf{n} \, d\Gamma = 0 \,. \tag{4.22}$$

The combination of (4.18) and (4.22) then yields that

$$
\begin{aligned}
&4\nu \int_{\Omega_\alpha} D(\mathbf{u}(\alpha)) : D(\widehat{\mathbf{u}}')\, d\Omega \\
&= \int_{\Gamma_b(\alpha)} \Big(2\nu - D(\mathbf{u}(\alpha)) : D(\boldsymbol{\mu}(\alpha)) + p(\alpha)\nabla\cdot\boldsymbol{\mu}(\alpha) + \mathbf{n}\cdot\nabla\boldsymbol{\mu}(\alpha)\cdot\mathbf{t}(\alpha) \\
&\qquad + \xi(\alpha)\nabla\cdot\mathbf{u}(\alpha) + \mathbf{n}\cdot\nabla\mathbf{u}(\alpha)\cdot\boldsymbol{\tau}(\alpha)\Big)\mathbf{V}(0,\cdot)\cdot\mathbf{n}\, d\Gamma\,.
\end{aligned}
$$

The substitution of this last result into (4.4) then yields that

$$
\begin{aligned}
d\mathcal{J}(\alpha;\vartheta) = \int_{\Gamma_b(\alpha)} \Big(&2\nu D(\mathbf{u}(\alpha)) : D(\mathbf{u}(\alpha)) - 2\nu D(\mathbf{u}(\alpha) : D(\boldsymbol{\mu}(\alpha)) \\
&+ p(\alpha)\nabla\cdot\boldsymbol{\mu}(\alpha) + \mathbf{n}\cdot\nabla\boldsymbol{\mu}(\alpha)\cdot\mathbf{t}(\alpha) + \xi(\alpha)\nabla\cdot\mathbf{u}(\alpha) \\
&+ \mathbf{n}\cdot\nabla\mathbf{u}(\alpha)\cdot\boldsymbol{\tau}(\alpha)\Big)\mathbf{V}(0,\cdot)\cdot\mathbf{n}\, d\Gamma\,,
\end{aligned}
\tag{4.23}
$$

where for the second term in the right-hand side of (4.4) we have again used the fact that $\mathbf{V}(0,\cdot) = \mathbf{0}$ on $\Gamma_\alpha - \Gamma_b(\alpha)$.

Note that, almost everywhere on $\Gamma_b(\alpha)$,

$$
\mathbf{n} = \Big(\frac{\alpha'}{\sqrt{1+\alpha'^2}}\ ,\ -\frac{1}{\sqrt{1+\alpha'^2}}\Big)
$$

and $d\Gamma = \sqrt{1+\alpha'^2}\, dx_1$, where $\alpha'(x_1) = \dfrac{d\alpha(x_1)}{dx_1}$. Since $\mathbf{V} = (0,\vartheta)^T$ on $\Gamma_b(\alpha)$, we have that (4.23) may be expressed as

$$
\begin{aligned}
d\mathcal{J}(\alpha;\vartheta) = -\int_{M_1}^{M_2} \Big(&2\nu D(\mathbf{u}) : D(\mathbf{u}) + 2\nu D(\mathbf{u}) : D(\boldsymbol{\mu}) + p\nabla\cdot\boldsymbol{\mu} \\
&+ \mathbf{n}\cdot\nabla\boldsymbol{\mu}\cdot\mathbf{t} + \xi\nabla\cdot\mathbf{u} + \mathbf{n}\cdot\nabla\mathbf{u}\cdot\boldsymbol{\tau}\Big)\Big|_{(x_1,\alpha(x_1))} \vartheta(x_1)\, dx_1\,.
\end{aligned}
\tag{4.24}
$$

Recall that $\mathbf{V}(0,\cdot)\cdot\mathbf{n}\, d\Gamma$ corresponds to $-\vartheta(x_1)\, dx_1$ along $\Gamma_b(\alpha)$. Hence, in the sense of (3.7), we may say that the shape gradient of the design functional $\mathcal{J}$ is given by

$$
\begin{aligned}
\operatorname{grad}\mathcal{J} = \Big(&2\nu D(\mathbf{u}) : D(\mathbf{u}) - 2\nu D(\mathbf{u}) : D(\boldsymbol{\mu}) + p\nabla\cdot\boldsymbol{\mu} \\
&+ \mathbf{n}\cdot\nabla\boldsymbol{\mu}\cdot\mathbf{t} + \xi\nabla\cdot\mathbf{u} + \mathbf{n}\cdot\nabla\mathbf{u}\cdot\boldsymbol{\tau}\Big)\Big|_{(x_1,\alpha(x_1))}
\end{aligned}
\tag{4.25}
$$

along $\Gamma_b(\alpha)$, and grad $\mathcal{J} = \mathbf{0}$ along the rest of the boundary $\Gamma_\alpha - \Gamma_b(\alpha)$.

It is useful to summarize the above discussions in the following theorem.

THEOREM 4.3. *Let* $(\alpha, \mathbf{u}(\alpha)) \in \mathcal{V}_{ad}$ *and let the design functional* $\mathcal{J}$ *be given by* (2.6). *Under reasonable regularity assumptions for the data, the shape gradient of* $\mathcal{J}$ *at* α *is given by* (4.25), *where* $(\mathbf{u}, p, \mathbf{t})$ *and* $(\boldsymbol{\mu}, \xi, \boldsymbol{\tau})$ *are solutions of the state equations* (2.3)–(2.5) *and adjoint equations* (4.11)–(4.13), *respectively, with respect to the domain* Ω_α.

REMARK 4.4. In (4.24) or (4.25), the curvature κ does not appear. This is due to the fact that either $\mathbf{V}(0, \cdot)$ or a trial function, e.g., $\mathbf{u}$ or $\boldsymbol{\eta}$, vanishes at every point of the boundary Γ_α.

The sensitivities $(\widehat{\mathbf{u}}', \widehat{p}\,', \widehat{\mathbf{t}}\,')$ may be determined from the system (4.8)–(4.10) which may be written in the following equivalent form. One seeks $\widehat{\mathbf{u}}' \in \mathbf{H}^1(\Omega_\alpha)$, $\widehat{p}\,' \in L^2_0(\Omega_\alpha)$, and $\widehat{\mathbf{t}}\,' \in \mathbf{H}^{-1/2}(\Gamma_\alpha)$ satisfying

$$\nu a_\alpha(\widehat{\mathbf{u}}', \mathbf{v}) + c_\alpha(\widehat{\mathbf{u}}', \mathbf{u}, \mathbf{v}) + c_\alpha(\mathbf{u}, \widehat{\mathbf{u}}', \mathbf{v}) + b_\alpha(\mathbf{v}, \widehat{p}') - < \widehat{\mathbf{t}}\,', \mathbf{v} >_{-1/2,\Gamma_\alpha}$$

$$\begin{aligned} &= \int_{M_1}^{M_2} \Big(2\nu D(\mathbf{u}) : D(\mathbf{v}) - p \nabla \cdot \mathbf{v} - \mathbf{f} \cdot \mathbf{v} \\ &-\mathbf{n} \cdot \nabla(\mathbf{t} \cdot \mathbf{v}) - \kappa \mathbf{t} \cdot \mathbf{v} \Big)\Big|_{(x_1, \alpha(x_1))} \vartheta(x_1)\, dx_1 \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega_\alpha)\,, \end{aligned} \tag{4.26}$$

$$b_\alpha(\widehat{\mathbf{u}}', q) = \int_{M_1}^{M_2} \Big(q \nabla \cdot \mathbf{u} \Big)\Big|_{(x_1, \alpha(x_1))} \vartheta(x_1)\, dx_1 \quad \forall q \in L^2_0(\Omega_\alpha)\,, \tag{4.27}$$

and for all $\mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_\alpha)$

$$< \mathbf{s}, \widehat{\mathbf{u}}' >_{-1/2,\Gamma_\alpha} = \int_{M_1}^{M_2} \Big((\mathbf{n} \cdot \nabla \mathbf{u}) \cdot \mathbf{s} \Big)\Big|_{(x_1, \alpha(x_1))} \vartheta(x_1)\, dx_1\,, \tag{4.28}$$

where we have used the facts that $\mathbf{u}$ vanishes on $\Gamma_0(\alpha)$ and that $\mathbf{V} = \mathbf{0}$ on $\Gamma_\alpha - \Gamma_b(\alpha)$. The sensitivities $(\widehat{\mathbf{u}}', \widehat{p}\,', \widehat{\mathbf{t}}\,')$, of course, are of interest in their own right and may also be used in (4.4) to compute the shape derivative of the functional $\mathcal{J}$. Instead, we have chosen to solve the adjoint equations (4.11)–(4.13) and then use the adjoint variables in (4.24) to compute that shape derivative. One advantage of the second approach is that (4.24), unlike (4.4), only involves integrals along $\Gamma_b(\alpha)$ and not also on Ω_α. Note that (4.28) implies that $\widehat{\mathbf{u}}' = \mathbf{0}$ on Γ_α except along $\Gamma_b(\alpha)$.

In the formal computations, we have implicitly assumed the existence of the material dervative $\dot{\mathbf{u}}$. To justify these computations, we need two hypotheses (see [20]):

(H1) the material derivative $\dot{\mathbf{u}}$ exists and $\mathbf{V} \longmapsto \dot{\mathbf{u}}(\Omega_\alpha;\mathbf{V})$ is linear and continuous; and

(H2) the feasible domains $\{\Omega_\alpha\}_{\alpha\in\mathcal{U}_{ad}}$ are sufficiently regular so that there exists a linear continuous extension $P_{\widehat{\Omega}} : \mathbf{H}^m(\Omega_\alpha) \longrightarrow \mathbf{H}^m(\widehat{\Omega})$ for m a positive integer.

(H2) is true for uniform Lipschitz domains by the Calderón's extension theorem; see [15]. The verification of (H1) is nontrivial. It may be verified by applying the implicit function theorem to resolve the nonlinear structure of the problem.

Another possible way to perform this task is to reverse the process. We first find a meaningful boundary value problem for $\widehat{\mathbf{u}}'\Big|_{\Omega_\alpha}$. From it, one may understand $\dot{\mathbf{u}}$ through the relation

$$\dot{\mathbf{u}} = \widehat{\mathbf{u}}' + \mathbf{V}\cdot\nabla\mathbf{u}\,.$$

The system (4.26)–(4.28) of course provides a mechanism for determining $\widehat{\mathbf{u}}'$. It can be shown that this system has a solution for almost all Reynolds numbers. Since this is a linear system of equations whose data, i.e., see the right-hand sides of (4.8)–(4.10), depend linearly on $\mathbf{V}(0,\cdot)\cdot\mathbf{n}$, we have that $\widehat{\mathbf{u}}'\Big|_{\Omega_\alpha}$ depends linearly on $\mathbf{V}(0,\cdot)\cdot\mathbf{n}$.

5. Regularity of solutions of state and adjoint systems

In section 4, we have used solutions of the state and adjoint equations to derive an expression for the shape gradient. The existence of solutions of the state equations is considered in Theorem 2.1. The adjoint equations are linear in the adjoint variables $(\boldsymbol{\mu},\xi,\boldsymbol{\tau})$ so that the existence of solutions of the adjoint equations can be shown in a manner similar to that for the Stokes problem; see [19] and [30].

The formal computations in section 4 can be justified if one has sufficient regularities for the domain (geometry) and solutions of the state and adjoint equations. Such regularity is not always available. For example, in [25], the driven cavity problem in a rectangular region Ω is considered. Suppose the flow is governed by the Navier–Stokes equations with uniform velocity along the top side. Then, the solution $\mathbf{u}$ does not even belong to $\mathbf{H}^1(\Omega)$. The main cause of this irregularity is the existence of jumps of the velocity components around corners. One may dispense with this situation by requiring $\mathbf{g}_i$ to have compact support on Γ_i for $i = 1, 2$. In our case, we can also impose that $\mathbf{g}_i = \mathbf{0}$ at the top and bottom of the channel; this is reasonable since $\mathbf{g}_i$ for $i = 1, 2$ can be set to a parabolic flow along the inflow and outflow boundaries. Then, at the corner points $\partial\Gamma_1$ and $\partial\Gamma_2$ the velocity field is continuous.

The basic requirement for the regularity of the external force field $\mathbf{f}$ is that it can be extended to an $\widehat{\mathbf{f}} \in \mathbf{L}^2(\widehat{\Omega})$. This requirement originates from the need to justify the existence of the material derivative involving $\mathbf{f}$. It also contributes to the regularity of solutions of the state and adjoint equations.

LEMMA 5.1. *Let* $\mathbf{f}$ *be extendible to a function belonging to* $\mathbf{L}^2(\widehat{\Omega})$. *Then,* $[\,0, \sigma) \ni t \longmapsto \mathbf{f} \circ \mathcal{F}_t \in \mathbf{L}^2(\widehat{\Omega})$ *is weakly differentiable in* $\mathbf{H}^{-1}(\widehat{\Omega})$.

Proof. When $\mathbf{V}$ is smooth enough, the proof can be found in [29]. In our case, the same principle can be applied since $\mathcal{F}_t \in C^{0,1}(\widehat{\Omega})$ so that $\mathcal{F}_t$ is differentiable almost everywhere. $\square$

Next, we consider the regularity of the domain. If the domain is non–smooth, rigorous computations of the boundary integral such as (3.9) may introduce extra jump states at the singular points of the boundary. For example, even though y and $\mathbf{V}$ are smooth in $\mathbb{R}^2$, if $\partial\Omega = \Gamma$ has a piecewise smooth boundary, i.e., Γ is smooth except at the points $\{\mathbf{a}_1, \cdots, \mathbf{a}_k\}$, then

$$\left(\frac{d}{dt}\int_{\Gamma_t} y\, d\Gamma_t\right)\bigg|_{t=0^+} = \int_\Gamma \left(\frac{\partial y}{\partial t} + \Big(\frac{\partial y}{\partial \mathbf{n}} + \kappa\, y\Big)\, \mathbf{V}(0,\cdot)\cdot\mathbf{n}\right) d\Gamma + \sum_{i=1}^{k} \mathbf{V}(0,\mathbf{a}_i)[\eta(\mathbf{a}_i)]\,,$$

where $\{[\eta(\mathbf{a}_i)]\}$ = jump states of the tangents at the singular points $\mathbf{a}_i$, $i = 1, \cdots, k$. However, if the domain has a Lipschitz boundary, a corner is no longer a singular point; see [27]. Hence, boundary integrals do not involve jump states. If $\Gamma_\alpha \in C^{0,1}([-M_1, M_1])$, an outward unit normal vector along the boundary Γ_α exists almost everywhere and $\mathbf{n} \in \mathbf{L}^\infty(\Gamma; \mathbb{R}^2)$. This implies that

$$\mathbf{n} = \left(\frac{\alpha'}{\sqrt{1+\alpha'^2}}, -\frac{1}{\sqrt{1+\alpha'^2}}\right) \in \mathbf{L}^\infty(\Gamma_b(\alpha); \mathbb{R}^2)$$

exists. Since $\mathbf{V} = (0, V_2)$ is continuous over $\overline{\Omega}_\alpha$ and

$$\nabla V_2 = \left(\frac{(p_2 - L)\vartheta'(\alpha - L) - (\alpha' - L)(p_2 - L)\vartheta}{(\alpha - L)^2}, \frac{\vartheta}{(\alpha - L)} \right)$$

if $M_1 \leq x_1 \leq M_2$ and $\nabla V_2 = \mathbf{0}$ otherwise, it follows that $\mathbf{V} \in \mathbf{H}^1(\Omega_\alpha)$. Thus, if $\mathbf{u}(\alpha)$ is sought in $\mathbf{H}^1(\Omega_\alpha)$, then, since $\nabla \mathbf{u} \cdot \mathbf{V}$ belongs to $\mathbf{L}^2(\Omega_\alpha)$, $\dot{\mathbf{u}}$ or $\widehat{\mathbf{u}}'\Big|_{\Omega_\alpha}$ will belong to $\mathbf{L}^2(\Omega_\alpha)$ as well. In this case, the computation discussed in section 3 may not be justified. However, if we take ϑ to be of class $C^{1,1}$ with smooth contacts at $\partial\Gamma_b(\alpha)$, i.e., at the points $(M_1, 0)$ and $(M_2, 0)$, then all the regularity needed to justify the computations is secured.

Now, let us turn to state and adjoint equations. These have been used to derive the shape gradient grad $\mathcal{J}$ that can be used in an optimization process. Using (1.1), we can replace the right-hand side of (4.14) by

$$-\nu\Delta\boldsymbol{\mu} - \mathbf{u} \cdot \nabla\boldsymbol{\mu} + \boldsymbol{\mu} \cdot (\nabla\mathbf{u})^T + \nabla\xi = 2(\mathbf{f} - \mathbf{u} \cdot \nabla\mathbf{u} - \nabla p) \quad \text{in } \Omega_\alpha \,.$$

This replacement makes it possible to obtain, in a straighforward manner, regularity results for solutions of the adjoint equations. Thus the state and adjoint equations may be written in the form

$$-\nu\Delta\mathbf{u} + \nabla p = \widetilde{\mathbf{f}} = \mathbf{f} - \mathbf{u} \cdot \nabla\mathbf{u} \quad \text{in } \Omega_\alpha \,, \tag{5.1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega_\alpha \,, \tag{5.2}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_\alpha \,. \tag{5.3}$$

$$-\nu\Delta\boldsymbol{\mu} + \nabla\phi = \widetilde{\widetilde{\mathbf{f}}} = \mathbf{u} \cdot \nabla\boldsymbol{\mu} - \boldsymbol{\mu} \cdot (\nabla\mathbf{u})^T + 2(\mathbf{f} - \mathbf{u} \cdot \nabla\mathbf{u}) \quad \text{in } \Omega_\alpha \,, \tag{5.4}$$

$$\nabla \cdot \boldsymbol{\mu} = 0 \quad \text{in } \Omega_\alpha \,, \tag{5.5}$$

and

$$\boldsymbol{\mu} = \mathbf{0} \quad \text{on } \Gamma_\alpha \,, \tag{5.6}$$

where $\phi = \xi + 2p$.

Now, consider the regularity of the solutions of the system (5.1)–(5.6). According to [2], the regularity of solution of these equations follows from that for solutions of the Stokes equations. Indeed, the way we have written these equations points out that they can be viewed as two Stokes systems.

THEOREM 5.2. *Suppose* $\mathbf{g} \in \mathbf{H}^{3/2}(\Gamma_\alpha)$ *with* $\mathbf{g}_i$ *having compact support on* Γ_i, $i = 1, 2$. *Let* $\mathbf{f} \in \mathbf{L}^2(\Omega_\alpha)$. *Suppose* α *is of class* $C^{1,1}$ *with smooth contacts at* $\partial\Gamma_b(\alpha)$. *If* $(\mathbf{u}, p, \boldsymbol{\mu}, \phi) \in \mathbf{H}^1(\Omega_\alpha) \times L_0^2(\Omega_\alpha) \times \mathbf{H}^1(\Omega_\alpha) \times L_0^2(\Omega_\alpha)$ *denotes a solution of* (5.1)–(5.6), *we have that*

$$(\mathbf{u}, p, \boldsymbol{\mu}, \phi) \in \mathbf{H}^2(\Omega_\alpha) \times H^1(\Omega_\alpha) \times \mathbf{H}^2(\Omega_\alpha) \times H^1(\Omega_\alpha)\,.$$

Proof. If the body force belongs to $\mathbf{L}^2(\Omega_\alpha)$ and the velocity boundary data satisfies, say, the hypotheses of the theorem, then, the allowable domains Ω_α such that solutions of the Stokes equations belong to $\mathbf{H}^2(\Omega_\alpha) \times (H^1(\Omega_\alpha) \cap L_0^2(\Omega_\alpha))$ are those that are piecewise $C^{1,1}$ with convex corners; see [12] or [27]. According to the hypotheses, the domain Ω_α satisfies these requirements. Thus, to obtain the result we first examine the right-hand side of (5.1), namely $\widetilde{\mathbf{f}} \equiv \mathbf{f} - (\mathbf{u} \cdot \nabla)\mathbf{u}$. Since $\mathbf{u} \in \mathbf{H}^1(\Omega_\alpha)$, $\mathbf{u} \in \mathbf{L}^6(\Omega_\alpha)$ and $\dfrac{\partial \mathbf{u}}{\partial x_j} \in \mathbf{L}^2(\Omega_\alpha)$ for $j = 1, 2$, and we have that $(\mathbf{u} \cdot \nabla)\mathbf{u} \in \mathbf{L}^{3/2}(\Omega_\alpha)$; hence $\widetilde{\mathbf{f}} \in \mathbf{L}^{3/2}(\Omega_\alpha) \subset \mathbf{H}^{-1/2}(\Omega_\alpha)$. Since $\mathbf{g} \in \mathbf{H}^{3/2}(\Gamma_\alpha)$, the Stokes system (5.1)–(5.3) has a solution $(\mathbf{u}, p) \in \mathbf{H}^{3/2}(\Omega_\alpha) \times (H^{1/2}(\Omega_\alpha) \cap L_0^2(\Omega_\alpha))$. With $\mathbf{f} \in \mathbf{L}^2(\Omega_\alpha)$, one can then show that $\widetilde{\mathbf{f}} \in \mathbf{L}^2(\Omega_\alpha)$, so that $(\mathbf{u}, p) \in \mathbf{H}^2(\Omega_\alpha) \times (H^1(\Omega_\alpha) \cap L_0^2(\Omega_\alpha))$. If we next consider the right hand side of (5.4)–(5.6), then standard results also lead to $(\boldsymbol{\mu}, \phi) \in \mathbf{H}^2(\Omega_\alpha) \times (H^1(\Omega_\alpha) \cap L_0^2(\Omega_\alpha))$. ☐

REMARK 5.3. By a bootstrap argument, one can show that if the domain Ω_α is smooth enough and $\mathbf{f} \in \mathbf{H}^m(\Omega_\alpha)$ and $\mathbf{g} \in \mathbf{H}^{m+3/2}(\Gamma_\alpha)$, then $(\mathbf{u}, p, \boldsymbol{\mu}, \phi) \in \mathbf{H}^{m+2}(\Omega_\alpha) \times H^{m+1}(\Omega_\alpha) \times \mathbf{H}^{m+2}(\Omega_\alpha) \times H^{m+1}(\Omega_\alpha)$. However, if we merely have that $\alpha \in \mathcal{C}^{0,1}([M_1, M_2])$, then even though $\mathbf{g} \in \mathbf{H}^{3/2}(\Gamma_\alpha)$, losing the convexity and regularity of the domain will result in $(\mathbf{u}, p) \in \mathbf{H}^{2-\delta}(\Omega_\alpha) \times (H^{1-\delta}(\Omega_\alpha) \cap L_0^2(\Omega_\alpha))$ for some $\delta > 0$ such that $0 < \delta < 1$.

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