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WEAK INJECTIVITY IN THE CATEGORY OF NORMAL FUZZY HYPERGROUPS

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Abstract. Based on injectivity, we introduce some definitions in the category **NFHG** of normal fuzzy hypergroups. And we show that a complete normal fuzzy hypergroup is a weakly injective object in **NFHG**. Also we investigate weak injectivity in the comma category **NFHG**$/K$.

1. Introduction

Banaschwski [1] investigated injectivity in the category **B** with some properties and Cagliari [3] investigated injectivity in the comma category $\mathbf{C}/A$. Also Sun [6] introduced some properties of the category of normal fuzzy hypergroups. In this paper, we introduce some definitions in **NFHG**. And we show that a complete normal fuzzy hypergroup is a weakly injective object in **NFHG**. Also we show that an object $f : X \to K$ in **NFHG** $/K$ is a weakly injective object if and only if $f^{-1}(k)$ is weakly injective in **NFHG** for all $k \in K$ and $< i, f >$ has a left inverse in **NFHG**$/K$.

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2. Preliminaries

In this section, we state some definitions and properties which will serve as the basic tools for the arguments used to prove our results.

Let H be a nonempty set and $F(H) = [0,1]^H$ be the set of all fuzzy subset of H and $F^*(H) = F(H) - \{\phi\}$. A fuzzy hyperoperation on H is a mapping $\star : H^2 \to F(H)$ and the couple $(H, \star)$ is called a partial fuzzy hypergroupoid. If the fuzzy hyperoperation $\star$ maps H^2 into $F^*(H)$, then $(H, \star)$ is called a fuzzy hypergroupoid.

Definition 2.1. (1) A *fuzzy semihypergroup* is a a fuzzy hypergroupoid $(H, \star)$ which satisfies the associative law.

(2) A *fuzzy quasihypergroup* is a a fuzzy hypergroupoid $(H, \star)$ which satisfies the reproductive law.

(3) A *fuzzy hypergroup* is a fuzzy semihypergroup which is also a fuzzy quasihypergroup

(4) A *fuzzy subhypergroup* $(A, \bullet)$ of a fuzzy hypergroup $(B, \bullet)$ is a nonempty subset $A \subseteq B$ such that for any $a \in A$, $a \bullet A = A = A \bullet a$.

Definition 2.2. A fuzzy hypergroup $(H, \star)$ is said to be *normal* if it satisfies the following three conditions:

(1) $(x \star x)(x) = 1$ for all $x \in H$;

(2) $x \star y = x \star x \cup y \star y$ for all $x, y \in H$;

(3) $(x \star x)(z) \geq (x \star x)(y) \wedge (y \star y)(z)$ for all $x, y, z \in H$.

Let **NFHG** be a category, where objects are normal fuzzy hypergroups and a morphism from $(H, \diamond)$ to $(K, \star)$ is a mapping $f : H \to K$ such that $f(a \diamond b) \subseteq f(a) \star f(b)$.

Definition 2.3. Let $(H, \star) \in \mathrm{Ob}(\mathbf{NFHG})$. $a \in (H, \star)$ is called a *complete element* if $x \star a = a \star x = H$ for all $x \in H$. And $(H, \star)$ is called a *complete normal fuzzy hypergroup* if there is a complete element in $(H, \star)$.

Definition 2.4. $I \in \mathrm{Ob}(\mathbf{NFHG})$ is said to be *weakly injective* if, for any monomorphism $m : A \to B$ with $m(a \circ b) = m(a) \circ m(b)$ and any morphism $n : A \to I$, there exists a morphism $f : B \to I$ such that $f \circ m = n$.

3. Weak Injectivity

Theorem 3.1. *A complete normal fuzzy hypergroup is weakly injective in* **NFHG.**

Proof. Let $m : A \to B$ be a monomorphism such that $m(a \circ b) = m(a) \circ m(b)$. For any $g : A \to C$ where C is a complete normal fuzzy hypergroup, there is a morphism $h : B \to C$ where

$$h(b) = \begin{cases} g(a), b = m(a) \\ c, \ \textit{otherwise} \end{cases}$$

with c is a complete element in C. Then $h \circ m = g$, that is, the following diagram

$$\begin{array}{ccc} A & \xrightarrow{m} & B \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ C & \xrightarrow[i]{} & C \end{array}$$

commutes, since $h \circ m(a) = h(m(a)) = g(a)$. And if $p, q \in Im(m)$, then we have that

$$\begin{aligned} h(p \circ q) &= h(m(u) \circ m(v)) = h(m(u \circ v)) = g(u \circ u) \\ &\subseteq g(u) \circ g(v) = h(m(u)) \circ h(m(v)) = h(p) \circ h(q). \end{aligned}$$

So we get $h(p \circ q) \subseteq h(p) \circ h(q)$.

Also, if p or q is not an element of $Im(m)$, that is, $h(p) = c$ or $h(q) = c$, then we have that $h(p) \circ h(q) = C$. So we get $h(p \circ q) \subseteq h(p) \circ h(q)$. □

Corollary 3.2. **NFHG** *has enough weakly injective objects.*

Proof. Let A be a normal fuzzy hypergroup. Then $A \cup \{c\}$, where c is a complete element, is the weakly injective object in **NFHG**. And there is a monomorphism $m : A \to (A \cup \{c\})$ such that $m(a \circ b) = m(a) \circ m(b)$ by [5]. □

Theorem 3.3. *Let $f : C \to K$ be an object such that $f(a \circ b) = f(a) \circ f(b)$ in* $\mathbf{NFHG}/K$. *Then f is weakly injective in* $\mathbf{NFHG}/K$ *if and only if every monomorphism $m : C \to D$ such that $m(a \circ b) = m(a) \circ m(b)$, where $g \circ m = f$ with $g : D \to K$, has a left inverse in* $\mathbf{NFHG}/K$.

Proof. For sufficiency part, by given condition we get the following commutative diagram:

$$\begin{array}{ccc} C & \xrightarrow{m} & D \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[f]{} & K \end{array}$$

By hypothesis, there is a morphism $n : D \to C$ such that $n \circ m = i$ and $f \circ n = g$. Thus $m : C \to D$ has a left inverse. For the necessary part, let $s : X \to Y$ be a monomorphism such that $s(a \circ b) = s(a) \circ s(b)$ where the following diagram

$$\begin{array}{ccc} X & \xrightarrow{s} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[f]{} & K \end{array}$$

commutes. Then f can be embedded into the weakly injective object $\pi_K : (C \cup \{c\}) \times K \to K$ with a monomorphism $< j, f >: C \to (C \cup \{c\}) \times K$ where $< j, f > (a) = (a, f(a))$ by [2, 5]. That is, the following diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & K \\ {\scriptstyle <j,f>}\downarrow & & \downarrow{\scriptstyle i} \\ (C \cup \{c\}) \times K & \xrightarrow[\pi_K]{} & K \end{array}$$

commutes. Since π_K is the weakly injective object, there is a morphism $n : Y \to (C \cup \{c\}) \times K$ such that $\pi_K \circ n = h$ and $n \circ s =< j, f > \circ g$. Also by hypothesis, there is a morphism $q : (C \cup \{c\}) \times K \to C$ such that $f \circ q = \pi_K$ and $q\circ < j, f >= i$. Now $q \circ n : Y \to C$ is a morphism with $f \circ (q \circ n) = \pi_K \circ n = h$ and $(q \circ n) \circ s = q \circ (< j, f > \circ g) = g$. □

Theorem 3.4. *$f : X \to K$ is weakly injective in* **NFHG**$/K$ *if and only if the following two conditions are satisfied.*

(1) $< i, f >: X \to X \times K$ has a left inverse in **NFHG**$/K$.

(2) the normal fuzzy subhypergroup $f^{-1}(k) = \{x \in X | f(x) = k\}$ is weakly injective in **NFHG** *for all $k \in K$.*

Proof. For sufficiency part, since $< i, f >: X \to X \times K$, where $i : X \to X$ and $f : X \to K$, is a regular monomorphism by [3], $< i, f >$

is also a monomorphism. By the definition of the product, we get the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{<i,f>} & X \times K \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \pi_K} \\ X & \xrightarrow[f]{} & K \end{array}$$

By hypothesis, there is a morphism $r : X \times K \to X$ such that $r\circ < i, f >= i$ and $f \circ r = \pi_K$. Thus $< i, f >: X \to X \times K$ has a left inverse in **NFHG**$/K$. Let $m : C \to D$ be a monomorphism with $m(a \circ b) = m(a) \circ m(b)$ and $g : C \to f^{-1}(k)$ be a morphism where $f : X \to K$. Since **NFHG**$/K$ is cartesian closed by [6], by adjunction, there is a morphism $h : C \times K \to X$ such that $f \circ h = \pi_K$. So we get the following commutative diagram:

$$\begin{array}{ccc} C \times K & \xrightarrow{(m\times i)} & D \times K \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \pi_K} \\ X & \xrightarrow[f]{} & K \end{array}$$

commutes. Since $f : X \to K$ is weakly injective in **NFHG**$/K$, there is a morphism $l : D \times K \to X$ such that $l \circ (m \times i) = h$ and $f \circ l = \pi_K$. Thus by adjunction, there is a morphism $n : D \to f^{-1}(k)$ such that $n \circ m = g$. That is, the diagram

$$\begin{array}{ccc} C & \xrightarrow{m} & D \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle n} \\ f^{-1}(k) & \xrightarrow[i]{} & f^{-1}(k) \end{array}$$

commutes. So $f^{-1}(k)$ is weakly injective in **NFHG**.

For the necessary part, since $f^{-1}(k) \subseteq X$ is weakly injective in **NFHG**, $\pi_K : f^{-1}(k) \times K \to K$ is weakly injective in **NFHG**$/K$. So for any monomorphism $m : C \to D$ with $m(u \circ v) = m(u) \circ m(v)$,

$g : C \to f^{-1}(k) \times K$ and $h : D \to K$ such that the diagram

$$\begin{array}{ccc} C & \xrightarrow{m} & D \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ f^{-1}(k) \times K & \xrightarrow[\pi_K]{} & K \end{array}$$

commutes, there is a morphism $n : D \to f^{-1}(k) \times K$ such that $h \circ m = \pi_K \circ g$, $n \circ m = g$ and $\pi_K \circ n = h$. Also for any morphism $< i, f >: X \to X \times K$, there is a morphism $s' : X \times K \to X$ such that $s' \circ < i, f >= i$. Let $s'_{|f^{-1}(k)\times K} = s$. So we get $s \circ < i, f > (x) = i(x)$ for all $x \in f^{-1}(k)$. That is, $s(x, f(x)) = x$ for all $x \in f^{-1}(k)$. Thus $f \circ s = \pi_K$, since $f \circ s(x, k) = f(x) = k = \pi_K(x, k)$ where $x \in f^{-1}(k)$ and $k \in K$. To show that $f : X \to K$ is weakly injective in **NFHG**$/K$, consider the diagram

$$\begin{array}{ccc} C & \xrightarrow{m} & D \\ {\scriptstyle s\circ g}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow[f]{} & K \end{array}$$

where $m : C \to D$ is a monomorphism with $m(u \circ v) = m(u) \circ m(v)$, $s \circ g : C \to X$ and $h : D \to K$. Since $f \circ s = \pi_K$ and $\pi_K \circ g = h \circ m$, we get $f \circ s \circ g = h \circ m$. Then there is a morphism $s \circ n : D \to X$ such that $s \circ n \circ m = s \circ g$ and $f \circ s \circ n = h$, since $n \circ m = g$, $f \circ s = \pi_K$ and $\pi_K \circ n = h$. Therefore $f : X \to K$ is weakly injective in **NFHG**$/K$. □

Corollary 3.5. *For $f : C \to K$ such that $f(a \circ b) = f(a) \circ f(b)$ in* **NFHG**$/K$*, the followings are equivalent.*

(1). f is weakly injective in **NFHG**$/K$*.*

(2). Every monomorphism $m : C \to D$ such that $m(a \circ b) = m(a) \circ m(b)$, where $g \circ m = f$ with $g : D \to K$, has a left inverse in **NFHG**$/K$*.*

(3). $< i, f >: C \to C \times K$ has a left inverse in **NFHG**$/K$ *and the normal fuzzy subhypergroup $f^{-1}(k) = \{x \in C | f(x) = k\}$ is weakly injective in* **NFHG**.

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