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MULTIPLICITY OF SOLUTIONS OF ELLIPTIC SYSTEM USING TWO CRITICAL POINT THEOREM

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Abstract. In this paper, we consider the system of three elliptic equations using two critical point theorem. We prove the existence of two solutions for suitable forcing terms, under a condition on the linear part which prevents resonance with eigenvalues of the operator.

1. Introduction

In this work we consider the problem

$$(1) \qquad \begin{cases} -\Delta u = au + bv + (v^+)^{p_1} + f_1 + t\phi_1 & \text{in } \Omega, \\ -\Delta v = bu + av + (u^+)^{p_2} + f_2 + r\phi_1 & \text{in } \Omega, \\ -\Delta w = cw + (w^+)^{p_3} + f_3 + s\phi_1 & \text{in } \Omega, \\ u = v = w = 0 & \text{on } \partial\Omega, \end{cases}$$

where $u^+ = \max\{0, u(x)\}$, $\phi_1 > 0$ is the first eigenfunction of the Laplacian with Dirichlet boundary conditions and $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain with $N \geq 2$.

The nonlinearities will be assumed both superlinear and subcritical, that is, $1 < p_1, p_2, p_3 < 2^* - 1$, where $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = \infty$ if $N = 2$.

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We may write (1) in vectorial form as

$$\begin{cases} -\Delta \begin{bmatrix} u \\ v \\ w \end{bmatrix} = A \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} (u^+)^{p_1} \\ (v^+)^{p_2} \\ (w^+)^{p_3} \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} + \begin{bmatrix} t \\ r \\ s \end{bmatrix} \phi_1 \text{ in } \Omega, \\ \qquad u = v = w = 0 \qquad\qquad \text{on } \partial\Omega, \end{cases}$$

where $A = \begin{bmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix}$; we will assume that A has real eigenvalues $\nu_{i,1} = a + b$, $\nu_{i,2} = a - b$ and $\nu_{i,3} = c$.

Throughout the paper, we will denote by $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_i \leq \ldots$ the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$ and by $\{\phi_i\}_{i\in\mathbb{N}}$ the corresponding eigenfunctions, taken orthogonal and normalized with $\|\phi_i\|_{L^2} = 1$ and $\phi_1 > 0$; by $\sigma(-\Delta)$ we will denote the spectrum of the Laplacian, that is, the set $\{\lambda_i : i \in \mathbb{N}\}$.

The results of our study are as follows.

THEOREM 1.1. *If A has real eigenvalues that are not $\sigma(-\Delta)$ and f_1, f_2, $f_3 \in L^n(\Omega)$ with $n > N \geq 2$ then there exits $(t_0, r_0, s_0) \in \mathbb{R}^3$ such that if*

$$(t, r, s)^T = (t_0, r_0, s_0)^T + (\lambda_1 I - A)(\tau, \rho, \sigma)^T$$

with $\tau, \rho, \sigma < 0$ then a negative solution $(u_{neg}, v_{neg}, w_{neg})$ of (1) exists.

THEOREM 1.2. *Let $a - b \notin \sigma(-\Delta)$, $a + b \notin \sigma(-\Delta)$ and $c \notin \sigma(-\Delta)$, f_1, f_2, $f_3 \in L^n(\Omega)$ with $n > N \geq 2$ and (t, r, s) as in Theorem 1.1; then there exists a second solution for system (1).*

2. The negative solution

In this section, we will look for negative solutions, in the sense that both components are negative: this is relatively simple since in this case the nonlinear term disappears in (1).

We will need the following.

LEMMA 2.1. *If A has real eigenvalues that are not in $\sigma(-\Delta)$ and f_1, f_2, $f_3 \in L^n(\Omega)$ with $n > N$ then there exists a unique solution (u_0, v_0, w_0)*

of the problem

$$(2)\qquad \begin{cases} -\Delta \begin{bmatrix} u \\ v \\ w \end{bmatrix} = A \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} & \text{in } \Omega, \\ u = v = w = 0 & \text{in } \partial\Omega. \end{cases}$$

Proof. For the matrix A eigenvalue-eigenvector pairs are

$$\nu_{i,1} = a+b, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \quad \nu_{i,2} = a-b, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}; \quad \nu_{i,3} = c, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hance A is diagonalizable, that is, $X^{-1}AX = D$ where

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} a+b & 0 & 0 \\ 0 & a-b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

Let $\begin{bmatrix} u \\ v \\ w \end{bmatrix} = X \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix}$ then we written the equation (2) as

$$(3)\qquad \begin{cases} -\Delta\tilde{u} = (a+b)\tilde{u} + \tilde{f}_1 & \text{in } \Omega, \\ -\Delta\tilde{v} = (a-b)\tilde{v} + \tilde{f}_2 & \text{in } \Omega, \\ -\Delta\tilde{w} = c\tilde{w} + \tilde{f}_3 & \text{in } \Omega, \\ \tilde{u} = \tilde{v} = \tilde{w} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \end{bmatrix} = X^{-1} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}f_1 + \frac{1}{2}f_2 \\ \frac{1}{2}f_1 - \frac{1}{2}f_2 \\ f_3 \end{bmatrix}.$

Since each real eigenvalue of A is not in $\sigma(-\Delta)$, equation (3) are uniquely solvable.

The hypothesis $f_1,\ f_2,\ f_3 \in L^n(\Omega)$ implies that $\tilde{f}_1,\ \tilde{f}_2,\ \tilde{f}_3 \in L^n(\Omega)$. By regularity theory and General Sobolev inequalities, $u_0,\ v_0,\ w_0 \in W^{2,n}(\Omega) \subseteq C^{1,\alpha}(\bar{\Omega})$ for $\alpha = 1 - \frac{N}{n}$. □

With this result we may obtain the negative solution:

Proof of Theorem 1.1. Let (u_0, v_0, w_0) be the corresponding solution for (2). Assuming that the problem

$$\begin{cases} -\Delta \begin{bmatrix} u \\ v \\ w \end{bmatrix} = A \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} t \\ r \\ s \end{bmatrix} \phi_1 & \text{in } \Omega, \\ u = v = w = 0 & \text{in } \partial\Omega \end{cases}$$

is looking for a solution of the form $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \phi_1$, the coefficients $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ satisfies the condition $(\lambda_1 I - A) \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} t \\ r \\ s \end{bmatrix}$. By the superposition principle,

$$(\lambda_1 I - A)^{-1} \begin{bmatrix} t \\ r \\ s \end{bmatrix} \phi_1 + \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} \tag{4}$$

is a solution of (1), provided it is nonpositive.

Since $u_0, v_0, w_0 \in C^{1,\alpha}$, we set

$$\begin{aligned} \alpha_0 &= \sup\{\alpha | \alpha\phi_1 + u_0 < 0\}, \\ \beta_0 &= \sup\{\beta | \beta\phi_1 + v_0 < 0\}, \\ \gamma_0 &= \sup\{\gamma | \gamma\phi_1 + w_0 < 0\}. \end{aligned}$$

If we set $(t_0, r_0, s_0)^T = (\lambda_1 I - A)(\alpha_0, \beta_0, \gamma_0)^T$ in the condition

$$\begin{bmatrix} t \\ r \\ s \end{bmatrix} = (\lambda_1 I - A) \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = (\lambda_1 I - A) \left(\begin{bmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{bmatrix} + \begin{bmatrix} \alpha - \alpha_0 \\ \beta - \beta_0 \\ \gamma - \gamma_0 \end{bmatrix} \right),$$

then $\tau = \alpha - \alpha_0 < 0$, $\rho = \beta - \beta_0 < 0$, $\sigma = \gamma - \gamma_0 < 0$ because $\alpha < \alpha_0$, $\beta < \beta_0$, and $\gamma < \gamma_0$. We get the condition in the claim. □

3. The second solution

We will find the second solution by using a minimax theorem due to Felmer [4].

3.1. The variational structure. We consider the Hilbert space $E = H_0^1 \times H_0^1 \times H_0^1$ equipped with the scalar product

$$\langle (u_1,v_1,w_1),(u_2,v_2,w_2)\rangle_E = \int_\Omega (\nabla u_1 \nabla u_2 + \nabla v_1 \nabla v_2 + \nabla w_1 \nabla w_2)dx,$$

the related norm $\|(u_1,v_1,w_1)\|_E$ and the bounded symmetric quadratic form

$$\begin{aligned} B((u_1,v_1,w_1),(u_2,v_2,w_2)) &= \int_\Omega (\nabla u_1 \nabla v_2 + \nabla v_1 \nabla u_2 + \nabla w_1 \nabla w_2)dx \\ &\quad -a\int_\Omega (u_1v_2 + v_1u_2)dx \\ &\quad -b\int_\Omega (u_1u_2 + v_1v_2)dx - c\int_\Omega w_1w_2dx. \end{aligned}$$

Let (t,r,s) be as in Theorem 1.1 and $(u_{neg},v_{neg},w_{neg})$ be the corresponding negative solution for (1), then we define the functional $F : E \to \mathbb{R}$ for $\mathbf{u} = (u,v,w) \in E$ by

$$F(\mathbf{u}) = \frac{1}{2}B(\mathbf{u},\mathbf{u}) - H(\mathbf{u}),$$

where

$$\begin{aligned} H(\mathbf{u}) &= \int_\Omega \frac{[(u+u_{neq})^+]^{p_1+1}}{p_1+1}dx \\ &\quad + \int_\Omega \frac{[(v+v_{neq})^+]^{p_2+1}}{p_2+1}dx + \int_\Omega \frac{[(w+w_{neq})^+]^{p_3+1}}{p_3+1}dx. \end{aligned}$$

Then it is simple to see that the functional F is $C^1(E;\mathbb{R})$ and its critical point (u,v,w) are such that $(u+u_{neq}, v+v_{neq}, w+w_{neq})$ are solutions of (1); in particular, the origin is a critical point at level zero and corresponds to the already found negative solution.

In order to find an orthogonal base for E which diagonalizes B, we consider, in a way similar to what was done in [1], the eigenvalue problem

$$B((u,v,w),(\phi,\varphi,\psi)) = \mu\langle (u,v,w),(\phi,\varphi,\psi)\rangle_E, \qquad \forall(\phi,\varphi,\psi) \in E.$$

Let u_i, v_i, and w_i be the Fourier's coefficients for u, v, and w. Then the above eigenvalue problem is summarized as

$$\begin{bmatrix} \mu\lambda_i + b & a-\lambda_i & 0 \\ a-\lambda_i & \mu\lambda_i + b & 0 \\ 0 & 0 & c-\lambda_i+\mu\lambda_i \end{bmatrix} \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} (i \in \mathbb{N}), \tag{5}$$

using $(\phi_i, 0, 0)$, $(0, \phi_i, 0)$, $(0, 0, \phi_i)$ as test function.

When the determinant of the above coefficient matrix is zero, we get nontrivial solutions. This is

$$(c - \lambda_i + \mu\lambda_i)(\mu\lambda_i + b)^2 - (c - \lambda_i + \mu\lambda_i)(a - \lambda_i)^2 = 0 \quad (i \in \mathbb{N})$$

and so

$$\mu_{i,1} = \frac{a - b - \lambda_i}{\lambda_i}, \quad \mu_{i,2} = \frac{-a - b + \lambda_i}{\lambda_i}, \quad \mu_{i,3} = \frac{-c + \lambda_i}{\lambda_i} \qquad (i \in \mathbb{N}).$$

From (5) we also get the related eigenvectors

$$\Phi_{i,1} = (\phi_i, -\phi_i, 0), \quad \Phi_{i,2} = (\phi_i, \phi_i, 0), \quad \Phi_{i,3} = (0, 0, \phi_i) \qquad (i \in \mathbb{N}).$$

Because that $\Phi_{i,j}$, $j = 1, 2, 3$ are orthogonal, we normalize to obtain $\Psi_{i,j}$, $j = 1, 2, 3$, that is , $\|\Psi_{i,j}\|_E = 1$:

$$\Psi_{i,1} = \frac{(\phi_i, -\phi_i, 0)}{\sqrt{2\lambda_i}}, \Psi_{i,2} = \frac{(\phi_i, \phi_i, 0)}{\sqrt{2\lambda_i}}, \Psi_{i,3} = \frac{(0, 0, \phi_i)}{\sqrt{\lambda_i}} \quad (i \in \mathbb{N}).$$

With this structure we have

$$\langle \Psi_{i,j}, \Psi_{k,l} \rangle_E = \begin{cases} 1 & i = k \text{ and } j = l \\ 0 & i \neq k \text{ or } j \neq l \end{cases},$$

$$B(\Psi_{i,j}, \Psi_{k,l}) = \begin{cases} \mu_{i,j} & i = k \text{ and } j = l \\ 0 & i \neq k \text{ or } j \neq l \end{cases},$$

so if we write $(u, v, w) = \sum_{i \in \mathbb{N}, j=1,2,3} C_{i,j} \Psi_{i,j}$, we get

$$\|(u, v, w)\|_E^2 = \sum_{i \in \mathbb{N}, j=1,2,3} C_{i,j}^2,$$

$$B((u, v, w), (u, v, w)) = \sum_{i \in \mathbb{N}, j=1,2,3} \mu_{i,j} C_{i,j}^2.$$

In view of this structure we may define

$$E^+ = \overline{\operatorname{span}\{\Psi_{i,j} : \mu_{i,j} > 0, i \in \mathbb{N}, j = 1, 2, 3\}},$$
$$E^- = \overline{\operatorname{span}\{\Psi_{i,j} : \mu_{i,j} < 0, i \in \mathbb{N}, j = 1, 2, 3\}},$$
$$E^0 = \operatorname{span}\{\Psi_{i,j} : \mu_{i,j} = 0, i \in \mathbb{N}, j = 1, 2, 3\},$$

and we have

Lemma 3.1. *There exists $\xi^* > 0$ such that*

(6) $$B(\mathbf{u}, \mathbf{u}) \geq 2\xi^* \|\mathbf{u}\|_E^2 \quad \textit{for} \quad \mathbf{u} \in E^+$$

(7) $$B(\mathbf{u}, \mathbf{u}) \leq -2\xi^* \|\mathbf{u}\|_E^2 \quad \textit{for} \quad \mathbf{u} \in E^-.$$

Moreover, if $a-b \notin \sigma(-\Delta)$*,* $a+b \notin \sigma(-\Delta)$ *and* $c \notin \sigma(-\Delta)$*, then* $E^0 = \{0\}$.

Proof. The claim is satisfied by setting

$$2\xi^* := \inf\{|\mu_{i,j}| : |\mu_{i,j}| > 0, i \in \mathbb{N}, j = 1, 2, 3\}$$

Since

$$\lim_{i\to\infty} \mu_{i,1} = -1, \qquad \lim_{i\to\infty} \mu_{i,2} = \lim_{i\to\infty} \mu_{i,3} = 1,$$

$2\xi^*$ is strictly positive.

The condition $a-b \notin \sigma(-\Delta)$, $a+b \notin \sigma(-\Delta)$ and $c \notin \sigma(-\Delta)$ implies $\mu_{i,j} \neq 0$ for any $i \in \mathbb{N}$, $j = 1, 2, 3$. □

For later use, we also define $\tilde{n}$ such that for $i \geq \tilde{n}$ we have $a - \lambda_i < b < -a + \lambda_i$, $c < \lambda_i$ and

$$E_h = \overline{\operatorname{span}\{\Psi_{i,j} : i \geq \tilde{n}, i \in \mathbb{N}, j = 1, 2, 3\}},$$
$$E_l = \operatorname{span}\{\Psi_{i,j} : i \leq \tilde{n}, i \in \mathbb{N}, j = 1, 2, 3\} :$$

we have the following

Lemma 3.2. $(u, v, w) \in E^+ \cap E_h$ *implies* $u = v$ *and* $(u, v, w) \in E^- \cap E_h$ *implies* $u + v = 0$*,* $w = 0$.

Proof. It follows readily from the fact that for $i \geq \tilde{n}$ we have $\mu_{i,1} < 0$, $\mu_{i,2} > 0$, $\mu_{i,3} > 0$ and that $\Psi_{i,1} = \frac{(\phi_i, -\phi_i, 0)}{\sqrt{2\lambda_i}}$, $\Psi_{i,2} = \frac{(\phi_i, \phi_i, 0)}{\sqrt{2\lambda_i}}$, $\Psi_{i,3} = \frac{(0,0,\phi_i)}{\sqrt{\lambda_i}}$. □

3.2. Estimates for the linking structure. In this section we will prove the estimates we need in order to apply the minimax theorem.

Lemma 3.3. *There exists* $\rho > 0$ *such that*

$$\text{if} \quad \|\mathbf{u}\|_E \leq \rho \quad \text{then} \quad F(\mathbf{u}) \geq 0 \quad \text{for} \quad \mathbf{u} \in E^+.$$

If $\|\mathbf{u}\|_E = \rho$*, then* $F(\mathbf{u}) > 0$.

Proof. Let $\mathbf{u}$ be as above. By the continuous embedding of H_0^1 in L^{p_1+1}, L^{p_2+1}, and L^{p_3+1} we get

$$\begin{aligned}\int_\Omega \frac{[(u+u_{neg})^+]^{p_1+1}}{p_1+1}dx &\le \int_\Omega \frac{|u|^{p_1+1}}{p_1+1}dx \le C_1\|u\|_{H_0^1}^{p_1+1},\\ \int_\Omega \frac{[(v+v_{neg})^+]^{p_2+1}}{p_2+1}dx &\le \int_\Omega \frac{|v|^{p_2+1}}{p_2+1}dx \le C_2\|v\|_{H_0^1}^{p_2+1},\\ \int_\Omega \frac{[(w+w_{neg})^+]^{p_3+1}}{p_3+1}dx &\le \int_\Omega \frac{|w|^{p_3+1}}{p_3+1}dx \le C_3\|w\|_{H_0^1}^{p_3+1},\end{aligned}$$

where C_1, C_2 and C_3 are positive constants. By (6) in Lemma 3.1,

$$\frac{1}{2}B(\mathbf{u},\mathbf{u}) \ge \xi^*\|\mathbf{u}\|_E^2 = \xi^*\left(\|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2 + \|w\|_{H_0^1}^2\right).$$

We get

$$\begin{aligned}F(\mathbf{u}) &\ge \xi^*\left(\|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2 + \|w\|_{H_0^1}^2\right)\\ &\quad -C\left(\|u\|_{H_0^1}^{p_1+1} + \|v\|_{H_0^1}^{p_2+1} + \|w\|_{H_0^1}^{p_3+1}\right)\\ &\ge \|u\|_{H_0^1}^2\left(\xi^* - C\rho^{p_1-1}\right) + \|v\|_{H_0^1}^2\left(\xi^* - C\rho^{p_2-1}\right)\\ &\quad +\|w\|_{H_0^1}^2\left(\xi^* - C\rho^{p_3-1}\right)\end{aligned}$$

where $C = \max\{C_1, C_2, C_3\}$ is a positive number. Since $p_1, p_2, p_3 > 1$, for $\rho > 0$ small enough we obtain $\xi^* - C\rho^{p_j-1} > 0$ $j = 1, 2, 3$. Let $C^* = \min\{\xi^* - C\rho^{p_j-1} : j = 1, 2, 3\} > 0$, then

$$F(\mathbf{u}) \ge C^*\|\mathbf{u}\|_E^2 \ge 0.$$

If $\|\mathbf{u}\|_E = \rho$, then

$$F(\mathbf{u}) \ge C^*\|\mathbf{u}\|_E^2 = C^*\rho^2 > 0.$$

□

LEMMA 3.4. *There exists* $\mathbf{g} = ((g_1, g_1, g_2) \in E^+ \cap E_h$ *with* $\|\mathbf{g}\|_E = 1$ *and* $\|(g_j)^+\|_{L^\infty} = +\infty$, *for* $j = 1, 2$.

Proof. Since H_0^1 is not embedded in L^∞ (here is where we need the condition $N \ge 2$), there exists $u_i \in H_0^1$ such that $\|(u_j)^+\|_{L^\infty} = +\infty$, for $j = 1, 2$; by removing the components of u in the directions of the eigenvectors ϕ_i with $i < \tilde{n}$ we maintain this property since we simply subtract a finite linear combination of regular functions, so we may assume that such components are zero.

Since $\mu_{i,2} > 0$, $\mu_{i,3} > 0$ and $\Psi_{i,2} = \frac{(\phi_i,\phi_i,0)}{\sqrt{2\lambda_i}}$, $\Psi_{i,3} = \frac{(0,0,\phi_i)}{\sqrt{\lambda_i}}$, for $i \geq \tilde{n}$, we have that $(u_1, u_1, u_2) \in E^+ \cap E_h$.

Finally, we obtain $\|(g_1, g_1, g_2)\|_E = 1$ by a suitable rescaling of (u_1, u_1, u_2). □

LEMMA 3.5. *Let* $\mathbf{g} = (g_1, g_1, g_2)$ *as in the lemma above. Then there exist* $R, \theta > 0$ *with* $R\theta > \rho$ *such that* $F(\mathbf{u}) \leq 0$ *for*

(a) $\mathbf{u} \in E^-$,
(b) $\mathbf{u} = \mathbf{w} + \tau\mathbf{g}$; $\quad \mathbf{w} \in E^-$, $\quad \|\mathbf{w}\|_E = R$, $\quad 0 \leq \tau \leq \theta R$,
(c) $\mathbf{u} = \mathbf{w} + \tau\mathbf{g}$; $\quad \mathbf{w} \in E^-$, $\quad \|\mathbf{w}\|_E \leq R$, $\quad \tau = \theta R$.

Proof. (a) Let $\mathbf{u} \in E^-$. By (7) in Lemma 3.1,

$$F(\mathbf{u}) \leq \frac{1}{2}B(\mathbf{u},\mathbf{u}) \leq -\xi^*\|\mathbf{u}\|_E^2 \leq 0.$$

(b) Let $\mathbf{w} \in E^-$ with $\|\mathbf{w}\|_E = R$ and $0 \leq \tau \leq \theta R$. Observe that $\mathbf{g}$ is orthogonal to $\mathbf{w}$, that is, $\langle \mathbf{w}, \mathbf{g}\rangle_E = 0 = B(\mathbf{w}, \mathbf{g})$; then we estimate, by using (7) in Lemma 3.1,

$$\begin{aligned} F(\mathbf{u}) &\leq \frac{1}{2}B(\mathbf{u},\mathbf{u}) = \frac{1}{2}B(\mathbf{w}+\tau\mathbf{g},\mathbf{w}+\tau\mathbf{g}) = \frac{1}{2}B(\mathbf{w},\mathbf{w}) + \frac{1}{2}\tau^2 B(\mathbf{g},\mathbf{g}) \\ &\leq -\xi^*\|\mathbf{w}\|_E^2 + \frac{1}{2}\tau^2 B(\mathbf{g},\mathbf{g}) = R^2\left(-\xi^* + \frac{1}{2}\left(\frac{\tau}{R}\right)^2 B(\mathbf{g},\mathbf{g})\right) \\ &\leq R^2\left(-\xi^* + \frac{1}{2}\theta^2 B(\mathbf{g},\mathbf{g})\right) \end{aligned}$$

Since $\|\mathbf{g}\|_E = 1$, $B(\mathbf{g},\mathbf{g}) \geq 2\xi^* > 0$(by (6) in Lemma 3.1) and then $0 < \frac{2\xi^*}{B(\mathbf{g},\mathbf{g})}$. By fixing $0 < \theta < \sqrt{\frac{2\xi^*}{B(\mathbf{g},\mathbf{g})}}$, such that last term is negative, the claim (b) is proved.

(c) Consider now $\|\mathbf{w}\|_E \leq R$, $\tau = \theta R$, and let

$$P_l\mathbf{w} = (\sigma_1, \sigma_2, \sigma_3)\,, \qquad P_h\mathbf{w} = (\delta_1, \delta_2, \delta_3)$$

where P_l and P_h are the orthogonal projections onto E_l and E_h, respectively. In this way, $P_h\mathbf{w} \in E^- \cap E_h$ and then it is of the form $P_h\mathbf{w} = (\delta_1, -\delta_1, 0)$, by Lemma 3.2.

Write now

$$\begin{aligned}\int_\Omega \left[(u+u_{neg})^+\right]^{p_1+1} dx &= \int_\Omega \left[(\sigma_1+\delta_1+\theta R g_1+u_{neg})^+\right]^{p_1+1} dx \\ &= R^{p_1+1}\int_\Omega \left[\left(\frac{\sigma_1+\delta_1+u_{neg}}{R}+\theta g_1\right)^+\right]^{p_1+1} dx \qquad (8)\end{aligned}$$

$$\begin{aligned}\int_\Omega \left[(v+v_{neg})^+\right]^{p_2+1} dx &= \int_\Omega \left[(\sigma_2-\delta_1+\theta R g_1+v_{neg})^+\right]^{p_2+1} dx \\ &= R^{p_2+1}\int_\Omega \left[\left(\frac{\sigma_2-\delta_1+v_{neg}}{R}+\theta g_1\right)^+\right]^{p_2+1} dx \qquad (9)\end{aligned}$$

$$\begin{aligned}\int_\Omega \left[(w+w_{neg})^+\right]^{p_3+1} dx &= \int_\Omega \left[(\sigma_3+\theta R g_2+w_{neg})^+\right]^{p_3+1} dx \\ &= R^{p_3+1}\int_\Omega \left[\left(\frac{\sigma_3+w_{neg}}{R}+\theta g_2\right)^+\right]^{p_3+1} dx \qquad (10)\end{aligned}$$

Since u_{neq}, v_{neq}, and w_{neq} are fixed and bounded, and $\sigma_1, \sigma_2, \sigma_3$ are linear combinations of a finite number of eigenvectors of the Laplacian(because of $P_l\mathbf{w} \in E_l$), there exists a constant C such that

$$|u_{neq}|, |v_{neq}|, |w_{neq}| < \frac{C}{2} \quad \text{and} \quad |\sigma_1|, |\sigma_2|, |\sigma_3| < \frac{C}{2},$$

so, for $R > 1$,

$$\frac{|\sigma_1+u_{neq}|}{R}, \frac{|\sigma_2+v_{neq}|}{R}, \frac{|\sigma_3+w_{neq}|}{R} < C.$$

Moreover, since $\mathbf{g}$ and θ have already been fixed and $\|(g_j)^+\|_{L^\infty} = \infty$, for $j = 1, 2$, we know that

$$\Omega^* = \{x \in \Omega : \theta g_1(x) > C+1 \quad \text{and} \quad \theta g_2(x) > C+1\}$$

has positive measure; we observe that $\|(g_1)^+\|_{L^\infty} = \infty$ implies

$$\max\{\theta g_1 + \delta_1/R\} > C+1 \quad \text{and} \quad \max\{\theta g_1 - \delta_1/R\} > C+1$$

for any bounded function δ_1 and any $R > 1$: then $\Omega^* \subset \Omega_1^* \cup \Omega_2^* \cup \Omega_3^*$, where

$$\begin{aligned}\Omega_1^* &= \{x \in \Omega : \theta g_1 + \delta_1/R > C + 1\},\\ \Omega_2^* &= \{x \in \Omega : \theta g_1 - \delta_1/R > C + 1\},\\ \Omega_3^* &= \{x \in \Omega : \theta g_2 > C + 1\}\end{aligned}$$

(observe that both Ω_i^* $(i = 1, 2)$ depend on $\mathbf{w}$ and R, but Ω^* does not).

Then $|\Omega_1^*| \geq |\Omega^*|/3$ or $|\Omega_2^*| \geq |\Omega^*|/3$ or $|\Omega_3^*| \geq |\Omega^*|/3$ and, as a consequence, for any $\mathbf{w}$ as assumed and $R > 1$, one of the following three cases hold:

(i) Let $|\Omega_1^*| \geq |\Omega^*|/3$.
For any $x \in \Omega_1^*$,

$$\frac{\sigma_1 + \delta_1 + u_{neq}}{R} + \theta g_1 > 1,$$

since $\theta g_1 + \delta_1/R > C + 1$ and $-C < \sigma_1/R + u_{neq}/R < C$.
We conclude from (8) that

$$\begin{aligned}H(\mathbf{u}) &\geq \int_\Omega \frac{[(u + u_{neg})^+]^{p_1+1}}{p_1 + 1} dx\\ &= \frac{R^{p_1+1}}{p_1 + 1} \int_\Omega \left[\left(\frac{\sigma_1 + \delta_1 + u_{neg}}{R} + \theta g_1\right)^+\right]^{p_1+1} dx\\ &\geq \frac{R^{p_1+1}}{p_1 + 1} \int_{\Omega_1^*} \left[\left(\frac{\sigma_1 + \delta_1 + u_{neg}}{R} + \theta g_1\right)^+\right]^{p_1+1} dx\\ &\geq \frac{R^{p_1+1}}{p_1 + 1}|\Omega_1^*| \geq \frac{|\Omega^*|R^{p_1+1}}{3(p_1 + 1)}\end{aligned}$$

(ii) Let $|\Omega_2^*| \geq |\Omega^*|/3$.
For any $x \in \Omega_2^*$,

$$\frac{\sigma_2 - \delta_1 + v_{neq}}{R} + \theta g_1 > 1,$$

since $\theta g_1 - \delta_1/R > C + 1$ and $-C < \sigma_2/R + v_{neq}/R < C$.

We conclude from (9) that

$$\begin{aligned}
H(\mathbf{u}) &\geq \int_\Omega \frac{[(v+v_{neg})^+]^{p_2+1}}{p_2+1}dx \\
&= \frac{R^{p_2+1}}{p_2+1}\int_\Omega \left[\left(\frac{\sigma_2-\delta_1+v_{neg}}{R}+\theta g_1\right)^+\right]^{p_2+1} dx \\
&\geq \frac{R^{p_2+1}}{p_2+1}\int_{\Omega_2^*} \left[\left(\frac{\sigma_2-\delta_1+v_{neg}}{R}+\theta g_1\right)^+\right]^{p_2+1} dx \\
&\geq \frac{R^{p_2+1}}{p_2+1}|\Omega_2^*| \geq \frac{|\Omega^*|R^{p_2+1}}{3(p_2+1)}
\end{aligned}$$

(iii) Let $|\Omega_3^*| \geq |\Omega^*|/3$.
For any $x \in \Omega_3^*$,

$$\frac{\sigma_3+w_{neq}}{R}+\theta g_2 > 1,$$

since $\theta g_2 > C+1$ and $-C < \sigma_3/R + w_{neq}/R < C$.
We conclude from (10) that

$$\begin{aligned}
H(\mathbf{u}) &\geq \int_\Omega \frac{[(w+w_{neg})^+]^{p_3+1}}{p_3+1}dx \\
&= \frac{R^{p_3+1}}{p_3+1}\int_\Omega \left[\left(\frac{\sigma_3+w_{neg}}{R}+\theta g_2\right)^+\right]^{p_3+1} dx \\
&\geq \frac{R^{p_3+1}}{p_3+1}\int_{\Omega_3^*} \left[\left(\frac{\sigma_3+w_{neg}}{R}+\theta g_2\right)^+\right]^{p_3+1} dx \\
&\geq \frac{R^{p_3+1}}{p_3+1}|\Omega_3^*| \geq \frac{|\Omega^*|R^{p_3+1}}{3(p_3+1)}
\end{aligned}$$

Let $\tilde{C} = \min\{\frac{|\Omega^*|}{3(p_i+1)} : i = 1,2,3\}$ then $\tilde{C} > 0$ does not depend on R and $\mathbf{w}$. And we conclude that $H(\mathbf{u}) \geq \tilde{C}R^{\min\{p_1,p_2,p_3\}+1}$.

Finally, by estimating the first terms as in point (b), we get

$$\begin{aligned} F(\mathbf{u}) &= \frac{1}{2}B(\mathbf{w}+\theta R\mathbf{g}, \mathbf{w}+\theta R\mathbf{g}) - H(\mathbf{u}) \\ &\le \frac{1}{2}B(\mathbf{w}+\theta R\mathbf{g}, \mathbf{w}+\theta R\mathbf{g}) - \tilde{C}R^{\min\{p_1,p_2,p_3\}+1} \\ &\le -\xi^*\|\mathbf{w}\|_E^2 + \frac{1}{2}\theta^2R^2B(\mathbf{g},\mathbf{g}) - \tilde{C}R^{\min\{p_1,p_2,p_3\}+1} \\ &\le R^2\left(\frac{1}{2}\theta^2B(\mathbf{g},\mathbf{g}) - \tilde{C}R^{\min\{p_1,p_2,p_3\}-1}\right): \end{aligned}$$

since $p_1, p_2, p_3 > 1$, we may choose $R > 1$(and also $R > \rho/\theta$) large enough to make the last expression negative; this concludes the proof of the claim (c). □

For $\mathbf{g} = ((g_1, g_1, g_2)$, θ and R in the lemma above, we set

$$\begin{aligned} S &= \{\mathbf{u} : \mathbf{u} \in E^+, \|\mathbf{u}\|_E \le \rho\}, \\ Q &= \{\mathbf{u} = \mathbf{w} + \tau\mathbf{g} : \mathbf{w} \in E^-, \|\mathbf{w}\|_E \le R, 0 \le \tau \le \theta R\}. \end{aligned}$$

Lemma 3.6. *We have*

$$\sup_Q F < +\infty.$$

Proof. By estimating the first terms as in point (b) in Lemma 3.5,

$$\begin{aligned} F(\mathbf{u}) &\le \frac{1}{2}B(\mathbf{u},\mathbf{u}) = \frac{1}{2}B(\mathbf{w}+\tau\mathbf{g}, \mathbf{w}+\tau\mathbf{g}) \\ &= \frac{1}{2}B(\mathbf{w},\mathbf{w}) + \frac{1}{2}\tau^2B(\mathbf{g},\mathbf{g}) \\ &\le -\xi^*\|\mathbf{w}\|_E^2 + \frac{1}{2}\tau^2B(\mathbf{g},\mathbf{g}) \\ &\le \frac{1}{2}\tau^2B(\mathbf{g},\mathbf{g}) \le \frac{1}{2}\theta^2R^2B(\mathbf{g},\mathbf{g}) < +\infty. \end{aligned}$$

□

3.3. The PS conditions. In this section we will prove that the PS condition holds, which was required for the application of the minimax theorem.

Lemma 3.7. *(PS condition). Under the considered hypotheses, the functional F satisfies the PS condition, that is, let ϵ_n be a sequence of* *positive reals converging to zero and* $\{\mathbf{u}_n\}_{n\in\mathbb{N}} \subseteq E$ *be such that*

$$|F(\mathbf{u}_n)| \le T, \tag{11}$$

$$|F'(\mathbf{u}_n)[\phi,\varphi,\psi]| \le \epsilon_n \|(\phi,\varphi,\psi)\|_E \qquad \forall(\phi,\varphi,\psi)\in E, \tag{12}$$

then $\{\mathbf{u}_n\}$ *admits a convergent subsequence.*

Proof. First, we want to prove that $\|\mathbf{u}_n\|_E$ is bounded: so we consider for the sake of contradiction a subsequence such that $\|\mathbf{u}_n\|_E \to \infty$ and we define

$$(U_n, V_n, W_n) = \frac{1}{\|\mathbf{u}_n\|_E}(u_n, v_n, w_n),$$

so that (up to a further subsequence) $(U_n, V_n, W_n) \to (U, V, W)$ weakly in E.

Applying the definition of the functional F,

$$\begin{aligned} F(\mathbf{u}_n) &= \frac{1}{2}B(\mathbf{u}_n,\mathbf{u}_n) - \int_\Omega \frac{[(u_n+u_{neq})^+]^{p_1+1}}{p_1+1}dx \\ &\quad - \int_\Omega \frac{[(v_n+v_{neq})^+]^{p_2+1}}{p_2+1}dx - \int_\Omega \frac{[(w_n+w_{neq})^+]^{p_3+1}}{p_3+1}dx \end{aligned}$$

and

$$\begin{aligned} F'(\mathbf{u}_n)\mathbf{u}_n &= B(\mathbf{u}_n,\mathbf{u}_n) - \int_\Omega [(u_n+u_{neq})^+]^{p_1} u_n dx \\ &\quad - \int_\Omega [(v_n+v_{neq})^+]^{p_2} v_n dx - \int_\Omega [(w_n+w_{neq})^+]^{p_3} w_n dx. \end{aligned}$$

Now observe that

$$\begin{aligned} \int_\Omega [(u_n+u_{neq})^+]^{p_1} u_n dx &= \int_\Omega [(u_n+u_{neq})^+]^{p_1+1} dx \\ &\quad + \int_\Omega [(u_n+u_{neq})^+]^{p_1} (-u_{neq}) dx \end{aligned}$$

(and an analogous relation holds for the term in v_n and w_n); then,

$$\begin{aligned} F'(\mathbf{u}_n)\mathbf{u}_n &= B(\mathbf{u}_n,\mathbf{u}_n) \\ &\quad - \int_\Omega [(u_n+u_{neq})^+]^{p_1+1} dx - \int_\Omega [(u_n+u_{neq})^+]^{p_1}(-u_{neq}) dx \\ &\quad - \int_\Omega [(v_n+v_{neq})^+]^{p_2+1} dx - \int_\Omega [(v_n+v_{neq})^+]^{p_2}(-v_{neq}) dx \\ &\quad - \int_\Omega [(w_n+w_{neq})^+]^{p_3+1} dx - \int_\Omega [(w_n+w_{neq})^+]^{p_3}(-w_{neq}) dx \end{aligned}$$

and by considering $F(\mathbf{u}_n) - \frac{1}{2}F(\mathbf{u}_n)\mathbf{u}_n$, we get

$$\begin{aligned}&\kappa_1 \int_\Omega [(u_n + u_{neq})^+]^{p_1+1} dx + \frac{1}{2}\int_\Omega [(u_n + u_{neq})^+]^{p_1}(-u_{neq}) dx \\ &\quad + \kappa_2 \int_\Omega [(v_n + v_{neq})^+]^{p_2+1} dx + \frac{1}{2}\int_\Omega [(v_n + v_{neq})^+]^{p_2}(-v_{neq}) dx \\ &\quad + \kappa_3 \int_\Omega [(w_n + w_{neq})^+]^{p_3+1} dx + +\frac{1}{2}\int_\Omega [(w_n + w_{neq})^+]^{p_3}(-w_{neq}) dx \\ &\leq T + \frac{1}{2}\epsilon_n \|\mathbf{u}_n\|_E;\end{aligned}$$

(where $\kappa_i = \frac{1}{2} - \frac{1}{p_i+1}$) by observing that each term in the expression above is nonnegative, we conclude that the estimate from above holds for each of them, and then

$$\frac{1}{\|\mathbf{u}_n\|_E}\int_\Omega [(u_n + u_{neq})^+]^{p_1+1} dx \to 0, \tag{13}$$

$$\frac{1}{\|\mathbf{u}_n\|_E}\int_\Omega [(v_n + v_{neq})^+]^{p_2+1} dx \to 0, \tag{14}$$

$$\frac{1}{\|\mathbf{u}_n\|_E}\int_\Omega [(w_n + w_{neq})^+]^{p_3+1} dx \to 0. \tag{15}$$

For any $(\phi, \varphi, \psi) \in E$ with $\|(\phi, \varphi, \psi)\|_E = 1$, we get

$$\begin{aligned}&\frac{1}{\|\mathbf{u}_n\|_E} F'(\mathbf{u}_n)[\phi, \varphi, \psi] \\ &\quad = B((U_n, V_n, W_n), (\phi, \varphi, \psi)) - \int_\Omega \frac{[(u_n + u_{neq})^+]^{p_1}}{\|\mathbf{u}_n\|_E}\phi dx \\ &\qquad - \int_\Omega \frac{[(v_n + v_{neq})^+]^{p_2}}{\|\mathbf{u}_n\|_E}\varphi dx - \int_\Omega \frac{[(w_n + w_{neq})^+]^{p_3}}{\|\mathbf{u}_n\|_E}\psi dx\end{aligned}$$

From (12) we get

$$\frac{1}{\|\mathbf{u}_n\|_E} F'(\mathbf{u}_n)[\phi, \varphi, \psi] \to 0$$

which, by using the weak convergence of (U_n, V_n, W_n) and (13), (14), (15), implies that

$$B((U, V, W), (\phi, \varphi, \psi)) = 0.$$

This means that (U, V, W) is a solution of

$$-\Delta(U, V, W)^T = A(U, V, W)^T,$$

where $A = \begin{bmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix}$. Since for the matrix B eigenvalue-eigenvector pairs are

$$\nu_{i,1} = a+b, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \quad \nu_{i,2} = a-b, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}; \quad \nu_{i,3} = c, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(U, V, W) is the unique solution and then it is zero if real eigenvalues of A are not in $\sigma(-\Delta)$.

This gives rise to a contradiction since by definition we have

$$\|(U, V, W)\|_E = 1.$$

We conclude that $\|\mathbf{u}_n\|_E$ is bounded.

It is now simple to see that $\mathbf{u}_n$ admits a convergent subsequence. In fact, up to a subsequence, $(u_n, v_n, w_n) \to (u, v, w)$ weakly in E, then we calculate the inner product of $(u_n, v_n, w_n) - (u, v, w)$ and $\Psi_{i,j}$ to obtain that the convergence is in fact strong. □

3.4. The second solution through the minimax theorem. Now, we will prove the main theorem.

PROPOSITION 3.1. *There exists a critical point $\mathbf{u} \in E$ for the functional F with $F(\mathbf{u}) > 0$(and then $\mathbf{u} \neq (0, 0, 0)$, so that it is a second solution).*

Proof. By using the estimates in Lemma 3.3, Lemma 3.5, Lemma 3.6 and the PS condition in Lamma 3.7., there exists $0 < \rho < R$ such that

$$\sup_{\partial Q} F \leq 0 < \inf_{\partial S} F,$$

and

$$\sup_{Q} F < +\infty, \qquad \inf_{S} F \geq 0 > -\infty.$$

By the two critical point theorem in [2], F has at least two critical values c_1 and c_2

$$\inf_{S} F \leq c_1 \leq \sup_{\partial Q} F < \inf_{\partial S} F \leq c_2 \leq \sup_{Q} F.$$

Since $\inf_S F \geq 0$ and $\sup_{\partial Q} F \leq 0$, $\inf_S F = c_1 = \sup_{\partial Q} F = 0$. and $c_2 > 0$. □

Finally, we may conclude the proof of Theorem 1.2.

Proof of Theorem 1.2. Theorem 1.1 and Proposition 3.1 imply Theorem 1.2.

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