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NONCONSTANT WARPING FUNCTIONS ON EINSTEIN WARPED PRODUCT MANIFOLDS WITH 2−DIMENSIONAL BASE

Soo-Young Lee

Abstract. In this paper, we study nonconstant warping functions on an Einstein warped product manifold $M = B \times_{f^2} F$ with a warped product metric $g = g_B + f(t)^2 g_F$. And we consider a $2-$dimensional base manifold B with a metric $g_B = dt^2 + (f'(t))^2 du^2$. As a result, we prove the following: if M is an Einstein warped product manifold with a $2-$dimensional base, then there exist generally nonconstant warping functions $f(t)$.

1. Introduction

In [1], A.L. Besse, the author studies an Einstein manifold. And he considers an Einstein warped product manifold $M = B \times_{f^2} F$ with a warped product metric $g = g_B + f(t)^2 g_F$. In addition to, he may also consider an Einstein warped product manifold with a $2-$dimensional base manifold.

In a recent study, we have various results on an Einstein warped product manifold by several authors([4, 6–10]). And we get results on an Einstein warped product manifold with a $2-$dimensional base by several authors([8–10]).

In this paper, we study the following question:

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Main Theorem : If $M = B \times_{f^2} F$ is an Einstein warped product manifold with a $2-$dimensional base, then there exist generally nonconstant warping functions $f(t)$.

Definition 1.1. Let (B, g_B) and (F, g_F) be two manifolds. Let g_B be metric tensors of B and g_F be metric tensors of F. We denote by π and σ the projections of $B \times F$ onto B and F, respectively.

For a positive smooth function f on B the warped product manifold $M = B \times_f F$ is the product manifold $M = B \times F$ furnished with the metric tensor g defined by $g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F)$. We denote by π^* and σ^* the pullback π and σ, respectively. Here B is called the base of M and F the fiber([1–3, 11]).

We denote by Ric_F is the Ricci curvature of (F, g_F) and Ric_B is the Ricci curvature of (B, g_B). We denote by Ric^B and Ric^F the lifts to M of Ricci curvatures of B and F, respectively. Let p be a dimension of F ([1–3, 11]).

Proposition 1.2. (See Proposition 9.106 in [1].) *The Ricci curvature Ric of the warped product manifold $M = B \times_{f^2} F$ satisfies*
(i) $Ric(V, W) = Ric^F(V, W) + g(V, W)[(\frac{\Delta f}{f} - (p-1)\frac{||df||^2}{f^2})\pi]$,
(ii) $Ric(X, V) = 0$,
(iii) $Ric(X, Y) = Ric^B(X, Y) - \frac{p}{f} H^f(X, Y)$
for any vertical vectors V, W and any horizontal vectors X, Y. We are defined by df is the gradient of f for g_B, H^f is the Hessian of f for g_B. We denote by Δf is the Laplacian of f for g_B and p is a dimension of F.

Corollary 1.3. (See Corollary 9.107 in [1].) *The warped product manifold $M = B \times_{f^2} F$ is an Einstein manifold (with $Ric = \lambda g$) if and only if g_F, g_B and f satisfy*
(i) (F, g_F) *is Einstein (with* $Ric_F = \lambda_0 g_F$*),*
(ii) $\frac{\Delta f}{f} - (p-1)\frac{\|df\|^2}{f^2} + \frac{\lambda_0}{f^2} = \lambda$,
(iii) $Ric_B - \frac{p}{f} H^f = \lambda g_B$.

Obviously, (i) gives a condition on (F, g_F) alone, whereas (ii) and (iii) are two differential equations for f on (B, g_B).

REMARK 1.4. (See 9.108 in [1].) Using Corollary 1.3 (ii) and (iii), we replace the unique equation

(1.1)
$$Ric_B - \frac{p}{f}H^f = \frac{1}{2}[\ s_B + 2p\frac{\Delta f}{f} - p(p-1)\frac{\|df\|^2}{f^2} + p\frac{\lambda_0}{f^2} - (p+q-2)\lambda\ ]g_B,$$

where $q = \dim B$.

REMARK 1.5. (See 9.116 in [1].) In a special case of a warped product manifold $M = B\times_{f^2}F$ over a 2−dimensional base. We denote by s_B is a Gaussian curvature of B, $Ric_B = \frac{1}{2}s_B g_B$, and $q = 2$. Hence equation (1.1) simplifies to

$$H^f = -\frac{1}{2}[\ 2\Delta f - (p-1)\frac{||df||^2}{f} + \frac{\lambda_0}{f} - \lambda f\ ]g_B. \tag{1.2}$$

LEMMA 1.6. (See Lemma 9.117 in [1].) *On a 2−dimensional manifold* (B, g_B) *the equation* $H^f = f''g_B$ *admits a nonconstant solution* f *if and only if, locally at points where* $df \neq 0$, *there exist local coordinates* (t, u) *such that* f *is a function of* t *alone and* $g_B = dt^2 + f'(t)^2du^2$.

REMARK 1.7. (See 9.117a in [1].) With the notations of the lemma, equation (1.2) becomes an ordinary differential equation for f in the variable t and $\|df\|^2 = (f')^2$. Then we have an equation

$$2f'' + (p-1)\frac{f'^2}{f} - \frac{\lambda_0}{f} + \lambda f = 0. \tag{1.3}$$

If we denote $f(t) = u(t)^{\frac{2}{p+1}}$, where $u(t)$ is a positive function and $\dim F = p > 1$, then equation (1.3) can be changed into

$$u''(t) = \frac{(p+1)\lambda_0}{4}\ u(t)^{1-\frac{4}{p+1}} - \frac{(p+1)\lambda}{4}\ u(t).$$

By multiplies both side $u'(t)$ and an integration gives, then we obtain

$$(u'(t))^2 = \frac{(p+1)^2\ \lambda_0}{4(p-1)}\ u(t)^{2-\frac{4}{p+1}} - \frac{(p+1)\lambda}{4}\ (u(t))^2. \tag{1.4}$$

We study generally nonconstant warping functions of equation (1.4) on an Einstein warped product manifold $M = B \times_{f^2} F$ with a 2−dimensional base B and the metric $g_B = dt^2 + f'(t)^2 du^2$.

2. Fiber manifold with $\lambda_0 = 0$

Let dim$F = p > 1$. In case that $\lambda_0 = 0$, we consider to the following theorem according to the signs of λ.

THEOREM 2.1. *In case that* $\lambda_0 = 0$. *If* λ *is a constant, then there exist solutions of equation (1.4).*
(i) For $\lambda = 0$, $u(t) = c$, *where* c *is a positive constant.*
(ii) For $\lambda > 0$, *there does not exist a solution of equation (1.4).*
(iii) For $\lambda < 0$, $u(t) = e^{\pm\sqrt{\frac{-(p+1)\lambda}{4}}\ t+c}$, *where* c *is a constant.*

Proof. For $\lambda_0 = 0$, equation (1.4) implies that

$$(u'(t))^2 = -\frac{(p+1)\lambda}{4}(u(t))^2. \tag{2.1}$$

(i) For $\lambda = 0$, equation (2.1) implies that $(u'(t))^2 = 0$ and $u'(t) = 0$. An integration gives $u(t) = c$, where c is a positive constant.
(ii) For $\lambda > 0$, equation (2.1) implies that $(u'(t))^2 < 0$, which is a contradiction. Hence there does not exist a solution of equation (1.4).
(iii) For $\lambda < 0$, then we get $u'(t) = \pm\sqrt{\frac{-(p+1)\lambda}{4}}\ u(t)$, where $u(t)$ is a positive function. Multiplying both sides of equation by $\frac{1}{u(t)}$ and an integration gives

$$\ln|u(t)| = \pm\sqrt{\frac{-(p+1)\lambda}{4}}\ t + c,$$

where c is a constant. Then we have $u(t) = e^{\pm\sqrt{\frac{-(p+1)\lambda}{4}}\ t+c}$, where c is a constant and $u(t)$ is a positive function.

Therefore we have $u(t) = e^{\pm\sqrt{\frac{-(p+1)\lambda}{4}}\ t+c}$, where c is a constant. □

From above Theorem 2.1, the following remark considers that equation (1.3) satisfies generally nonconstant warping function $f(t)$.

REMARK 2.2. For $\lambda_0 = 0$, there exist generally nonconstant warping functions $f(t)$ of equation (1.3) according to the signs of λ :
(i) For $\lambda = 0$, $f(t) = c$, where c is a positive constant. Because c is not nonconstant, thus $f(t) = c$ is not our nonconstant solution.
(ii) For $\lambda > 0$, there does not exist a solution of equation (1.3).
(iii) For $\lambda < 0$, $f(t) = e^{\pm\sqrt{\frac{-\lambda}{p+1}}\ t+\frac{2c}{p+1}}$, where c is a constant.

The following example shows that our results are satisfied with well-known special cases of equation (1.3) besides the constant cases.

EXAMPLE 2.3. Let dimF $= p > 1$. From Remark 2.2, for the well-known special cases λ_0 and λ, we have a nonconstant warping function of equation (1.3). For $\lambda_0 = 0$ and $\lambda = -(p+1)$, then $f(t) = e^{\pm t}$ when $c = 0$.

3. Fiber manifold with $\lambda_0 > 0$

Let dim$F = p > 1$. In case that $\lambda_0 > 0$, we consider to the following theorem according to the signs of λ.

THEOREM 3.1. *In case that $\lambda_0 > 0$. If λ is a constant, then there exist solutions $u(t)$ of equation (1.4):*
(i) For $\lambda = 0$, $u(t) = \left(\pm\sqrt{\frac{\lambda_0}{p-1}}\ t + \frac{2c}{p+1}\right)^{\frac{p+1}{2}}$,
(ii) for $\lambda > 0$, $u(t) = \left(\sqrt{\frac{(p-1)\lambda}{(p+1)\lambda_0}}\sec(\pm\sqrt{\frac{\lambda}{p+1}}\ t - \sqrt{\frac{\lambda}{p+1}}\ c\ )\right)^{-\frac{p+1}{2}}$,
(iii) for $\lambda < 0$, $u(t) = \left(\sqrt{\frac{-(p+1)\lambda_0}{(p-1)\lambda}}\sinh(\pm\sqrt{\frac{-\lambda}{p+1}}\ t + \sqrt{\frac{-\lambda}{p+1}}\ c\ )\right)^{\frac{p+1}{2}}$,
where c is a constant.

Proof. (i) For $\lambda = 0$, equation (1.4) implies that we have equation

$$(u'(t))^2 = \frac{(p+1)^2\lambda_0}{4(p-1)}\ u(t)^{2-\frac{4}{p+1}}.$$

Here we get $u'(t) = \pm\sqrt{\dfrac{(p+1)^2\lambda_0}{4(p-1)}}\ u(t)^{1-\frac{2}{p+1}}$. Multiplying both sides of equation by $u(t)^{-1+\frac{2}{p+1}}$ and an integration gives

$$\frac{p+1}{2} u(t)^{\frac{2}{p+1}} = \pm\sqrt{\frac{(p+1)^2\lambda_0}{4(p-1)}}\ t + c,$$

where c is a constant. Then we obtain $u(t)^{\frac{2}{p+1}} = \pm\sqrt{\frac{\lambda_0}{p-1}}\ t + \frac{2c}{p+1}$, where c is a constant. Therefore we have

$$u(t) = \left(\pm\sqrt{\frac{\lambda_0}{p-1}}\ t + \frac{2c}{p+1}\right)^{\frac{p+1}{2}},$$

where c is a constant.
(ii) For $\lambda > 0$, first of all, equation (1.4) simplifies that we rewritten as

$$\int \frac{1}{u(t)\ \sqrt{\frac{(p+1)^2\ \lambda_0}{4(p-1)}\ u(t)^{-\frac{4}{p+1}} - \frac{(p+1)\lambda}{4}}}\ du = \pm\int dt. \tag{3.1}$$

Putting $\frac{(p+1)^2\lambda_0}{4(p-1)} = a > 0$ and $\frac{(p+1)\lambda}{4} = b > 0$, then we have

$$\int \frac{1}{u(t)\ \sqrt{a\ u(t)^{\frac{-4}{p+1}} - b}}\ du = \pm\int dt.$$

By using trigonometric substitution, $u(t)^{\frac{-2}{p+1}} = \frac{\sqrt{b}}{\sqrt{a}}\sec\theta$, then we get

$$\int -\frac{p+1}{2}\frac{1}{\sqrt{b}}\ d\theta = \pm\int dt.$$

Upon integration, we obtain $-\frac{p+1}{2}\frac{1}{\sqrt{b}}\ \theta = \pm t + c$, where c is a constant. Now we have

$$u(t)^{\frac{-2}{p+1}} = \frac{\sqrt{b}}{\sqrt{a}}\sec(\pm\frac{2\sqrt{b}}{p+1}\ t - \frac{2\sqrt{b}}{p+1}\ c\ ),$$

where c is a constant. Then we become

$$u(t) = \left(\ \frac{\sqrt{b}}{\sqrt{a}}\sec(\pm\frac{2\sqrt{b}}{p+1}\ t - \frac{2\sqrt{b}}{p+1}\ c\ )\right)^{-\frac{p+1}{2}},$$

where c is a constant. Therefore we have

$$u(t) = \left(\sqrt{\frac{(p-1)\lambda}{(p+1)\lambda_0}}\ \sec(\pm\sqrt{\frac{\lambda}{p+1}}\ t - \sqrt{\frac{\lambda}{p+1}}\ c\ ) \right)^{-\frac{p+1}{2}},$$

where c is a constant.
(iii) For $\lambda < 0$, by a proof similar to Theorem 3.1 (ii), putting $\dfrac{(p+1)^2\lambda_0}{4(p-1)} = a > 0$ and $-\dfrac{(p+1)\lambda}{4} = b > 0$, equation (3.1) implies that we have the equation

$$\int \frac{1}{u(t)\ \sqrt{a\ u(t)^{\frac{-4}{p+1}} + b}} du = \pm \int dt.$$

By using trigonometric substitution, $u(t)^{\frac{-2}{p+1}} = \dfrac{\sqrt{b}}{\sqrt{a}} \tan\theta$, then we obtain

$$\int -\frac{p+1}{2}\frac{1}{\sqrt{b}}\ \csc\theta d\theta = \pm \int dt.$$

Upon integration, we become $\ln|\csc\theta + \cot\theta| = \pm\dfrac{2\sqrt{b}}{p+1}\ t + \dfrac{2\sqrt{b}}{p+1}\ c$, where c is a constant. Here we obtain

$$\ln|\ \frac{\sqrt{a + b\ u(t)^{\frac{4}{p+1}}}}{\sqrt{a}} + \frac{\sqrt{b}\ u(t)^{\frac{2}{p+1}}}{\sqrt{a}}\ | = \pm\frac{2\sqrt{b}}{p+1}\ t + \frac{2\sqrt{b}}{p+1}\ c\ ,$$

where c is a constant. Now we get

$$|\sqrt{a + b\ u(t)^{\frac{4}{p+1}}} + \sqrt{b}\ u(t)^{\frac{2}{p+1}}| = e^{\pm\frac{2\sqrt{b}}{p+1}\ t + \frac{2\sqrt{b}}{p+1}\ c + ln\sqrt{a}},$$

where c is a constant and $e^{\pm\frac{2\sqrt{b}}{p+1}t+\frac{2\sqrt{b}}{p+1}c+ln\sqrt{a}}$ are positive functions.
Hence we have

$$u(t)^{\frac{2}{p+1}} = \frac{\sqrt{a}}{\sqrt{b}} \sinh(\pm\frac{2\sqrt{b}}{p+1}\ t + \frac{2\sqrt{b}}{p+1}\ c\ ),$$

where c is a constant. Therefore we have

$$u(t) = \left(\sqrt{\frac{-(p+1)\lambda_0}{(p-1)\lambda}} \sinh(\pm\sqrt{\frac{-\lambda}{p+1}}\ t + \sqrt{\frac{-\lambda}{p+1}}\ c\ ) \right)^{\frac{p+1}{2}},$$

where c is a constant. □

REMARK 3.2. For $\lambda_0 > 0$, there exist generally nonconstant warping functions $f(t)$ of the equation (1.3) according to the signs of λ:

(i) For $\lambda = 0$, then $f(t) = \pm\sqrt{\frac{\lambda_0}{p-1}}\ t + \frac{2c}{p+1}$,

(ii) for $\lambda > 0$, then $f(t) = \sqrt{\frac{(p+1)\lambda_0}{(p-1)\lambda}} \cos(\pm\sqrt{\frac{\lambda}{p+1}}\ t - \sqrt{\frac{\lambda}{p+1}}\ c\ )$,

(iii) for $\lambda < 0$, then $f(t) = \sqrt{\frac{-(p+1)\lambda_0}{(p-1)\lambda}} \sinh(\pm\sqrt{\frac{-\lambda}{p+1}}\ t + \sqrt{\frac{-\lambda}{p+1}}\ c\ )$,

where c is a constant.

EXAMPLE 3.3. Let dimF $= p > 1$. From the Remark 3.2, we have nonconstant warping functions $f(t)$ of equation (1.3) depending on well-known special constants λ_0 and λ when $c = 0$.
(i) For $\lambda_0 = p-1$ and $\lambda = 0$, then $f(t) = \pm t$.
(ii) For $\lambda_0 = p-1$ and $\lambda = p+1$, then $f(t) = \cos t$.
(iii) For $\lambda_0 = p-1$ and $\lambda = -(p+1)$, then $f(t) = \pm \sinh t$.

4. Fiber manifold with $\lambda_0 < 0$

Let dim$F = p > 1$. In case that $\lambda_0 < 0$, we consider to the following theorem according to the signs of λ.

THEOREM 4.1. *In case that $\lambda_0 < 0$. If λ is a constant, then there exists a solution equation (1.4):*
(i) For $\lambda \geq 0$, there does not exist a solution of equation (1.4).
(ii) For $\lambda < 0$, $u(t) = \left(\sqrt{\frac{(p+1)\lambda_0}{(p-1)\lambda}}\ \cosh(\pm\sqrt{\frac{-\lambda}{p+1}}\ t + \sqrt{\frac{-\lambda}{p+1}}\ c\ ) \right)^{\frac{p+1}{2}}$,
where c is a constant.

Proof. (i) For $\lambda \geq 0$, equation (1.4) implies that $(u'(t))^2 < 0$. Therefore there does not exist a solution of equation (1.4).
(ii) For $\lambda < 0$, by a proof similar to Theorem 3.1 (ii) and (iii), putting $-\dfrac{(p+1)^2\lambda_0}{4(p-1)} = a > 0$ and $-\dfrac{(p+1)\lambda}{4} = b > 0$, equation (3.1) implies that we get the equation

$$\int \frac{1}{u(t)\ \sqrt{-a\ u(t)^{\frac{-4}{p+1}} + b}} du = \pm \int dt.$$

By using trigonometric substitution, $u(t)^{\frac{-2}{p+1}} = \dfrac{\sqrt{b}}{\sqrt{a}} \sin\theta$, then we obtain

$$\int -\frac{p+1}{2}\frac{1}{\sqrt{b}}\ \csc\theta d\theta = \pm \int dt.$$

Upon integration, we have $\ln|\csc\theta + \cot\theta| = \pm\dfrac{2\sqrt{b}}{p+1}\ t + \dfrac{2\sqrt{b}}{p+1}\ c$, where c is a constant. Here we become

$$\ln|\ \frac{\sqrt{b}\ u(t)^{\frac{2}{p+1}}}{\sqrt{a}} + \frac{\sqrt{-a + b\ u(t)^{\frac{4}{p+1}}}}{\sqrt{a}}\ | = \pm\frac{2\sqrt{b}}{p+1}\ t + \frac{2\sqrt{b}}{p+1}\ c\ ,$$

where c is a constant. Here we get

$$|\ \sqrt{b}\ u(t)^{\frac{2}{p+1}} + \sqrt{-a + b\ u(t)^{\frac{4}{p+1}}}\ | = e^{\pm\frac{2\sqrt{b}}{p+1}\ t + \frac{2\sqrt{b}}{p+1}\ c + ln\sqrt{a}},$$

where c is a constant and $e^{\pm\frac{2\sqrt{b}}{p+1}\ t + \frac{2\sqrt{b}}{p+1}\ c + ln\sqrt{a}}$ are positive functions. Then we have

$$u(t)^{\frac{2}{p+1}} = \frac{\sqrt{a}}{\sqrt{b}}\cosh(\pm\frac{2\sqrt{b}}{p+1}\ t + \frac{2\sqrt{b}}{p+1}\ c\ ),$$

where c is a constant. Therefore we have

$$u(t) = \left(\sqrt{\frac{(p+1)\lambda_0}{(p-1)\lambda}}\cosh(\pm\sqrt{\frac{-\lambda}{p+1}}\ t + \sqrt{\frac{-\lambda}{p+1}}\ c\ )\right)^{\frac{p+1}{2}},$$

where c is a constant. □

Remark 4.2. For $\lambda_0 < 0$, there exists generally nonconstant warping function $f(t)$ of equation (1.3) for the signs of λ:
(i) For $\lambda \geq 0$, there does not exist a positive solution of equation (1.3).
(ii) For $\lambda < 0$, then $f(t) = \sqrt{\dfrac{(p+1)\lambda_0}{(p-1)\lambda}} \cosh(\pm\sqrt{\dfrac{-\lambda}{p+1}}\ t + \sqrt{\dfrac{-\lambda}{p+1}}\ c\ )$,
where c is a constant.

Example 4.3. Let dimF $= p > 1$. From the Remark 4.2, for the well-known special cases $\lambda_0 = -(p-1)$, $\lambda = -(p+1)$, and $c = 0$, then $f(t) = \cosh(t)$.

From above Remark 2.2, Remark 3.2, and Remark 4.2, the following remark shows that equation (1.3) satisfies nonconstant warping functions $f(t)$.

Remark 4.4. If $M = B \times_{f^2} F$ is an Einstein warped product manifold with a 2−dimensional base B and a metric $g_B = dt^2 + f'(t)^2\ du^2$, then there exist generally nonconstant warping functions $f(t)$ of equation (1.3). The behavior of the solutions depends on the signs of λ_0 and λ. We are reduced to the following five cases (besides the constant case), where $c = 0$:
(i) For $\lambda_0 = 0$ and $\lambda < 0$, then $f(t) = e^{\pm\sqrt{\frac{-\lambda}{p+1}}\ t}$.
(ii) For $\lambda_0 > 0$ and $\lambda = 0$, then $f(t) = \pm\sqrt{\frac{\lambda_0}{p-1}}\ t$.
(iii) For $\lambda_0 > 0$ and $\lambda > 0$, then $f(t) = \sqrt{\frac{(p+1)\lambda_0}{(p-1)\lambda}} \cos(\pm\sqrt{\frac{\lambda}{p+1}}\ t\ )$.
(iv) For $\lambda_0 > 0$ and $\lambda < 0$, then $f(t) = \sqrt{\frac{-(p+1)\lambda_0}{(p-1)\lambda}} \sinh(\pm\sqrt{\frac{-\lambda}{p+1}}\ t\ )$.
(v) For $\lambda_0 < 0$ and $\lambda < 0$, then $f(t) = \sqrt{\frac{(p+1)\lambda_0}{(p-1)\lambda}} \cosh(\pm\sqrt{\frac{-\lambda}{p+1}}\ t\ )$.

Thus main theorem is verified.

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Soo-Young Lee
Department of Mathematics
Chosun University
Kwangju, 61452, Republic of Korea.
E-mail: skdmlskan@hanmail.net