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WEAK FUZZY EQUIVALENCE RELATIONS AND WEAK FUZZY CONGRUENCES

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Abstract. We define a weak fuzzy equivalence relation and a weak fuzzy congruence and develop some properties of the weak fuzzy equivalence relations and the weak fuzzy congruences on semigroups.

1. Introduction

The concept of a fuzzy relation was first proposed by Zadeh ([7]). Subsequently, many researchers ([3], [4], [5]) studied fuzzy relations in various contexts. The standard definition of a reflexive fuzzy relation μ on a set X is $\mu(x, x) = 1$ for all $x \in X$ and that of a symmetric fuzzy relation μ is $\mu(x, y) = \mu(y, x)$ for all $x, y \in X$. These definitions have seemed to be too strong. We suggest a weak reflexive fuzzy relation μ in a set X as $\mu(x, x) \geq \epsilon > 0$ for all $x \in X$ and $\inf_{t \in X} \ \mu(t, t) \geq \mu(y, z)$ for all $y \neq z \in X$ and suggest a weak symmetric fuzzy relation as $\min\,[\mu(x, y), \mu(y, x)] > 0$ or $\mu(x, y) = \mu(y, x) = 0$ for all x, y in X such that $x \neq y$. We define a weak fuzzy equivalence relation and a weak fuzzy congruence using the weak reflexive and symmetric conditions and develop some properties of those relations and those congruences on semigroups.

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In section 2 we define a weak fuzzy equivalence relation and recall some basic properties of fuzzy relations which will be used in next sections. In section 3 we discuss some properties of the weak fuzzy equivalence relations, characterize the weak fuzzy equivalence relation generated by a fuzzy relation on a set, and develop some lattice theoretic properties of the fuzzy equivalence relations. In section 4 we develop some properties of the weak fuzzy congruences, characterize the weak fuzzy congruence generated by the fuzzy relation on a semigroup, find the largest weak fuzzy congruence contained in the given weak fuzzy congruence on a group, and give some lattice theoretic properties of the weak fuzzy congruences on semigroups.

2. Preliminaries

We define a weak fuzzy equivalence relation and recall some basic properties of fuzzy relations, weakly reflexive fuzzy relations, and weakly symmetric fuzzy relations, which will be used in next sections.

Definition 2.1. A function ν from a set X to the closed unit interval [0, 1] in $\mathbb{R}$ is called a *fuzzy set* in X. A function μ from a set $S \times S$ to [0, 1] is called a *fuzzy relation* in S.

The standard definition of a reflexive fuzzy relation μ in a set X is $\mu(x, x) = 1$ for all $x \in X$ and that of a symmetric fuzzy relation μ is $\mu(x, y) = \mu(y, x)$ for all $x, y \in X$. We redefine a fuzzy equivalence relation by weakening reflexive and symmetric conditions.

Definition 2.2. Let μ be a fuzzy relation in a set X. Then μ is *weakly reflexive* (briefly, *w-reflexive*) iff $\mu(x, x) \geq \epsilon > 0$ and $\inf_{t \in X} \ \mu(t, t) \geq \mu(x, y)$ for all $x, y \in X$ such that $x \neq y$. μ is *weakly symmetric* (briefly, *w-symmetric*) iff min $[\mu(x, y), \mu(y, x)] > 0$ or $\mu(x, y) = \mu(y, x) = 0$ for all x, y in X such that $x \neq y$. The composition $\lambda \circ \mu$ of two fuzzy relations λ, μ in X is the fuzzy subset of $X \times X$ defined by

$$(\lambda \circ \mu)(x, y) = \sup_{z \in X} \ \min(\lambda(x, z), \mu(z, y)).$$

A fuzzy relation μ is *transitive* iff $\mu \circ \mu \subseteq \mu$. A fuzzy relation μ in X is called a *weak fuzzy equivalence relation* iff μ is w-reflexive, w-symmetric, and transitive.

DEFINITION 2.3. Let μ be a fuzzy relation in a set X. μ^{-1} is defined as a fuzzy relation in X by $\mu^{-1}(x, y) = \mu(y, x)$.

It is easy to see that $(\mu \circ \nu)^{-1} = \nu^{-1} \circ \mu^{-1}$ for fuzzy relations μ and ν.

PROPOSITION 2.4. *Let μ and each ν_i be fuzzy relations in a set X for all $i \in I$. Then $\mu \circ (\cap_{i\in I}\nu_i) \subseteq \cap_{i\in I}(\mu \circ \nu_i)$ and $(\cap_{i\in I}\nu_i) \circ \mu \subseteq \cap_{i\in I}(\nu_i \circ \mu)$.*

Proof. Straightforward. □

PROPOSITION 2.5. *Let μ be a fuzzy relation on a set X. Then $\cup_{n=1}^{\infty} \mu^n$ is the smallest transitive fuzzy relation on X containing μ, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.*

Proof. See Proposition 2.3 of [6]. □

PROPOSITION 2.6. *Let μ be a fuzzy relation on a set X. If μ is w-reflexive, then so is $\cup_{n=1}^{\infty} \mu^n$, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.*

Proof. See the proof of Theorem 3.4 in [1]. □

PROPOSITION 2.7. *Let μ and ν be w-symmetric fuzzy relations on a set X. Then $\mu \cap \nu$ and $\mu \cup \nu$ are w-symmetric fuzzy relations.*

Proof. Let $x, y \in X$ with $x \neq y$. If $\mu(x, y) > 0$ and $\nu(x, y) > 0$, then $\mu(y, x) > 0$ and $\nu(y, x) > 0$, and hence min $[(\mu \cap \nu)(x, y),\ (\mu \cap \nu)(y, x)] > 0$. If $\mu(x, y) = \nu(x, y) = 0$, then $\mu(y, x) = \nu(y, x) = 0$, and hence $(\mu \cap \nu)(x, y) = (\mu \cap \nu)(y, x) = 0$. If $\mu(x, y) > 0$ and $\nu(x, y) = 0$, then $\mu(y, x) > 0$ and $\nu(y, x) = 0$, and hence $(\mu \cap \nu)(x, y) = (\mu \cap \nu)(y, x) = 0$. Similarly, if $\mu(x, y) = 0$ and $\nu(x, y) > 0$, then $(\mu \cap \nu)(x, y) = (\mu \cap \nu)(y, x) = 0$. Thus $\mu \cap \nu$ is w-symmetric. Similarly we may show that $\mu \cup \nu$ is w-symmetric. □

PROPOSITION 2.8. *Let μ be a fuzzy relation on a set X. If μ is w-symmetric, then so is $\cup_{n=1}^{\infty} \mu^n$, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.*

Proof. See the proof of Lemma 6 in [2]. □

3. Weak fuzzy equivalence relations

In this section we discuss some basic properties of the weak fuzzy equivalence relations, characterize the weak fuzzy equivalence relation generated by a fuzzy relation on a set, and develop some lattice theoretic properties of the weak fuzzy equivalence relations.

PROPOSITION 3.1. *Let μ and ν be weak fuzzy equivalence relations in a set X. Then $\mu \cap \nu$ is a weak fuzzy equivalence relation.*

Proof. Clearly $\mu \cap \nu$ is w-reflexive. By Proposition 2.7, $\mu \cap \nu$ is w-symmetric. By Proposition 2.4, $[(\mu\cap\nu)\circ(\mu\cap\nu)] \subseteq [\mu\circ(\mu\cap\nu)]\cap[\nu\circ(\mu\cap\nu)] \subseteq [(\mu\circ\mu)\cap(\mu\circ\nu)]\cap[(\nu\circ\mu)\cap(\nu\circ\nu)] \subseteq [\mu\cap(\mu\circ\nu)]\cap[(\nu\circ\mu)\cap\nu] \subseteq \mu\cap\nu$. That is, $\mu \cap \nu$ is transitive. Thus $\mu \cap \nu$ is a weak fuzzy equivalence relation. □

It is easy to see that even though μ and ν are weak fuzzy equivalence relations, $\mu \cup \nu$ is not necessarily a weak fuzzy equivalence relation. We find the weak fuzzy equivalence relation generated by $\mu \cup \nu$.

THEOREM 3.2. *Let μ and ν be weak fuzzy equivalence relations in a set X. Then the weak fuzzy equivalence relation generated by $\mu \cup \nu$ is $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$.*

Proof. See the proof of Theorem 7 in [2]. □

We now turn to the characterization of the weak fuzzy equivalence relation generated by a fuzzy relation in a set.

THEOREM 3.3. *Let μ be a fuzzy relation in a set X. Then the weak fuzzy equivalence relation in X generated by μ is $\cup_{n=1}^{\infty}(\mu \cup \rho \cup \theta)^n$. Here θ is a fuzzy relation in X such that $\theta(x,y) \le \mu(x,y)$ for all $x, y \in X$ with $x \ne y$ and $\theta(t,t) = \max\,[\epsilon,\ \sup_{x\ne y,\ x,y\in X} \mu(x,y)]$ for all $t \in X$, and ρ is a fuzzy relation in X such that $\rho(z,z) = 0$ for all $z \in X$ and for all $x, y \in X$ such that $x \ne y$,*

(1) *if $\mu(x,y) = \mu(y,x) = 0$, then $\rho(x,y) = \rho(y,x) = 0$,*
(2) *if $\mu(x,y) > 0$ and $\mu(y,x) = 0$, then $\rho(x,y) = 0$ and $\rho(y,x) = \min\,[\mu(x,y),\ \delta]$ for some $\delta > 0$,*
(3) *if $\mu(x,y) > 0$ and $\mu(y,x) > 0$, then $\rho(x,y) = \mu(x,y)$ and $\rho(y,x) = \mu(y,x)$.*

Proof. Let $\mu_1 = \mu \cup \rho \cup \theta$. Then $\mu_1(x,x) = \max[\mu(x,x), \rho(x,x), \theta(x,x)] \geq \theta(x,x) \geq \epsilon > 0$. Let $x, y \in X$ with $x \neq y$. Since $\theta(x,y) \leq \mu(x,y) \leq \theta(t,t)$ and $\rho(x,y) \leq \theta(t,t)$ for all $t \in X$,

$$\inf_{t\in X} \mu_1(t,t) \geq \inf_{t\in X} \theta(t,t) \geq \max[\mu(x,y), \rho(x,y), \theta(x,y)] = \mu_1(x,y).$$

Thus μ_1 is w-reflexive. By Proposition 2.6, $\cup_{n=1}^{\infty}\mu_1^n$ is w-reflexive. Let ν be a weak fuzzy equivalence relation containing μ.
(i) We consider the case of $\mu(x,y) = 0$ and $\mu(y,x) = 0$. Since $\theta(x,y) \leq \mu(x,y)$, $\theta(x,y) = \theta(y,x) = 0$. Since $\rho(y,x) = \rho(x,y) = 0$, $\mu_1(x,y) = \mu_1(y,x) = 0$. That is, μ_1 is w-symmetric. Since $\mu_1(x,y) = \mu_1(y,x) = 0$, $\mu_1(x,y) \leq \nu(x,y)$ and $\mu_1(y,x) \leq \nu(y,x)$.
(ii) We consider the case of $\mu(x,y) > 0$ and $\mu(y,x) = 0$. Since $\mu(x,y) \leq \nu(x,y)$, $\nu(x,y) > 0$. Since ν is w-symmetric, $\nu(y,x) > 0$. That is, there exists $\delta \in \mathbb{R}$ such that $\nu(y,x) > \delta > 0$. Since $\rho(x,y) = 0$ and $\theta(x,y) \leq \mu(x,y)$, $\mu_1(x,y) = \mu(x,y) > 0$. Since $\rho(y,x) = \min\ [\mu(x,y),\ \delta]$ and $\theta(y,x) \leq \mu(y,x)$, $\mu_1(y,x) = \rho(y,x) > 0$. Thus $\min\ [\mu_1(x,y),\ \mu_1(y,x)] > 0$. That is, μ_1 is w-symmetric. Clearly $\mu_1(x,y) = \mu(x,y) \leq \nu(x,y)$ and $\mu_1(y,x) = \rho(y,x) \leq \delta < \nu(y,x)$.
(iii) We consider the case of $\mu(x,y) > 0$ and $\mu(y,x) > 0$. Since $\rho(x,y) = \mu(x,y)$ and $\theta(x,y) \leq \mu(x,y)$, $\mu_1(x,y) = \mu(x,y) > 0$. Since $\rho(y,x) = \mu(y,x)$ and $\theta(y,x) \leq \mu(y,x)$, $\mu_1(y,x) = \mu(y,x) > 0$. Thus $\min\ [\mu_1(x,y),\ \mu_1(y,x)] > 0$. That is, μ_1 is w-symmetric. Clearly $\mu_1(x,y) = \mu(x,y) \leq \nu(x,y)$ and $\mu_1(y,x) = \mu(y,x) \leq \nu(y,x)$.
From (i), (ii), and (iii), μ_1 is w-symmetric. By Proposition 2.8, $\cup_{n=1}^{\infty}\mu_1^n$ is w-symmetric. By Proposition 2.5, $\cup_{n=1}^{\infty}\mu_1^n$ is transitive. Thus $\cup_{n=1}^{\infty}\mu_1^n$ is a weak fuzzy equivalence relation containing μ. From (i), (ii), and (iii), $\mu_1(x,y) \leq \nu(x,y)$ for all $x, y \in X$ such that $x \neq y$. Since $\mu(x,y) \leq \nu(x,y) \leq \nu(t,t)$, $\sup_{x\neq y,\ x,y\in X} \mu(x,y) \leq \nu(t,t)$ for all $t \in X$, and hence $\theta(t,t) \leq \nu(t,t)$. Clearly $\rho(t,t) \leq \nu(t,t)$. That is, $\mu_1(t,t) \leq \nu(t,t)$. Thus $\mu_1 \subseteq \nu$. Suppose that $\mu_1^k \subseteq \nu$. Then

$$\begin{aligned}\mu_1^{k+1}(a,b) = (\mu_1 \circ \mu_1^k)(a,b) &= \sup_{z\in X} \min[\mu_1(a,z), \mu_1^k(z,b)] \\ &\leq \sup_{z\in X} \min[\nu(a,z), \nu(z,b)] = (\nu \circ \nu)(a,b).\end{aligned}$$

Since ν is transitive, $\mu_1^{k+1} \subseteq \nu \circ \nu \subseteq \nu$. By the mathematical induction, $\mu_1^n \subseteq \nu$ for all natural numbers n. Thus $\cup_{n=1}^{\infty} \mu_1^n = \mu_1 \cup (\mu_1 \circ \mu_1) \cup (\mu_1 \circ \mu_1 \circ \mu_1) \cdots \subseteq \nu$. Thus $\cup_{n=1}^{\infty}\mu_1^n$ is the weak fuzzy equivalence relation generated by μ. □

We now turn to the lattice theoretic properties of weak fuzzy equivalence relations. Let $E(X)$ be the collection of all weak fuzzy equivalence relations on a set X.

THEOREM 3.4. *$(E(X), \leq)$ is a complete lattice, where $\leq$ is a relation on the set of all weak fuzzy equivalence relations on X defined by $\mu \leq \nu$ iff $\mu(x, y) \leq \nu(x, y)$ for all $x, y \in X$.*

Proof. See the proof of Theorem 9 in [2]. $\square$

We define an addition on $E(X)$ by $\mu + \nu =< \mu \cup \nu >$ and a multiplication on $E(X)$ by $\mu \cdot \nu = \mu \cap \nu$, where $< \mu \cup \nu >$ is the weak fuzzy equivalence relation generated by $\mu \cup \nu$. Then for $\mu, \nu \in E(X)$, $\mu \cdot \nu \in E(X)$ by Proposition 3.1 and $\mu + \nu \in E(X)$ by Theorem 3.2.

THEOREM 3.5. *Let $E_k(X) = \{\mu \in E(X) : \mu = \mu^{-1}$ and $\mu(c, c) = k$ for all $c \in X\}$. Then $E_k(X)$ is a sublattice of $(E(X), +, \cdot)$ for $0 < \epsilon \leq k \leq 1$.*

Proof. Let $\mu, \nu \in E_k(X)$. Since $\mu = \mu^{-1}$ and $\nu = \nu^{-1}$, $(\mu \cap \nu)^{-1} = \mu^{-1} \cap \nu^{-1} = \mu \cap \nu$. Clearly $(\mu \cap \nu)(c, c) = k$. Thus $\mu \cdot \nu \in E_k(X)$. The weak fuzzy equivalence relation generated by $\mu \cup \nu$ is $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ by Theorem 3.2. That is, $\mu + \nu$ is $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$. Since $\mu = \mu^{-1}$ and $\nu = \nu^{-1}$, $(\mu \cup \nu)^{-1} = \mu^{-1} \cup \nu^{-1} = \mu \cup \nu$. Let $\zeta = \mu \cup \nu$. Then $\zeta = \zeta^{-1}$. Suppose $(\zeta^k)^{-1} = (\zeta^{-1})^k$. Then

$$
\begin{aligned}
(\zeta^{-1})^{k+1}(x, y) &= [\zeta^{-1} \circ (\zeta^{-1})^k](x, y) = \sup_{z \in X} \min\, [\zeta^{-1}(x, z),\ (\zeta^{-1})^k(z, y)] \\
&= \sup_{z \in X} \min\, [\zeta(z, x),\ (\zeta^k)^{-1}(z, y)] = \sup_{z \in X} \min\, [\zeta^k(y, z),\ \zeta(z, x)] \\
&= (\zeta^k \circ \zeta)(y, x) = \zeta^{k+1}(y, x) = (\zeta^{k+1})^{-1}(x, y).
\end{aligned}
$$

By the mathematical induction, $(\zeta^n)^{-1} = (\zeta^{-1})^n$ for all natural numbers n. Thus

$$
\begin{aligned}
[\cup_{n=1}^{\infty}\zeta^n]^{-1}(x, y) &= [\cup_{n=1}^{\infty}\zeta^n](y, x) = [\cup_{n=1}^{\infty}(\zeta^{-1})^n](y, x) \\
&= [\cup_{n=1}^{\infty}(\zeta^n)^{-1}](y, x) = [\cup_{n=1}^{\infty}\zeta^n](x, y).
\end{aligned}
$$

That is, $\mu + \nu = \cup_{n=1}^{\infty}(\mu \cup \nu)^n = [\cup_{n=1}^{\infty}(\mu \cup \nu)^n)]^{-1} = (\mu + \nu)^{-1}$. Clearly $\zeta(c, c) = k$ for all $c \in X$ and $\zeta(a, b) \leq k$ for all $a, b \in X$ such that $a \neq b$. Suppose $\zeta^p(a, b) \leq k$ for all $a, b \in X$ such that $a \neq b$. Then $\zeta^{p+1}(a, b) = \sup_{z \in X} \min\, [\zeta^p(a, z),\ \zeta(z, b)] \leq k$. By the mathematical induction, $\zeta^n(a, b) \leq k$ for all natural numbers n. Suppose

$\zeta^m(c,c) = k$. Since $\zeta(c,c) \geq \zeta(a,b)$ and $\zeta^m(c,c) = k \geq \zeta^m(a,b)$ for all $a,b,c \in X$ such that $a \neq b$, $\zeta^{m+1}(c,c) = \sup_{z\in X} \min[\zeta^m(c,z), \zeta(z,c)] = \min[\zeta^m(c,c), \zeta(c,c)] = k$. By the mathematical induction, $\zeta^n(c,c) = k$ for all natural numbers n. Thus $(\mu+\nu)(c,c) = [\cup_{n=1}^{\infty}\zeta^n](c,c) = k$. Hence $\mu + \nu \in E_k(X)$. □

Definition 3.6. A lattice $(L, +, \cdot)$ is called *modular* if $(x+y)\cdot z \leq x + (y \cdot z)$ for all $x, y, z \in L$ with $x \leq z$.

Theorem 3.7. *Let μ and ν be weak fuzzy equivalence relations in a set X. Suppose that $\mu(c,c) = \nu(c,c)$ for all $c \in X$ and $\mu \circ \nu = \nu \circ \mu$. Then $\mu \circ \nu$ is a weak fuzzy equivalence relation.*

Proof. We may show that $\mu\circ\nu$ is w-reflexive (see the proof of Theorem 4.3 in [1]). Suppose that $(\mu\circ\nu)(x,y) = 0$. Then $\sup_{z\in X} \min[\mu(x,z), \nu(z,y)] = 0$. That is, $\min[\mu(x,z), \nu(z,y)] = 0$ for all $z \in X$. Thus $\mu(x,z) = 0$ or $\nu(z,y) = 0$ for all $z \in X$. Since μ and ν are w-symmetric, $\mu(z,x) = 0$ or $\nu(y,z) = 0$ for all $z \in X$. Since $\mu\circ\nu = \nu\circ\mu$, $(\mu\circ\nu)(y,x) = (\nu\circ\mu)(y,x) = \sup_{z\in S} \min[\nu(y,z), \mu(z,x)] = 0$. That is,

$$\text{if } (\mu\circ\nu)(x,y) = 0, \text{ then } \min[(\mu\circ\nu)(x,y),\ (\mu\circ\nu)(y,x)] = 0.$$

Suppose that $(\mu\circ\nu)(x,y) > 0$. Let $(\mu\circ\nu)(x,y) = \sup_{z\in X} \min[\mu(x,z), \nu(z,y)] = p > 0$. Then for any $\alpha > 0$, there exists $v \in X$ such that $\min[\mu(x,v), \nu(v,y)] > p - \alpha$. Since $\frac{p}{2} > 0$, there exists $u \in X$ such that

$$\min[\mu(x,u), \nu(u,y)] > p - \frac{p}{2} = \frac{p}{2} > 0.$$

That is, $\mu(x,u) > 0$ and $\nu(u,y) > 0$. Since μ and ν are w-symmetric, $\mu(u,x) > 0$ and $\nu(y,u) > 0$. Thus
$(\mu\circ\nu)(y,x) = (\nu\circ\mu)(y,x) = \sup_{z\in X} \min[\nu(y,z), \mu(z,x)] \geq \min[\nu(y,u), \mu(u,x)] > 0$.
That is,

$$\text{if } (\mu\circ\nu)(x,y) > 0, \text{ then } \min[(\mu\circ\nu)(x,y),\ (\mu\circ\nu)(y,x)] > 0.$$

Thus $\mu\circ\nu$ is w-symmetric. Since μ and ν are transitive and the operation $\circ$ is associative,

$$(\mu\circ\nu)\circ(\mu\circ\nu) = \mu\circ(\nu\circ\mu)\circ\nu = \mu\circ(\mu\circ\nu)\circ\nu = (\mu\circ\mu)\circ(\nu\circ\nu) \subseteq \mu\circ\nu.$$

Hence $\mu\circ\nu$ is a weak fuzzy equivalence relation. □

Lemma 3.8. *Let μ and ν be weak fuzzy equivalence relations in a set X such that $\mu(c,c) = \nu(c,c)$ for all $c \in X$. If $\mu \circ \nu = \nu \circ \mu$, then $\mu \circ \nu$ is the weak fuzzy equivalence relation in X generated by $\mu \cup \nu$.*

Proof. By Theorem 3.7, $\mu \circ \nu$ is a weak fuzzy equivalence relation. We may show that $\mu \cup \nu \subseteq \mu \circ \nu$ (see the proof of Lemma 4.3 in [1]). That is, $\mu \circ \nu$ is a weak fuzzy equivalence relation containing $\mu \cup \nu$. Let λ be a weak fuzzy equivalence relation in X containing $\mu \cup \nu$. Since λ is transitive, $\mu \circ \nu \subseteq (\mu \cup \nu) \circ (\mu \cup \nu) \subseteq \lambda \circ \lambda \subseteq \lambda$. Thus $\mu \circ \nu$ is the weak fuzzy equivalence relation generated by $\mu \cup \nu$. $\square$

It is well known that if μ and ν are equivalence relations on a set X and $\mu \circ \nu = \nu \circ \mu$, then $\mu \circ \nu$ is the equivalence relation on X generated by $\mu \cup \nu$. Lemma 3.8 may be considered as a generalization of this in the weak fuzzy equivalence relations.

Theorem 3.9. *Let X be a set and let H be a sublattice of $(E_k(X), +, \cdot)$ such that $\mu \circ \nu = \nu \circ \mu$ for all $\mu, \nu \in H$. Then H is a modular lattice for k such that $0 < \epsilon \leq k \leq 1$.*

Proof. Let $\mu, \nu, \rho \in H$ with $\mu \leq \rho$. We may show that $(\mu \circ \nu) \cdot \rho \leq \mu \circ (\nu \cdot \rho)$ (see the proof of Theorem 4.4 in [1]). Since $\mu, \nu \in E_k(X)$, $\mu(c,c) = \nu(c,c) = k$ for all $c \in X$. By Lemma 3.8, $\mu \circ \nu$ is the weak fuzzy equivalence relation generated by $\mu \cup \nu$. That is, $\mu + \nu = \mu \circ \nu$. Since $\mu, \nu \cdot \rho \in H$, $\mu \circ (\nu \cdot \rho) = (\nu \cdot \rho) \circ \mu$. Clearly $\mu(c,c) = (\nu \cdot \rho)(c,c) = k$ for all $c \in X$. By Lemma 3.8, $\mu \circ (\nu \cdot \rho)$ is the weak fuzzy equivalence relation generated by $\mu \cup (\nu \cdot \rho)$. That is, $\mu + (\nu \cdot \rho) = \mu \circ (\nu \cdot \rho)$. Thus $(\mu + \nu) \cdot \rho \leq \mu + (\nu \cdot \rho)$. Hence H is modular. $\square$

Corollary 3.10. *If X is a group and $0 < \epsilon \leq k \leq 1$, then $(E_k(X), +, \cdot)$ is a modular lattice.*

Proof. It is easy to see that $\mu \circ \nu = \nu \circ \mu$ for all $\mu, \nu \in E_k(X)$ since X is a group (see the proof of Proposition 4.3 in [6]). By Theorem 3.9, $(E_k(X), +, \cdot)$ is modular. $\square$

4. Weak fuzzy congruences on semigroups

In this section we recall some basic properties of the weak fuzzy congruences, characterize the weak fuzzy congruence generated by a fuzzy relation on a semigroup, find the largest weak fuzzy congruence contained in the given weak fuzzy congruence on a group, and develop some lattice theoretic properties of the weak fuzzy congruences on semigroups.

Definition 4.1. Let μ be a fuzzy relation in a set X. μ is called *fuzzy left (right) compatible* if $\mu(x,y) \leq \mu(zx,zy)$ $(\mu(x,y) \leq \mu(xz,yz))$ for all $x,y,z \in X$. A weak fuzzy equivalence relation on X is called a *weak fuzzy left congruence (right congruence)* if it is fuzzy left compatible (right compatible). A weak fuzzy equivalence relation on X is called a *weak fuzzy congruence* if it is a weak fuzzy left and right congruence.

Proposition 4.2. *If μ is a fuzzy relation on a semigroup S that is fuzzy left and right compatible, then so is $\cup_{n=1}^{\infty} \mu^n$, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.*

Proof. See Proposition 3.6 of [6]. □

Proposition 4.3. *Let μ and ν be weak fuzzy congruences in a set X. Then $\mu \cap \nu$ is a weak fuzzy congruence.*

Proof. Clearly $\mu \cap \nu$ is fuzzy left and right compatible. By Proposition 3.1, $\mu \cap \nu$ is a weak fuzzy congruence. □

It is easy to see that even though μ and ν are weak fuzzy congruences, $\mu \cup \nu$ is not necessarily a weak fuzzy congruence. We find the weak fuzzy congruence generated by $\mu \cup \nu$ in the following proposition.

Proposition 4.4. *Let μ and ν be weak fuzzy congruences on a semigroup S. Then the weak fuzzy congruence generated by $\mu \cup \nu$ in S is $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$.*

Proof. It is easy to see that $\mu \cup \nu$ is fuzzy left and right compatible. By Proposition 4.2, $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is fuzzy left and right compatible. We may show that $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is the weak fuzzy congruence generated by $\mu \cup \nu$ by the same way as shown in Theorem 3.2. □

We now turn to the characterization of the weak fuzzy congruence generated by a fuzzy relation on a semigroup.

Definition 4.5. Let μ be a fuzzy relation on a semigroup S and let $S^1 = S \cup \{e\}$, where e is the identity of S. We define the fuzzy relation μ^* on S as

$$\mu^*(x,y) = \bigcup_{c,d\in S^1, cad=x, cbd=y} \mu(a,b) \quad \text{for all } x,y \in S.$$

Proposition 4.6. *Let μ and ν be two fuzzy relations on a semigroup S. Then*

(1) $\mu \subseteq \mu^*$
(2) $(\mu^*)^{-1} = (\mu^{-1})^*$
(3) *If* $\mu \subseteq \nu$, *then* $\mu^* \subseteq \nu^*$
(4) $(\mu \cup \nu)^* = \mu^* \cup \nu^*$
(5) $\mu = \mu^*$ *if and only if* μ *is fuzzy left and right compatible*
(6) $(\mu^*)^* = \mu^*$

Proof. See Proposition 3.5 of [6]. □

Theorem 4.7. *Let μ be a fuzzy relation on a semigroup S. Then the weak fuzzy congruence in S generated by μ is $\cup_{n=1}^{\infty}(\mu^* \cup \rho^* \cup \theta^*)^n$, where θ is a fuzzy relation in S such that $\theta(x,y) \leq \mu(x,y)$ for all $x,y \in S$ with $x \neq y$ and $\theta(t,t) = \max\,[\epsilon,\ \sup_{x\neq y,\ x,y\in S} \mu(x,y)]$ for all $t \in S$, and ρ is a fuzzy relation in S such that $\rho(z,z) = 0$ for all $z \in S$ and for all $x,y \in S$ such that $x \neq y$,*

(1) *if* $\mu(x,y) = \mu(y,x) = 0$, *then* $\rho(x,y) = \rho(y,x) = 0$,
(2) *if* $\mu(x,y) > 0$ *and* $\mu(y,x) = 0$, *then* $\rho(x,y) = 0$ *and* $\rho(y,x) = \min\,[\mu(x,y),\ \delta]$ *for some* $\delta > 0$,
(3) *if* $\mu(x,y) > 0$ *and* $\mu(y,x) > 0$, *then* $\rho(x,y) = \mu(x,y)$ *and* $\rho(y,x) = \mu(y,x)$.

Here μ^*, ρ^*, θ^* are fuzzy relations defined in Definition 4.5.

Proof. Let $\mu_1 = \mu \cup \rho \cup \theta$. By (4) of Proposition 4.6,

$$\mu_1^* = (\mu \cup \rho \cup \theta)^* = \mu^* \cup \rho^* \cup \theta^*.$$

Since $\mu_1(x,x) \geq \theta(x,x) \geq \epsilon > 0$, $\mu_1^*(x,x) \geq \epsilon$ by (1) of Proposition 4.6. Let $x,y \in X$ with $x \neq y$ and let $S^1 = S \cup \{e\}$, where e is the identity of S. Since $x \neq y$ implies $a \neq b$ in Definition 4.5,

$$\mu^*(x,y) \leq \sup_{x\neq y\in S} \mu(x,y) \leq \theta(t,t)$$

for all $t \in S$. Since $\theta(x,y) \leq \mu(x,y)$, $\theta^*(x,y) \leq \mu^*(x,y)$ by (3) of Proposition 4.6. That is, $\theta^*(x,y) \leq \theta(t,t)$. Since $\rho(a,b) \leq \sup_{x \neq y \in S} \ \mu(x,y)$ for all $a, b \in S$,

$$\rho^*(x,y) \leq \sup_{x \neq y \in S} \ \mu(x,y) \leq \theta(t,t)$$

for all $t \in S$. Thus $\mu_1^*(x,y) \leq \theta(t,t)$, and hence

$$\inf_{z \in S} \ \mu_1^*(z,z) \geq \inf_{z \in S} \ \theta^*(z,z) \geq \theta(t,t) \geq \mu_1^*(x,y).$$

Thus μ_1^* is w-reflexive. By Proposition 2.6, $\cup_{n=1}^{\infty} \ (\mu_1^*)^n$ is w-reflexive.
(i) Suppose that $\mu_1^*(x,y) = \max[\mu^*(x,y), \rho^*(x,y), \theta^*(x,y)] = 0$. Then $\mu^*(x,y) = 0$ and $\rho^*(x,y) = 0$. Since $\rho^*(x,y) = 0$, $\rho(\alpha,\beta) = 0$ for every $\alpha, \beta \in S$ such that $c\alpha d = x$ and $c\beta d = y$ for $c, d \in S^1$. Since $\mu^*(x,y) = 0$, $\mu(\alpha,\beta) = 0$ for every $\alpha, \beta \in S$ such that $c\alpha d = x$ and $c\beta d = y$ for $c, d \in S^1$. By the hypothesis, $\mu(\beta,\alpha) = \rho(\beta,\alpha) = 0$ for every $\alpha, \beta \in S$ such that $c\alpha d = x$ and $c\beta d = y$ for $c, d \in S^1$. That is, $\mu^*(y,x) = \rho^*(y,x) = 0$. Since $\theta^*(y,x) \leq \mu^*(y,x)$, $\mu_1^*(y,x) = 0$. Thus $\min\,[\mu_1^*(x,y), \ \mu_1^*(y,x)] = 0$.
(ii) Suppose that $\mu_1^*(x,y) > 0$. Since $\theta^*(x,y) \leq \mu^*(x,y)$, $\max\,[\mu^*(x,y), \rho^*(x,y)] > 0$. Thus $\mu^*(x,y) > 0$ or $\rho^*(x,y) > 0$. If $\mu^*(x,y) > 0$, then $\mu(\alpha,\beta) > 0$ for some $\alpha, \beta \in S$ such that $c\alpha d = x$ and $c\beta d = y$ for $c, d \in S^1$. By the hypothesis, $\rho(\beta,\alpha) > 0$ for some $\alpha, \beta \in S$ such that $c\alpha d = x$ and $c\beta d = y$ for $c, d \in S^1$. That is, $\rho^*(y,x) > 0$, and hence $\mu_1^*(y,x) > 0$. If $\rho^*(x,y) > 0$, then $\rho(\alpha,\beta) > 0$ for some $\alpha, \beta \in S$ such that $c\alpha d = x$ and $c\beta d = y$ for $c, d \in S^1$. By the hypothesis, $\mu(\beta,\alpha) > 0$ for some $\alpha, \beta \in S$ such that $c\alpha d = x$ and $c\beta d = y$ for $c, d \in S^1$. That is, $\mu^*(y,x) > 0$, and hence $\mu_1^*(y,x) > 0$. Thus $\min\,[\mu_1^*(x,y), \ \mu_1^*(y,x)] > 0$.
From (i) and (ii), μ_1^* is w-symmetric, and hence $\cup_{n=1}^{\infty}(\mu_1^*)^n$ is w-symmetric by Proposition 2.8. By Proposition 2.5, $\cup_{n=1}^{\infty}(\mu_1^*)^n$ is transitive. Thus $\cup_{n=1}^{\infty}(\mu_1^*)^n$ is a weak fuzzy equivalence relation containing μ. By (4) and (6) of Proposition 4.6,

$$(\mu_1^*)^* = (\mu^* \cup \rho^* \cup \theta^*)^* = (\mu^*)^* \cup (\rho^*)^* \cup (\theta^*)^* = \mu^* \cup \rho^* \cup \theta^* = \mu_1^*.$$

By (5) of Proposition 4.6, μ_1^* is fuzzy left and right compatible. By Proposition 4.2, $\cup_{n=1}^{\infty}(\mu_1^*)^n$ is fuzzy left and right compatible. Thus $\cup_{n=1}^{\infty}(\mu_1^*)^n$ is a weak fuzzy congruence containing μ. Let ν be a weak fuzzy congruence containing μ.
(i)$'$ We consider the case of $\mu(x,y) = \mu(y,x) = 0$. Since $\rho(x,y) = \rho(y,x) = 0$, $\mu_1(x,y) \leq \nu(x,y)$ for all $x, y \in S$ such that $x \neq y$.

(ii)$'$ We consider the case of $\mu(x,y) > 0$ and $\mu(y,x) = 0$. Since $\mu(x,y) > 0$, $\nu(x,y) > 0$. Since ν is w-symmetric, $\nu(y,x) > 0$. Thus $\nu(y,x) > \delta > 0$ for some $\delta > 0$. Since $\rho(x,y) = 0$ and $\rho(y,x) = \min\,[\mu(x,y),\ \delta]$, $\rho(y,x) < \nu(y,x)$, and hence $\mu_1(x,y) \leq \nu(x,y)$ for all $x, y \in S$ such that $x \neq y$.
(iii)$'$ We consider the case of $\mu(x,y) > 0$ and $\mu(y,x) > 0$. Since $\rho(x,y) = \mu(x,y)$ and $\rho(y,x) = \mu(y,x)$, $\mu_1(x,y) \leq \nu(x,y)$ for all $x, y \in S$ such that $x \neq y$.
From (i)$'$, (ii)$'$, and (iii)$'$, $\mu_1(x,y) \leq \nu(x,y)$ for all $x, y \in S$ such that $x \neq y$. Since $\mu(x,y) \leq \nu(x,y) \leq \nu(t,t)$, $\sup_{x\neq y,\ x,y\in X} \mu(x,y) \leq \nu(t,t)$ for all $t \in X$, and hence $\theta(t,t) \leq \nu(t,t)$. Clearly $\rho(t,t) \leq \nu(t,t)$. That is, $\mu_1(t,t) \leq \nu(t,t)$. Thus $\mu_1 \subseteq \nu$. By (3) of Proposition 4.6, $\mu_1^* \subseteq \nu^*$. Since ν is fuzzy left and right compatible, $\nu^* = \nu$ by (5) of Proposition 4.6. That is, $\mu_1^* \subseteq \nu$. Suppose that $(\mu_1^*)^k \subseteq \nu$. Then

$$\begin{aligned}(\mu_1^*)^{k+1}(x,y) = (\mu_1^* \circ (\mu_1^*)^k)(x,y) &= \sup_{z\in X}\ \min[\mu_1^*(x,z), (\mu_1^*)^k(z,y)] \\ &\leq \sup_{z\in X}\ \min[\nu(x,z), \nu(z,y)] = (\nu \circ \nu)(x,y).\end{aligned}$$

Since ν is transitive, $(\mu_1^*)^{k+1} \subseteq \nu\circ\nu \subseteq \nu$. By the mathematical induction, $(\mu_1^*)^n \subseteq \nu$ for all natural numbers n. Thus

$$\cup_{n=1}^{\infty}\ (\mu_1^*)^n = \mu_1^* \cup (\mu_1^* \circ \mu_1^*) \cup (\mu_1^* \circ \mu_1^* \circ \mu_1^*) \cdots \subseteq \nu.$$

□

In next theorem, we find the largest weak fuzzy congruence contained in the given weak fuzzy congruence on a group.

THEOREM 4.8. *Let μ be a weak fuzzy congruence on a group S. Then the function $\nu : S \times S \to \mathbb{R}$ defined by $\nu(a,b) = \inf_{x,y\in S}\ \mu(xay, xby)$ is the largest weak fuzzy congruence on S contained in μ.*

Proof. Let $\nu(a,b) = \inf_{x,y\in S}\ \mu(xay, xby)$. Clearly $\nu(a,a) \geq \epsilon > 0$. Since S is a group, $c \neq d$ implies $xcy \neq xdy$ for $c, d, x, y \in S$. Thus $\inf_{t\in S}\ \nu(t,t) = \inf_{t\in S}\ \inf_{x,y\in S}\ \mu(xty, xty) = \inf_{x,y\in S}\ \inf_{t\in S}\ \mu(xty, xty) \geq \inf_{x,y\in S}\ \mu(xcy, xdy) = \nu(c,d)$ for $c, d \in S$ such that $c \neq d$. That is, ν is weakly reflexive. Since μ is weakly symmetric,

$$\min\,[\nu(a,b),\ \nu(b,a)] = \min[\inf_{x,y\in S}\ \mu(xay, xby),\ \inf_{x,y\in S}\ \mu(xby, xay)] > 0$$

or $\nu(a,b)=\nu(b,a)=0$. That is, ν is weakly symmetric. Since

$$\sup_{c\in S}\ \min[\inf_{x,y\in S}\ \mu(xay,xcy),\ \inf_{x,y\in S}\ \mu(xcy,xby)]$$

is a lower bound for the set $\{\ \sup_{c\in S}\ \min\ [\mu(xay,xcy),\ \mu(xcy,xby)]:\ x,y\in S\}$,

$$\begin{aligned}\sup_{c\in S}\ \min\ [\inf_{x,y\in S}\ \mu(xay,xcy),\ \inf_{x,y\in S}\ \mu(xcy,xby)]\\ \leq \inf_{x,y\in S}\ \sup_{c\in S}\ \min\ [\mu(xay,xcy),\ \mu(xcy,xby)].\end{aligned}$$

Since μ is transitive,

$$\begin{aligned}(\nu\circ\nu)(a,b)&\leq \inf_{x,y\in S}\ \sup_{c\in S}\ \min\ [\mu(xay,xcy),\ \mu(xcy,xby)]\\ &\leq \inf_{x,y\in S}\ \mu(xay,xby)=\nu(a,b).\end{aligned}$$

That is, ν is transitive. Thus ν is a weak fuzzy equivalence relation in S. Since μ is compatible,

$$\begin{aligned}\mu(a,b)\leq\mu(pa,pb)&\leq \inf_{y\in S}\ \mu(pay,pby)\\ &\leq \inf_{x,y\in S}\ \mu(xpay,xpby)=\nu(pa,pb).\end{aligned}$$

It is easy to see that $\nu(a,b)=\inf_{x,y\in S}\ \mu(xay,xby)\leq\mu(a,b)$. Thus $\nu(a,b)\leq\nu(pa,pb)$. Similarly $\nu(ap,bp)\geq\nu(a,b)$. That is, ν is a weak fuzzy congruence on S. Let λ be a weak fuzzy congruence on S such that $\lambda\subseteq\mu$. Then $\lambda(a,b)\leq\lambda(xa,xb)\leq\lambda(xay,xby)\leq\mu(xay,xby)$ for all $x,y\in S$. That is, $\lambda(a,b)\leq\inf_{x,y\in S}\ \mu(xay,xby)=\nu(a,b)$. Clearly $\nu\subseteq\mu$. Thus ν is the largest weak fuzzy congruence on S contained in μ. $\square$

We now turn to the lattice theoretic properties of the weak fuzzy congruences on semigroups. Let $C(S)$ be the collection of all weak fuzzy congruences on a semigroup S. Then it is easy to see that $(C(S),\leq)$ is a complete lattice, where $\leq$ is a relation on the set of all weak fuzzy congruences on S defined by $\mu\leq\nu$ iff $\mu(x,y)\leq\nu(x,y)$ for all $x,y\in S$. We define an addition on $C(S)$ by $\mu+\nu=<\mu\cup\nu>$ and a multiplication on $C(S)$ by $\mu\cdot\nu=\mu\cap\nu$, where $<\mu\cup\nu>$ is the weak fuzzy congruence generated by $\mu\cup\nu$. Then for $\mu,\nu\in C(S)$, $\mu\cdot\nu\in C(S)$ and $\mu+\nu\in C(S)$ by Proposition 4.3 and Proposition 4.4, respectively. Let $C_k(S)=\{\mu\in C(S):\mu=\mu^{-1}$ and $\mu(c,c)=k$ for all $c\in S\}$.

THEOREM 4.9. *$C_k(S)$ is a sublattice of $(C(S),+,\cdot)$ for $0<\epsilon\leq k\leq 1$.*

Proof. We may show that $C_k(S)$ is a sublattice of $(C(S), +, \cdot)$ for $0 < \epsilon \leq k \leq 1$ by the same way as shown in the proof of Theorem 3.5. $\square$

LEMMA 4.10. *Let μ and ν be weak fuzzy congruences on a semigroup S such that $\mu(c,c) = \nu(c,c)$ for all $c \in S$. If $\mu \circ \nu = \nu \circ \mu$, then $\mu \circ \nu$ is the weak fuzzy congruence on S generated by $\mu \cup \nu$.*

Proof. By Theorem 3.7, $\mu \circ \nu$ is a weak fuzzy equivalence relation. Since S is a semigroup,

$$\begin{aligned}(\mu \circ \nu)(x,y) &= \sup_{a \in S} \min[\mu(x,a), \nu(a,y)] \leq \sup_{za \in S} \min[\mu(zx,za), \nu(za,zy)] \\ &\leq \sup_{t \in S} \min[\mu(zx,t), \nu(t,zy)] = (\mu \circ \nu)(zx,zy).\end{aligned}$$

Thus $\mu \circ \nu$ is fuzzy left compatible. Similarly we may show $\mu \circ \nu$ is fuzzy right compatible. Hence $\mu \circ \nu$ is a weak fuzzy congruence on S. We may show that $\mu \circ \nu$ is the weak fuzzy congruence generated by $\mu \cup \nu$ by the same way as shown in the proof of Lemma 3.8. $\square$

It is well known that if μ and ν are congruences on a semigroup S and $\mu \circ \nu = \nu \circ \mu$, then $\mu \circ \nu$ is the congruence on S generated by $\mu \cup \nu$. Lemma 4.10 may be considered as a generalization of this in the weak fuzzy congruences.

THEOREM 4.11. *Let S be a semigroup and let H be a sublattice of $(C_k(S), +, \cdot)$ such that $\mu \circ \nu = \nu \circ \mu$ for all $\mu, \nu \in H$. Then H is a modular lattice for $0 < \epsilon \leq k \leq 1$.*

Proof. Let $\mu, \nu, \rho \in H$ with $\mu \leq \rho$. Then we may show that $(\mu \circ \nu) \cdot \rho \leq \mu \circ (\nu \cdot \rho)$ by the same way as shown in the proof of Theorem 3.9. Since $\mu, \nu \in C_k(S)$, $\mu(c,c) = \nu(c,c) = k$ for all $c \in S$. By Lemma 4.10, $\mu \circ \nu$ is the fuzzy congruence generated by $\mu \cup \nu$. That is, $\mu + \nu = \mu \circ \nu$. Since $\mu, \nu \cdot \rho \in H$, $\mu \circ (\nu \cdot \rho) = (\nu \cdot \rho) \circ \mu$. Clearly $\mu(c,c) = (\nu \cdot \rho)(c,c) = k$ for all $c \in S$. By Lemma 4.10, $\mu \circ (\nu \cdot \rho)$ is the weak fuzzy congruence generated by $\mu \cup (\nu \cdot \rho)$. That is, $\mu + (\nu \cdot \rho) = \mu \circ (\nu \cdot \rho)$. Thus $(\mu + \nu) \cdot \rho \leq \mu + (\nu \cdot \rho)$. Hence H is modular. $\square$

COROLLARY 4.12. *If S is a group and $0 < \epsilon \leq k \leq 1$, then $(C_k(S), +, \cdot)$ is a modular lattice.*

Proof. It is easy to see that $\mu \circ \nu = \nu \circ \mu$ for all $\mu, \nu \in C_k(S)$ since S is a group (see the proof of Proposition 4.3 in [6]). By Theorem 4.11, $(C_k(S), +, \cdot)$ is modular. □

Open Problems. In this note, we defined weak fuzzy equivalence relations (or fuzzy congruences) and developed some crucial properties of those relations (or congruences). It is an open problem to find weaker fuzzy equivalence relations (or fuzzy congruences) which still have so many nice properties as those relations (or congruences) in this note. In Theorem 4.12, we found the largest weak fuzzy congruence contained in a given weak fuzzy congruence on a group. We suggest a problem of finding the largest weak fuzzy congruence contained in a given weak fuzzy equivalence relation (or congruence) on a semigroup.

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