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PROPERTIES OF INDUCED INVERSE POLYNOMIAL MODULES OVER A SUBMONOID

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Abstract. Let M be a left R-module and R be a ring with unity, and $S = \{0, 2, 3, 4, \cdots\}$ be a submonoid. Then $M[x^{-s}] = \{a_0 + a_2x^{-2} + a_3x^{-3} + \cdots + a_nx^{-n} \mid a_i \in M\}$ is an $R[x^s]$-module. In this paper we show some properties of $M[x^{-s}]$ as an $R[x^s]$-module.

Let $f : M \longrightarrow N$ be an R-linear map and $\overline{M}[x^{-s}] = \{a_2x^{-2} + a_3x^{-3} + \cdots + a_nx^{-n} \mid a_i \in M\}$ and define $N + \overline{M}[x^{-s}] = \{b_0 + a_2x^{-2} + a_3x^{-3} + \cdots + a_nx^{-n} \mid b_0 \in N, a_i \in M\}$. Then $N + \overline{M}[x^{-s}]$ is an $R[x^s]$-module.

We show that given a short exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ of R-modules, $0 \longrightarrow L \longrightarrow M[x^{-s}] \longrightarrow N+\overline{M}[x^{-s}] \longrightarrow 0$ is a short exact sequence of $R[x^s]$-module. Then we show $E_1 + \overline{E_0}[x^{-s}]$ is not an injective left $R[x^s]$-module, in general.

1. Introduction

Let M be a left R-module and R be a ring with unity and $S = \{0, 2, 3, 4, \cdots\}$ be a submonoid. Then $M[x^s] = \{a_0 + a_2x^2 + a_3x^3 + \cdots + a_nx^n \mid a_i \in M\}$ is an $R[x^s]$-module defined by

$$x^k(a_0 + a_2x^2 + a_3x^3 + \cdots + a_nx^n) = a_0x^k + a_2x^{2+k} + a_3x^{3+k} + \cdots + a_nx^{n+k}$$

for $x^k \in R[x^s]$.

Also $M[x^{-s}] = \{a_0 + a_2x^{-2} + a_3x^{-3} + \cdots + a_nx^{-n} \mid a_i \in M\}$ is an $R[x^s]$-module defined by

$$x^k(a_0+a_2x^{-2}+a_3x^{-3}+\cdots+a_nx^{-n}) = a_k+a_{k+2}x^{-2}+a_{k+3}x^{-3}+\cdots+a_nx^{-n+k}$$

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for $x^k \in R[x^s]$. For example,

$$x^3(a_0+a_2x^{-2}+a_3x^{-3}+\cdots+a_nx^{-n} = a_3+a_5x^{-2}+a_6x^{-3}+\cdots+a_nx^{-n+3}.$$

Induced inverse polynomial module was introduced in ([8]). Let M and N be left R-modules and $f : M \longrightarrow N$ be an R-linear map. Then an induced polynomial $N + x^{-1}M[x^{-1}]$ is a left $R[x]$-module defined by

$$x(b_0 + a_1x^{-1} + \cdots + a_nx^{-n}) = b_1 + a_2x^{-1} + \cdots + a_nx^{-n+1},$$

where $f(a_1) = b_1, b_0 \in N$, and $a_i \in M$. In this paper we generalized induced polynomial modules over a submonoid.

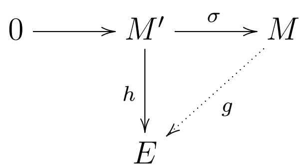
Let $f : M \longrightarrow N$ be an R-linear map and let $\overline{M}[x^{-s}] = \{a_2x^{-2} + a_3x^{-3} + \cdots + a_nx^{-n} \mid a_i \in M\}$ and define $N + \overline{M}[x^{-s}] = \{b_0 + a_2x^{-2} + a_3x^{-3} + \cdots + a_nx^{-n} \mid b_0 \in N, a_i \in M\}$. Then $N + \overline{M}[x^{-s}]$ is an $R[x^s]$-module defined by

$$\begin{aligned} &x^k(b_0 + a_2x^{-2} + a_3x^{-3} + \cdots + a_nx^{-n}) \\ &\quad = f(a_k) + a_{k+2}x^{-2} + a_{k+3}x^{-3} + \cdots + a_nx^{-n+k}. \end{aligned}$$

For example,

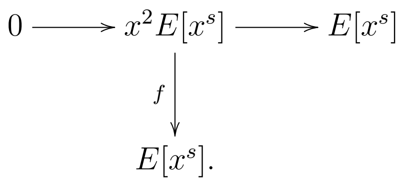
$$x^2(b_0+a_2x^{-2}+a_3x^{-3}+\cdots+a_nx^{-n}) = f(a_2)+a_4x^{-2}+a_5x^{-3}+\cdots+a_nx^{-n+2}.$$

A left R-module E is said to be injective if given any injective linear map $\sigma : M' \longrightarrow M$ and any linear map $h : M' \longrightarrow E$, there is a linear map $g : M \longrightarrow E$ such that $g \circ \sigma = h$. That is,



can always be completed to a commutative diagram([9]).

The map $f : x^2E[x^s] \longrightarrow E[x^s]$ defined by $f(ex^2) = e$ is an $R[x^s]$-linear map. So consider the following diagram



Then we easily see that the above diagram can not be completed for any $R[x^s]$-linear map. Thus $E[x^s]$ is not an injective left $R[x^s]$-module with $E \neq 0$.

Northcott([3]) defined inverse polynomial modules and used inverse polynomial modules to study the properties of injective modules and he studied $K[x^{-1}]$ as $K[x]$-module on field K. And McKerraw([2]) showed that if R is a left noetherian ring and E is an injective left R-module, then $E[x^{-1}]$ is an injective envelope of $M[x^{-1}]$ as $R[x]$-module. Inverse polynomial modules were studied in ([4]),([5]) and recently in ([1]), ([5]), ([6]), ([7]).

2. inverse polynomial modules over a submonoid

Through this paper we let $S = \{0, 2, 3, 4, \cdots\}$ a submonoid.

Definition 2.1. *Let M be a left R-module and R be a ring with unity and $S = \{0, 2, 3, 4, \cdots\}$ be a submonoid. Then $M[x^{-s}] = \{a_0 + a_2x^{-2} + a_3x^{-3} + \cdots + a_nx^{-n} \mid a_i \in M\}$ is an $R[x^s]$-module defined by*

$$r(a_0+a_2x^{-2}+a_3x^{-3}+\cdots+a_nx^{-n}) = ra_0+ra_2x^{-2}+ra_3x^{-3}+\cdots+ra_nx^{-n}$$

and

$$x^k(a_0+a_2x^{-2}+a_3x^{-3}+\cdots+a_nx^{-n}) = a_k+a_{k+2}x^{-2}+a_{k+3}x^{-3}+\cdots+a_nx^{-n+k}$$

where $r \in R$ and $x^k \in R[x^s]$.

Proposition 2.2. *Let $\phi : M[x^{-s}] \longrightarrow M[x^{-s}]$ be an $R[x^s]$-linear map. Then $\phi(M) \subset (M + Mx^{-2}) \cap (M + Mx^{-3}) = M$.*

Proof. Suppose $m \in M$ and $\phi(m) = a_0 + a_2x^{-2} + a_3x^{-3} + \cdots + a_nx^{-n}$. Then for $x^2 \in R[x^s]$, $\phi(x^2m) = \phi(0) = 0$, and $x^2\phi(m) = a_2 + a_4x^{-2} + \cdots + a_nx^{-n+2} = 0$ implies $a_2 = a_4 = \cdots = a_n = 0$. Therefore, $\phi(M) \subset (M + Mx^{-2}) \cap (M + Mx^{-3}) = M$. □

If N is a submodules of M, then we easily see

$$\frac{M[x^{-s}]}{N[x^{-s}]} \cong \frac{M}{N}[x^{-s}]$$

as $R[x^s]$-module.

Proposition 2.3. *Let M be a left R-module, then*

$$\frac{M[x^{-s}]}{M+Mx^{-3}} \cong M[x^{-s}].$$

Proof. Define $\phi : M[x^{-s}] \longrightarrow M[x^{-s}]$ by

$$\begin{aligned} &\phi(a_0 + a_2x^{-2} + a_3x^{-3} + \cdots + a_nx^{-n}) \\ &= \phi(a_0 + a_2x^{-2} + a_3x^{-3} + \cdots + a_nx^{-n}) \\ &= a_2 + a_4x^{-2} + \cdots + a_nx^{-n+2}. \end{aligned}$$

Then we easily see that ϕ is an $R[x^s]$-linear map.

Let $a_0 + a_2x^{-2} + a_3x^{-3} + \cdots + a_nx^{-n} \in M[x^{-s}]$, then

$$\phi(a_0x^{-2}+a_2x^{-4}+a_3x^{-5}+\cdots+a_nx^{-n-2}) = a_0+a_2x^{-2}+a_3x^{-3}+\cdots+a_nx^{-n}.$$

Thus ϕ is surjective.

Let $a_0+a_3x^{-3} \in M[x^{-s}]$, then $\phi(a_0+a_3x^{-3}) = 0$, so that $M+Mx^{-3} \subset ker(\phi)$. Let $a_0 + a_2x^{-2} + a_3x^{-3} + \cdots + a_nx^{-n} \in ker(\phi)$, then

$$\phi(a_0x^{-2}+a_2x^{-4}+a_3x^{-5}+\cdots+a_nx^{-n-2}) = a_2+a_4x^{-2}+\cdots+a_nx^{-n+2} = 0,$$

so that $a_2 = a_4 = a_5 = \cdots = a_n = 0$, so that $ker(\phi) \subset M+Mx^{-3}$. Thus $ker(\phi) = M + Mx^{-3}$. Therefore,

$$\frac{M[x^{-s}]}{M+Mx^{-3}} \cong M[x^{-s}].$$

□

Proposition 2.4. *Let M be a left R-module, then*

$$\sigma : \frac{M[[x^{-s}]]}{M[x^{-s}]} \longrightarrow \frac{M[[x^{-s}]]}{M[x^{-s}]}$$

by $\sigma(f + M[x^{-s}]) = x^2(f + M[x^{-s}])$ is an isomorphism.

Proof. Let $f+M[x^{-s}] \in ker(\sigma)$ and let $f = a_0+a_2x^{-2}+a_3x^{-3}+\cdots$. Then

$$\begin{aligned} \sigma(f + M[x^{-s}]) &= x^2(f + M[x^{-s}]) \\ &= a_2 + a_4x^{-2} + a_5x^{-3} + \cdots + M[x^{-s}] \\ &= M[x^{-s}]. \end{aligned}$$

So $a_kx^{-k+2} + a_{k+1}x^{-k+1} + \cdots = 0$, $a_k = a_{k+1} = \cdots = 0$, for some k. Thus $f + M[x^{-s}] = M[x^{-s}]$. Therefore, σ is injective.

Let $f + M[x^{-1}] = (a_0 + a_2x^{-2} + a_3x^{-3} + \cdots) + M[x^{-s}] \in \frac{M[[x^{-s}]]}{M[x^{-s}]}$. Then there exists

$$g + M[x^{-s}] = (a_0x^{-2} + a_2x^{-4} + a_3x^{-5} + \cdots + a_kx^{-k-2} + \cdots) + M[x^{-s}]$$

such that $\sigma(g+M[x^{-s}]) = f+M[x^{-s}]$. Therefore, σ is surjective. Hence, σ is an isomorphism. □

3. Induced polynomial modules over a submonoid

Definition 3.1. *Let $f : M \longrightarrow N$ be an R-linear map and $S = \{0, 2, 3, 4, \cdots\}$ be a submonoid. Then $N + \overline{M}[x^{-s}]$ is an $R[x^s]$-module, where $\overline{M}[x^{-s}] = \{a_2x^{-2} + a_3x^{-3} + \cdots + a_nx^{-n} \mid a_i \in M\}$ be defined by*

$$r(b_0+a_2x^{-2}+a_3x^{-3}+\cdots+a_nx^{-n}) = rb_0+ra_2x^{-2}+ra_3x^{-3}+\cdots+ra_nx^{-n}$$

and

$$x^k(b_0+a_2x^{-2}+a_3x^{-3}+\cdots+a_nx^{-n}) = f(a_k)+a_{k+2}x^{-2}+a_{k+3}x^{-3}+\cdots+a_nx^{-n+k}$$

for $x^k \in R[x^s], b_0 \in N$. Similarly, we can define

$$N + \overline{M}[x^{-s}] = \{b_0 + a_2x^{-2} + a_3x^{-3} + \cdots + a_nx^{-n} \mid b_0 \in N, a_i \in M\}$$

as a left $R[x^s]$-module.

Theorem 3.2. *If $0 \to L \to M \to N \to 0$ is a short exact sequence of R-modules, then*

$$0 \to L \to M[x^{-s}] \to N + \overline{M}[x^{-s}] \to 0$$

is a short exact sequence of $R[x^s]$-module.

Proof. Let $f[x^{-s}] : L \longrightarrow M$ be defined by $f[x^{-s}](l) = f(l)$ for $l \in L$. Then easily $f[x^{-s}]$ is an injective $R[x^s]$-linear map. Let $g[x^{-s}] : M[x^{-s}] \to N + \overline{M}[x^{-s}]$ be defined by

$$g[x^{-s}](a_0+a_2x^{-2}+a_3x^{-3}+\cdots+a_nx^{-n}) = g(a_0)+a_2x^{-2}+a_3x^{-3}+\cdots+a_nx^{-n}$$

Then easily $g[x^{-s}]$ is an $R[x^s]$-linear map.

Let $b_0 + a_2x^{-2} + a_3x^{-3} + \cdots + a_nx^{-n} \in N + \overline{M}[x^{-s}]$. Then since g is a surjective R-linear map, there exists $a_0 \in M$ such that $g(a_0) = b_0$. Thus

$$g[x^{-s}](a_0+a_2x^{-2}+a_3x^{-3}+\cdots+a_nx^{-n}) = b_0+a_2x^{-2}+a_3x^{-3}+\cdots+a_nx^{-n}.$$

So $g[x^{-s}]$ is a surjective $R[x^s]$-linear map.

Now $(g[x^{-s}] \circ f[x^{-s}])(l) = g[x^{-s}](f(l)) = g(f(l)) = 0$, so that

$$Im(f[x^{-s}]) \subseteq Ker(g[x^{-s}]).$$

And if $a_0 + a_2x^{-2} + a_3x^{-3} + \cdots + a_nx^{-n}) \in Ker(g[x^{-s}])$, then

$$\begin{aligned} &g[x^{-s}](a_0 + a_2x^{-2} + a_3x^{-3} + \cdots + a_nx^{-n}) \\ &= g(a_0) + a_2x^{-2} + a_3x^{-3} + \cdots + a_nx^{-n} \\ &= 0, \end{aligned}$$

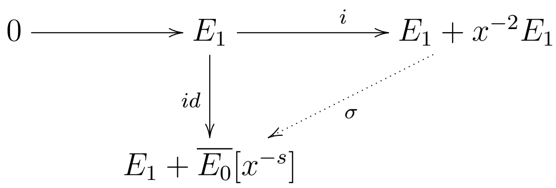
implies $g(a_0) = 0$, and $a_2 = a_3 = \cdots = a_n = 0$. Thus $a_0 \in Ker(g) = Im(f)$, so that $Ker(g[x^{-s}]) \subseteq Im(f[x^{-s}])$. Therefore, $Im(f[x^{-s}]) \subseteq Ker(g[x^{-s}])$. Hence,

$$0 \to L \to M[x^{-s}] \to N + \overline{M}[x^{-s}] \to 0$$

is a short exact sequence of $R[x^s]$-modules. □

THEOREM 3.3. *Let* $0 \longrightarrow N \xrightarrow{f} E_0 \xrightarrow{g} E_1 \longrightarrow 0$ *be a short exact sequence of R-modules, with $injdim_R N = 1$, where E_0, E_1 are injective left R-modules. Then $E_1 + \overline{E_0}[x^{-s}]$ is not an injective left $R[x^s]$-module.*

Proof. Suppose $E_1 + \overline{E_0}[x^{-s}]$ is an injective left $R[x^s]$-module. Then we can complete the following diagram



as commutative diagram by an $R[x^s]$-linear map σ. Then there exists an R-linear map $h : E_1 \longrightarrow E_0$ such that $g \circ h = id_{E_1}$. This contradicts the fact that the short exact sequence $0 \longrightarrow N \xrightarrow{f} E_0 \xrightarrow{g} E_1 \longrightarrow 0$ is not split. Hence, $E_1 + \overline{E_0}[x^{-s}]$ is not an injective left $R[x^s]$-module.

□

DEFINITION 3.4. *Let M be a left R-module and $S = \{0, 2, 3, 4, \cdots\}$ be a submonoid, then $M[x^s, x^{-s}] = \{a_0 + a_2x^2 + a_3x^3 + \cdots + a_ix^i + b_2x^{-2} + b_3x^{-3} + \cdots + b_jx^{-j} \mid a_n, b_m \in M\}$ is a left $R[x^s]$-module be defined by*

$$\begin{aligned} &r(a_0 + a_2x^2 + a_3x^3 + \cdots + a_ix^i + b_2x^{-2} + b_3x^{-3} + \cdots + b_jx^{-j}) \\ &= ra_0 + ra_2x^2 + ra_3x^3 + \cdots + ra_ix^i + rb_2x^{-2} + rb_3x^{-3} + \cdots + rb_jx^{-j} \end{aligned}$$

and

$$x^k(a_0 + a_2x^2 + a_3x^3 + \cdots + a_ix^i + b_2x^{-2} + b_3x^{-3} + \cdots + b_jx^{-j})$$
$$= a_0x^k + a_2x^{2+k} + \cdots + ra_ix^{i+k} + b_2x^{-2+k} + \cdots + b_jx^{-j+k}$$

for $x^k \in R[x^s]$.

Similarly we can define $M[[x^s, x^{-s}]]$ *as an* $R[x^s]$*-module.*

THEOREM 3.5. *For any nonzero left* R*-module* E, $E[x^s, x^{-s}]$ *is not an injective left* $R[x^s]$*-module.*

Proof. Define $f : (1 + x^2) \longrightarrow E[x^s, x^{-s}]$ by $f(1 + x^2) = e$ for $e \in E$ and consider the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & (1+x^2) & \longrightarrow & R[x^s] \\ & & \Big\downarrow f & & \\ & & E[x^s, x^{-s}]. & & \end{array}$$

Then we easily see that the above diagram can not be completed for any $R[x^s]$-linear map. Thus $E[x^s, x^{-s}]$ is not an injective left $R[x^s]$-module. □

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