Korean J. Math. **20** (2012), No. 3, pp. 315–321

SOME PROPERTIES OF GR-MULTIPLICATION MODULES

Seungkook Park

Abstract. In this paper, we provide the necessary and sufficient conditions for a faithful graded module to be a graded multiplication module and for a graded submodule of a faithful gr-multiplication to be gr-essential.

1. Introduction

Let R be a commutative ring with identity $1 \neq 0$ and M a unital R-module. M is called a *multiplication module* provided for each submodule N of M, there exists an ideal I of R such that $N = IM$ [2]. Let G be a multiplicative group with identity e. A ring R is said to be a *graded ring of type* G if there is a family of additive subgroups of R, say $\{R_i \mid i \in G\}$, such that $R = \bigoplus_{i \in G} R_i$ and $R_i R_j \subseteq R_{ij}$ for all $i,\ j \in G$, where $R_i R_j$ is the set of all finite sums of products $r_i r_j$ with $r_i \in R_i$ and $r_j \in R_j$. The elements of $h(R) = \bigcup_{i \in G} R_i$ are called the homogeneous elements of R. Any nonzero $r \in R$ has a unique expression as a sum of homogeneous elements, that is, $r = \sum_{i \in G} r_i$ where r_i is nonzero for a finite number of i in G. The nonzero elements r_i in the decomposition of r are called the homogeneous components of r. Let R be a graded ring of type G then R-module M is said to be a *graded R-module* if there is a family $\{M_i \mid i \in G\}$ of additive subgroups of M such that $M = \bigoplus_{i \in G} M_i$ and $R_i M_j \subseteq M_{ij}$ for all $i,\ j \in G$. Elements of $h(M) = \bigcup_{i \in G} M_i$ are called the homogeneous elements of M. A submodule N of M is a graded submodule if $N = \bigoplus_{i \in G}(N \cap M_i)$, or equivalently, if for any $x \in N$, the homogeneous components of x are again in N. Properties

Received July 22, 2012. Revised August 22, 2012. Accepted September 5, 2012.
2010 Mathematics Subject Classification: 16W50, 13A02.
Key words and phrases: gr-multiplication module, multiplication module.
This Research was supported by the Sookmyung Women's University Research Grants 2012.

of multiplication module have been studied by many mathematicians [1], [2], [3], [5], [6], [7], [8], [9], [10]. In this paper, we generalize some of the properties of the multiplication modules to graded multiplication modules.

2. Gr-multiplication modules

In this Section we state the definition of the gr-multiplication module and introduce a basic theorem which will be a main tool used to provide proofs of the theorems in the following sections.

DEFINITION 2.1. Let R be a graded ring and let M be a graded R-module. Then M is called a *gr-multiplication module* if for any graded submodule N of M, there exists a graded ideal I of R such that $N = IM$.

For any graded submodule N of M, we denote $(N : M)_g$ the graded ideal of R generated by $(h(N) : h(M)) = \{r \in h(R) \mid rh(M) \subseteq h(N)\}$. Note that $(N : M)_g$ is the graded ideal of R generated by $(N : M) \cap h(R)$ and that $(N : M)_g = (N : M)$, where $(N : M) = \{r \in R \mid rM \subseteq N\}$. Note that if M is a graded R-module and N is a submodule of M, then $(N : M)$ is a graded ideal of R [4].

PROPOSITION 2.2. *Let R be a graded ring and let M be a graded R-module. Then M is a gr-multiplication R-module if and only if for any graded submodule N of M, $N = (N : M)_g M$.*

Proof. Suppose that M is a gr-multiplication module and let N be a graded submodule. Then $N = IM$ for some graded ideal I of R. Since $I \subseteq (N : M) = (N : M)_g$, $N = IM \subseteq (N : M)_g M \subseteq N$. Thus $N = (N : M)_g M$. The other direction of the proof is clear by taking $(N : M)_g = I$. This completes the proof. □

REMARK. If M is a graded module and a multiplication module, then M is a gr-multiplication module. However, a gr-multiplication module may not be a multiplication module. An example of a gr-multiplication module which is not a multiplication module is given in [4].

PROPOSITION 2.3. *Let R be a graded ring and let M be a graded R-module. Then M is a gr-multiplication module if and only if for each $m \in h(M)$, there exists a graded ideal I of R such that $Rm = IM$.*

Proof. Suppose that M is a gr-multiplication module. Let $m \in h(M)$. Since $Rm \simeq R$ as an R-module, Rm is a graded submodule of M. Hence there exists a graded ideal I of R such that $Rm = IM$.

Conversely, suppose that for each $m \in h(M)$, there exists a graded ideal I of R such that $Rm = IM$. Let N be a submodule of M. For each $x \in h(N)$ there exists a graded ideal I_x such that $Rx = I_xM$. Let $I = \sum_{x\in h(N)} I_x$. Then $N = IM$. Therefore M is a gr-multiplication module. □

Let M be a graded R-module. If P is a gr-maximal ideal of R, then we define $T_P(h(M)) = \{m \in h(M) \mid (1-p)m = 0 \text{ for some } p \in P\}$.

LEMMA 2.4. *Let M be a gr-multiplication R-module and let P be a gr-maximal ideal of R. Then $M = PM$ if and only if $h(M) = T_P(h(M))$.*

Proof. Suppose that $M = PM$. Let $m \in h(M)$. Then $Rm = IM$ for some graded ideal I of R. Hence $Rm = IM = IPM = PIM = Pm$ and $m = pm$ for some $p \in P$. Thus $(1-p)m = 0$ and $m \in T_P(h(M))$. If follows that $h(M) = T_P(h(M))$.

Conversely, suppose $h(M) = T_P(h(M))$. Let $m \in M$. Then $m = m_{\sigma_1} + \cdots + m_{\sigma_n}$ for some $m_{\sigma_i} \in M_{\sigma_i}$. Since $h(M) = T_P(h(M))$, $m_{\sigma_i} \in T_P(h(M))$ and hence $m = p_{\sigma_1}m_{\sigma_1} + \cdots + p_{\sigma_n}m_{\sigma_n}$ for some $p_{\sigma_i} \in P$. Thus $m \in PM$. It follows that $M = PM$. □

The following theorem can be found in [4]. For our purpose we modify the statement and provide the proof of the theorem for completeness of the paper.

THEOREM 2.5. *Let R be a graded ring. Then a graded R-module M is a gr-multiplication module if and only if for every gr-maximal ideal P of R either $h(M) = T_P(h(M))$ or there exist $p \in P$ and $m \in h(M)$ such that $(1-p)M \subseteq Rm$.*

Proof. Let M be a gr-multiplication module and let P be a gr-maximal ideal of R. Suppose $M = PM$. Then $h(M) = T_P(h(M))$ by Lemma 2.4. Now suppose $M \neq PM$. Let $m \in h(M)$ with $m \notin PM$. Then there exists a graded ideal I of R such that $Rm = IM$. If $I \subseteq P$ then $Rm = IM \subseteq PM$ which gives a contradiction that $m \in PM$. Therefore $I \nsubseteq P$. Since $R = P + I$, $1 = p + i$ for some $p \in P$ and $i \in I$. Hence $1 - p \in I$. Thus $(1-p)M \subseteq IM = Rm$.

Conversely, let N be a graded submodule of M and let $I = (N : M)_g$. Then $IM \subseteq N$. Let $n \in h(N)$ and let $K = \{r \in R \mid rn \in IM\}$

be a graded ideal of R. Suppose $K \neq R$. Then there exists a gr-maximal ideal P of R such that $K \subseteq P$. If $h(M) = T_P(h(M))$, then $(1-p)n = 0$ for some $p \in P$. Hence $1-p \in K \subseteq P$ which implies $1 \in P$. This is a contradiction. Thus by hypothesis, there exist $q \in P$ and $m \in h(M)$ such that $(1-q)M \subseteq Rm$. It follows that $(1-q)N$ is a graded submodule of Rm and hence $(1-q)N = JRm = Jm$ where $J = \{r \in R \mid rm \in (1-q)N\}$ is a graded ideal of R. Note that $(1-q)JM = J(1-q)M \subseteq Jm \subseteq N$ and hence $(1-q)J \subseteq I$. It follows that $(1-q)^2 n \in (1-q)^2 N = (1-q)Jm \subseteq IM$. But this gives the contradiction $(1-q)^2 \in K \subseteq P$. Thus $K = R$ and $n \in IM$. Hence $h(N) \subseteq IM$. It follows that $N = IM$ and hence M is a gr-multiplication module. □

Corollary 2.6. *Let M be a graded R-module such that $M = \sum_{\lambda\in\Lambda} Rm_\lambda$ for some elements $m_\lambda \in h(M)$ ($\lambda \in \Lambda$). Then M is a gr-multiplication module if and only if there exist graded ideals I_λ of R such that $Rm_\lambda = I_\lambda M$ for all $\lambda \in \Lambda$.*

Proof. The necessity is clear.

Conversely, suppose that there exist graded ideals I_λ of R such that $Rm_\lambda = I_\lambda M$ for all $\lambda \in \Lambda$. Let P be a gr-maximal ideal of R. Suppose $I_\mu \nsubseteq P$ for some $\mu \in \Lambda$. Then there exist $p \in P$ such that $1-p \in I_\mu$. Thus $(1-p)M \subseteq I_\mu M = Rm_\mu$. Now suppose that $I_\lambda \subseteq P$ for all $\lambda \in \Lambda$. Then $Rm_\lambda \subseteq PM$ for all $\lambda \in \Lambda$ and hence $M = PM$. But for any $\lambda \in \Lambda$, this implies $Rm_\lambda = I_\lambda M = I_\lambda PM = PI_\lambda M = PRm_\lambda = Pm_\lambda$ and hence $m_\lambda \in T_P(h(M))$. It follows that $h(M) = T_P(h(M))$. By the Theorem 2.5, M is a gr-multiplication module. □

3. Main Results

Definition 3.1. An R-module M is *faithful* if, whenever $r \in R$ is such that $rM = 0$, then $r = 0$.

The next proposition gives the conditions for a faithful graded module to be gr-multiplication module.

Theorem 3.2. *Let R be a graded ring and let M be a faithful graded R-module. Then M is a gr-multiplication module if and only if*

(i) $\cap_{\lambda\in\Lambda}(I_\lambda M) = (\cap_{\lambda\in\Lambda} I_\lambda)M$ *for any non-empty collection of graded ideals I_λ ($\lambda \in \Lambda$) of R, and*

(ii) *for any graded submodule N of M and graded ideal A of R such that $N \subsetneq AM$ there exists an ideal B with $B \subsetneq A$ and $N \subseteq BM$.*

Proof. Suppose M is a gr-multiplication module. Let I_λ $(\lambda \in \Lambda)$ be a non-empty collection of graded ideals of R. Let $I = \cap_{\lambda\in\Lambda} I_\lambda$. Then $IM \subseteq \cap_{\lambda\in\Lambda}(I_\lambda M)$. Let $x \in h\left(\cap_{\lambda\in\Lambda}(I_\lambda M)\right)$ and let $K = \{r \in R \mid rx \in IM\}$ be a graded ideal of R. Suppose $K \neq R$. Then there exists a gr-maximal ideal P of R such that $K \subseteq P$. Then $x \notin T_P(h(M))$ and hence there exist $p \in P$ and $m \in h(M)$ such that $(1-p)M \subseteq R_m$. Then $(1-p)x \in (1-p)I_\lambda M = I_\lambda(1-p)M \subseteq I_\lambda m$ for all $\lambda \in \Lambda$. Thus $(1-p)x \in \cap_{\lambda\in\Lambda}(I_\lambda m)$. For each $\lambda \in \Lambda$, there exists $a_\lambda \in I_\lambda$ such that $(1-p)x = a_\lambda m$. Choose $\alpha \in \Lambda$. For each $\lambda \in \Lambda$, $a_\alpha m = a_\lambda m$ so that $(a_\alpha - a_\lambda)m = 0$. Now $(1-p)(a_\alpha - a_\lambda)M = (a_\alpha - a_\lambda)(1-p)M \subseteq (a_\alpha - a_\lambda)R_m = 0$ implies $(1-p)(a_\alpha - a_\lambda) = 0$. Therefore $(1-p)a_\alpha = (1-p)a_\lambda \in I_\lambda$ $(\lambda \in \Lambda)$ and hence $(1-p)a_\alpha \in I$. Thus $(1-p)^2x = (1-p)a_\alpha m \in IM$. It follows that $(1-p)^2 \in K \subseteq P$, which is a contradiction. Thus $K = R$ and $x \in IM$. Hence $h\left(\cap_{\lambda\in\Lambda}(I_\lambda M)\right) \subseteq IM$. This shows that $\cap_{\lambda\in\Lambda}(I_\lambda M) \subseteq IM$ and (i) is proved. Now let N be a graded submodule of M and A a graded ideal of R such that $N \subsetneq AM$. There exists a graded ideal C of R such that $N = CM$. Let $B = A \cap C$. Clear $B \subsetneq A$ and $N = AM \cap CM = (A \cap C)M = BM$ by (i). This proves (ii).

Conversely, suppose that (i) and (ii) hold. Let N be a graded submodule of M. Let $S = \{I \mid I \text{ is a graded ideal of } R \text{ and } N \subseteq IM\}$. Clearly $R \in S$. Let I_λ $(\lambda \in \Lambda)$ be any non-empty collection of graded ideals in S. By (i), $\cap_{\lambda\in\Lambda} I_\lambda \in S$. By Zorn's Lemma, S has a minimal member, say A. Then $N \subseteq AM$. Suppose that $N \neq AM$. By (ii), there exists a graded ideal B of R with $B \subsetneq A$ and $N \subseteq BM$. In this case $B \in S$, contradicting the choice of A. Thus $N = AM$. If follows that M is a gr-multiplication module. □

A graded R-module M is called *finitely gr-cogenerated* provided for every non-empty collection of graded submodules N_λ $(\lambda \in \Lambda)$ of M with $\cap_{\lambda\in\Lambda} N_\lambda = 0$ there exists a finite subset Λ' of Λ such that $\cap_{\lambda\in\Lambda'} N_\lambda = 0$. The graded ring R is called finitely gr-cogenerated provided it is finitely gr-cogenerated as an R-module.

Corollary 3.3. *Let M be a faithful gr-multiplication R-module. Then M is finitely gr-cogenerated if and only if R is finitely gr-cogenerated.*

Proof. Suppose that M is a finitely gr-cogenerated. Let I_λ $(\lambda \in \Lambda)$ be a non-empty collection of graded ideals of R such that $\cap_{\lambda\in\Lambda} I_\lambda = 0$. Then $\cap_{\lambda\in\Lambda}(I_\lambda M) = 0$ by Theorem 3.2. Since M is finitely gr-cogenerated, it follows that there exists a finite subset Λ' of Λ such that $\cap_{\lambda\in\Lambda'}(I_\lambda M) = 0$. Thus $(\cap_{\lambda\in\Lambda'} I_\lambda)M = 0$ and, because M is faithful, $\cap_{\lambda\in\Lambda'} I_\lambda = 0$. It follows that R is finitely gr-cogenerated.

Conversely, let N_γ $(\gamma \in \Gamma)$ be a non-empty collection of graded submodules of M such that $\cap_{\gamma\in\Gamma} N_\gamma = 0$. For each $\gamma \in \Gamma$, there exists a graded ideal I_γ of R such that $N_\gamma = I_\gamma M$. Then $0 = \cap_{\gamma\in\Gamma} N_\gamma = \cap_{\gamma\in\Gamma}(I_\gamma M) = (\cap_{\gamma\in\Gamma} I_\gamma)M$. Thus $\cap_{\gamma\in\Gamma} I_\gamma = 0$ and by hypothesis, there exists a finite subset Γ' of Γ such that $\cap_{\gamma\in\Gamma'} I_\gamma = 0$. By Theorem 3.2, $\cap_{\gamma\in\Gamma'} N_\gamma = \cap_{\gamma\in\Gamma'}(I_\gamma M) = (\cap_{\gamma\in\Gamma} I_\gamma)M = 0$. Hence M is finitely gr-cogenerated. □

A graded ideal P of R (i.e., a graded R-submodule of R) is called *gr-prime* if $P \neq R$ and whenever $rs \in P$ $(r, s \in h(R))$ then $r \in P$ or $s \in P$.

PROPOSITION 3.4. *Let P be a gr-prime ideal of R and M a faithful gr-multiplication R-module. Let $a \in h(R)$ and $x \in h(M)$ satisfy $ax \in PM$. Then $a \in P$ or $x \in PM$.*

Proof. Suppose $a \notin P$. Let $K = \{r \in R \mid rx \in PM\}$. Suppose $K \neq R$. Then there exists a gr-maximal ideal Q of R such that $K \subseteq Q$. Clearly $x \notin T_Q(h(M))$. By Theorem 2.5, there exist $q \in Q$ and $m \in h(M)$ such that $(1-q)M \subseteq Rm$. In particular, $(1-q)x = sm$ for some $s \in R$ and $(1-q)ax = pm$ for some $p \in P$. Thus $(as - p)m = 0$. Now $[(1-q)\operatorname{ann}(m)]M = 0$ implies $(1-q)\operatorname{ann}(m) = 0$, because M is faithful, and hence $(1-q)(as-p) = 0$. Then $(1-q)as = (1-q)p \in P$. But $P \subseteq K \subseteq Q$ so that $(1-q) \notin P$. Thus $s \in P$ and $(1-q)x = sm \in PM$. Thus $1 - q \in K \subseteq Q$, which is a contradiction. It follows that $K = R$ and $x \in PM$, as required. □

DEFINITION 3.5. A graded submodule N of a graded R-module M is called *gr-essential* provided $N \cap K \neq 0$ for every nonzero graded submodule K of M. A *gr-essential ideal of R* is just a gr-essential submodule of the graded R-module R.

THEOREM 3.6. *Let R be a graded ring and M a faithful gr-multiplication R-module. A graded submodule N of M is gr-essential if and only if there exists a gr-essential ideal E of R such that $N = EM$.*

Proof. Suppose that N is a gr-essential submodule of M. There exists a graded ideal A of R such that $N = AM$. Suppose $A \cap B = 0$ for some graded ideal B of R. By Theorem 3.2, we have $N \cap (BM) = (AM) \cap (BM) = (A \cap B)M = 0$, and hence $BM = 0$. Since M is faithful, $B = 0$. Hence A is a gr-essential ideal of R.

Conversely, suppose that E is gr-essential ideal of R. Let K be a graded submodule of M such that $(EM) \cap K = 0$. There exists a graded ideal C of R with $K = CM$ and hence $(E \cap C)M = (EM) \cap K = 0$. Since M is faithful, it follows that $E \cap C = 0$ and hence $C = 0$. Therefore $K = 0$ and thus EM is a gr-essential submodule of M. □

References

[1] M.M. Ali and D.J. Smith, *Some remarks on multiplication and projective modules*, Comm. Algebra **32** (10) (2004), 3897–3909.

[2] A. Barnard, *Multiplication modules*, J. Algebra **71** (1) (1981), 174–178.

[3] Z.A. El-Bast and P.F. Smith, *Multiplication modules*, Comm. Algebra **16**(4) (1988), 755–779.

[4] J. Escoriza and B. Torrecillas, *Multiplication objects in commutative Grothendieck categories*, Comm. Algebra **26** (6) (1998), 1867–1883.

[5] S.C. Lee, S. Kim, and S. Chung, *Ideals and submodules of multiplication modules*, J. Korean Math. Soc. **42** (5) (2005), 933–948.

[6] G.M. Low and P.F. Smith, *Multiplication modules and ideals*, Comm. Algebra **18** (12) (1990), 4353–4375.

[7] A.G. Naoum, *Flat modules and multiplication modules*, Period. Math. Hungar. **21** (4) (1990), 309–317.

[8] Y.S. Park and C.W. Choi, *Multiplication modules and characteristic submodules*, Bull. Korean Math. Soc. **32** (2) (1995), 321–328.

[9] Ünsal Tekir, *A note on multiplication modules*, Int. J. Pure Appl. Math. **27** (1) (2006), 107–111.

[10] A.A. Tuganbaev, *Flat and multiplication modules*, J. Math. Sci. (N. Y.) **128** (3) (2005), 2998–3004.

Department of Mathematics
Sookmyung Women's University
Seoul 140-742, Korea
E-mail: skpark@sookmyung.ac.kr