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THE HARMONIC ANALYSIS ASSOCIATED TO THE HECKMAN-OPDAM'S THEORY AND ITS APPLICATION TO A ROOT SYSTEM OF TYPE BC_d

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Abstract. In the five first sections of this paper we define and study the hypergeometric transmutation operators V_k^W and ${}^tV_k^W$ called also the trigonometric Dunkl intertwining operator and its dual corresponding to the Heckman-Opdam's theory on $\mathbb{R}^d$. By using these operators we define the hypergeometric translation operator $\mathcal{T}_x^W, x \in \mathbb{R}^d$, and its dual ${}^t\mathcal{T}_x^W, x \in \mathbb{R}^d$, we express them in terms of the hypergeometric Fourier transform $\mathcal{H}^W$, we give their properties and we deduce simple proofs of the Plancherel formula and the Plancherel theorem for the transform $\mathcal{H}^W$. We study also the hypergeometric convolution product on W-invariant $L^p_{\mathcal{A}_k}$-spaces, and we obtain some interesting results. In the sixth section we consider a some root system of type BC_d (see [17]) of whom the corresponding hypergeometric translation operator is a positive integral operator. By using this positivity we improve the results of the previous sections and we prove others more general results.

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1. Introduction

In [2] I. Cherednik introduced a family of differential-difference operators $T_j, j = 1, 2, ..., d$, on $\mathbb{R}^d$, associated with a root system $\mathcal{R}$ and a multiplicity function k. These operators play a crucial role in Heckman-Opdam's theory of hypergeometric functions, which generalizes the Harish-Chandra's theory of spherical fucntions on Riemannian symmetric spaces (see [3, 6, 7, 14, 15, 19]).

The Heckman-Opdam's theory is based on the Heckman-Opdam hypergeometric function $F_\lambda, \lambda \in \mathbb{C}^d$, which is the unique analytic solution of the system

$$\begin{cases} p(T)u(x) & = p(i\lambda)u(x), \lambda \in \mathbb{C}^d, x \in \mathbb{R}^d, \\ u(0) & = 1, \end{cases}$$

for all polynomials p which are invariant with respect to the Weyl group W associated with $\mathcal{R}$, and $p(T) = p(T_1, T_2, ..., T_d)$.
We have

$$\forall\ \lambda \in \mathbb{C}^d, F_\lambda(x) = \langle K_x^W, e^{i\langle \lambda,.\rangle} \rangle,$$

where K_x^W is a W-invariant distribution on $\mathbb{R}^d$ with compact support. By using this distribution we define in the five first sections of this paper, the hypergeometric transmutation operator V_k^W on $\mathcal{E}(\mathbb{R}^d)^W$ (the space of C^∞-functions on $\mathbb{R}^d$, which are W-invariant) by

$$\forall\ x \in \mathbb{R}^d, V_k^W(f)(x) = \langle K_x^W, f \rangle, f \in \mathcal{E}(\mathbb{R}^d)^W.$$

We define also its dual ${}^tV_k^W$ on $\mathcal{D}(\mathbb{R}^d)^W$ (the space of C^∞-functions on $\mathbb{R}^d$ with compact support and which are W-invariant) by

$$\int_{\mathbb{R}^d} {}^tV_k^W(f)(y)g(y)dy$$
$$= \int_{\mathbb{R}^d} V_k^W(g)(x)f(x)\mathcal{A}_k(x)dx, \quad f \in \mathcal{D}(\mathbb{R}^d)^W, \quad g \in \mathcal{E}(\mathbb{R}^d)^W,$$

with

$$\forall\ x \in \mathbb{R}^d, \mathcal{A}_k(x) = \prod_{\alpha \in \mathcal{R}_+} |2 \sinh\langle \frac{\alpha}{2}, x \rangle|^{2k(\alpha)},$$

and $\mathcal{R}_+$ is a positive subsystem of $\mathcal{R}$.

The operators V_k^W and ${}^tV_k^W$ are called also the trigonometric Dunkl intertwining operator and its dual.

We study the properties of the operators V_k^W and ${}^tV_k^W$ and we use them to define and study the hypergeometric translation operator $\mathcal{T}_x^W, x \in \mathbb{R}^d$, and its dual ${}^t\mathcal{T}_x^W, x \in \mathbb{R}^d$, by

$$\forall\, y \in \mathbb{R}^d, \mathcal{T}_x^W(f)(y) = (V_k^W)_x(V_k^W)_y[(V_k^W)^{-1}(f)(x+y)], f \in \mathcal{E}(\mathbb{R}^d)^W,$$

$$\forall\, y \in \mathbb{R}^d, {}^t\mathcal{T}_x^W(f)(y) = (V_k^W)_x({}^tV_k^W)_y^{-1}[({}^tV_k^W)(f)(y-x)], f \in \mathcal{D}(\mathbb{R}^d)^W.$$

From these relations we deduce the product formula for the function $F_\lambda, \lambda \in \mathbb{C}^d$, and the relation between the operators $\mathcal{T}_x^W, x \in \mathbb{R}^d$, and ${}^t\mathcal{T}_x^W, x \in \mathbb{R}^d$:

$$\forall\, y \in \mathbb{R}^d,\ \ \mathcal{T}_x^W(F_\lambda)(y) = F_\lambda(x).F_\lambda(y),$$

$$\forall\, y \in \mathbb{R}^d,\ \ \mathcal{T}_x^W(f)(y) = {}^t\mathcal{T}_x^W(\check{f})(-y), f \in \mathcal{D}(\mathbb{R}^d)^W,$$

where $\check{f}$ is the function given by

$$\forall\, x \in \mathbb{R}^d,\ \ \check{f}(x) = f(-x).$$

We express the operators $\mathcal{T}_x^W, x \in \mathbb{R}^d$, and ${}^t\mathcal{T}_x^W, x \in \mathbb{R}^d$, by using the hypergeometric Fourier transform $\mathcal{H}^W$ given on $\mathcal{D}(\mathbb{R}^d)^W$ by

$$\forall\, \lambda \in \mathbb{C}^d, \mathcal{H}^W(f)(\lambda) = \int_{\mathbb{R}^d} f(x)F_{-\lambda}(x)\mathcal{A}_k(x)dx = \int_{\mathbb{R}^d} f(x)F_\lambda(-x)\mathcal{A}_k(x)dx,$$

and we deduce simple proofs for the Plancherel formula and the Plancherel theorem for the transform $\mathcal{H}^W$ (See [14]).

Next we consider the spaces $L^p_{\mathcal{A}_k}(\mathbb{R}^d)^W$, $p \in [1, +\infty[$, of W-invariant functions on $\mathbb{R}^d$ which are pth integrable on $\mathbb{R}^d$ with respect to the measure $\mathcal{A}_k(x)dx$. We define first the operator $\mathcal{T}_x^W, x \in \mathbb{R}^d$, on $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and we prove that it is continuous from $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ into itself. This result permits to study the hypergeometric convolution product on the spaces $L^p_{\mathcal{A}_k}(\mathbb{R}^d)^W$, $p \in [1, 2]$, and to establish the Kunze and Stein's phenomenon for the Heckman-Opdam's theory, and Paley-Wiener's theorem for the hypergeometric Fourier transform $\mathcal{H}^W$ on $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$.

We remark that in this harmonic analysis we don't know if the hypergeometric translation operator $\mathcal{T}_x^W, x \in \mathbb{R}^d$, is positive or not.

In the sixth section we consider a root system of type BC_d (see [17]) of whom the corresponding hypergeometric translation operator denoted by $\mathcal{T}_x^{W,\mu}, x \in \mathbb{R}^d$, μ a positive real parameter, is a positive integral operator given by a probability measure. We prove first that this measure is absolutely continuous with respect to the Lebesgue measure. Next by using this positivity we improve the results of the previous sections and we prove the following others results.

- The Heckman-Opdam's hypergeometric function denoted by $F^{\mu}_{BC}(\lambda, x)$ satisfies for all λ in $\sum = \mathbb{R}^d + i\text{conv}(W.\rho)$, where $\text{conv}(W.\rho)$ is the convex hull for the Weyl group orbit $W.\rho$, the estimate

$$\sup_{x\in\mathbb{R}^d} |F^{\mu}_{BC}(\lambda, x)| = 1.$$

- The hypergeometric translation operator $\mathcal{T}^{W,\mu}_x, x \in \mathbb{R}^d$, is continuous from $L^p_{\mathcal{A}_k}(\mathbb{R}^d)^W$ into itself for $p \in [1, +\infty]$.

- We determine the maximal ideal space of the commutative Banach algebra $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$ for the hypergeometric convolution product $*^{\mu}_{\mathcal{H}^W}$.

- By applying the same proof as in [17], we show that $(\mathbb{R}^d, *^{\mu}_{\mathcal{H}^W})$ is a commutative hypergroup.

I point out that the harmonic analysis given in this paper is very useful to study many applications relating to the Heckman-opdam theory, by examples the multiplier operators, the Calderon's reproducing formula related to the cherednik operators and inverse formulas by using cherednick wavelets.

2. The Cherednik's operators and their eigenfunctions

We consider $\mathbb{R}^d$ with the standard basis $\{e_i, i = 1, 2, ..., d\}$ and the inner product $\langle ., .\rangle$ for which this basis is orthonormal. We extend this inner product to a complex bilinear form on $\mathbb{C}^d$.

2.1. The root system, the multiplicity function and the Cherednik's operators. Let $\alpha \in \mathbb{R}^d\backslash\{0\}$ and $\breve{\alpha} = \dfrac{2}{\|\alpha\|^2}\alpha$. We denote by

$$r_\alpha(x) = x - \langle \breve{\alpha}, x\rangle\alpha, \quad x \in \mathbb{R}^d, \tag{2.1}$$

the reflection in the hyperplan $H_\alpha \subset \mathbb{R}^d$ orthogonal to α.

A finite set $\mathcal{R} \subset \mathbb{R}^d\backslash\{0\}$ is called a root system if $\mathcal{R} \cap \mathbb{R}_\alpha = \{\pm\alpha\}$ and $r_\alpha\mathcal{R} = \mathcal{R}$, for all $\alpha \in \mathcal{R}$. For a given root system $\mathcal{R}$ the reflections $r_\alpha, \alpha \in \mathcal{R}$, generate a finite group $W \subset O(d)$, called the Weyl group associated with $\mathcal{R}$. For a given $\beta \in \mathbb{R}^d$ which belongs to no hyperplane $H_\alpha = \{x \in \mathbb{R}^d, \langle\alpha, x\rangle = 0\}, \alpha \in \mathcal{R}$, we fix the positive subsystem $\mathcal{R}_+ = \{\alpha \in \mathcal{R}, \langle\alpha, \beta\rangle > 0\}$, then for each $\alpha \in \mathcal{R}$ either $\alpha \in \mathcal{R}_+$ or $-\alpha \in \mathcal{R}_+$. We denote by $\mathcal{R}^0_+$ the set of positive indivisible roots.

Let

$$\mathfrak{a}^+ = \{x \in \mathbb{R}^d, \forall\ \alpha \in \mathcal{R}, \langle\alpha, x\rangle > 0\} \tag{2.2}$$

be the positive Weyl chamber. We denote by $\overline{\mathfrak{a}^+}$ its closure. Let also $\mathbb{R}^d_{reg} = \mathbb{R}^d \backslash \cup_{\alpha\in\mathcal{R}} H_\alpha$ be the set of regular elements in $\mathbb{R}^d$.

A function $k : \mathcal{R} \to [0, +\infty[$ on the root system $\mathcal{R}$ is called a multiplicity function if it is invariant under the action of the reflection group W. We introduce the index

$$\gamma = \gamma(\mathcal{R}) = \sum_{\alpha\in\mathcal{R}_+} k(\alpha). \tag{2.3}$$

Moreover, let $\mathcal{A}_k$ be the weight function

$$\forall\, x \in \mathbb{R}^d,\;\; \mathcal{A}_k(x) = \prod_{\alpha\in\mathcal{R}_+} |2\sinh\langle\frac{\alpha}{2}, x\rangle|^{2k(\alpha)}, \tag{2.4}$$

which is W-invariant.

The Cherednik's operators $T_j, j = 1, 2, ..., d$, on $\mathbb{R}^d$ associated with the reflection group W and the multiplicity function k are defined for f of class C^1 on $\mathbb{R}^d$ and $x \in \mathbb{R}^d_{reg}$ by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha\in\mathbb{R}_+} \frac{k(\alpha)\alpha^j}{1 - e^{-\langle\alpha,x\rangle}}\{f(x) - f(r_\alpha x)\} - \rho_j f(x), \tag{2.5}$$

where

$$\rho_j = \frac{1}{2}\sum_{\alpha\in\mathcal{R}_+} k(\alpha)\alpha^j, \text{ and } \alpha^j = \langle\alpha, e_j\rangle. \tag{2.6}$$

In the case $k(\alpha) = 0$, for all $\alpha \in \mathcal{R}_+$, the operators $T_j, j = 1, 2, ...d$, reduce to the corresponding partial derivatives. We suppose in the following that $k \neq 0$.

The Cherednik's operators form a commutative system of differential-difference operators.

For f of class C^1 on $\mathbb{R}^d$ with compact support, and g of class C^1 on $\mathbb{R}^d$, we have for $j = 1, 2, ..., d$:

$$\int_{\mathbb{R}^d} T_j f(x) g(x)\mathcal{A}_k(x)dx = -\int_{\mathbb{R}^d} f(x)(T_j + S_j)g(x)\mathcal{A}_k(x)dx, \tag{2.7}$$

with

$$\forall\, x \in \mathbb{R}^d, S_j g(x) = \sum_{\alpha\in\mathcal{R}_+} k(\alpha)\alpha^j g(r_\alpha x). \tag{2.8}$$

2.2. The Opdam-Cherednik's kernel and the Heckman-Opdam's hypergeometric function (see [3, 6, 14, 19]). We denote by $G_\lambda, \lambda \in \mathbb{C}^d$, the eigenfunction of the operators $T_j, j = 1, 2, ..., d$. It is the unique analytic function on $\mathbb{R}^d$ which satisfies the differential-difference system

$$\begin{cases} T_j G_\lambda(x) &= i\lambda_j G_\lambda(x), \quad j = 1, 2, ..., d, x \in \mathbb{R}^d, \\ G_\lambda(0) &= 1. \end{cases} \tag{2.9}$$

It is called the Opdam-Cherednik kernel.

We consider the function F_λ defined by

$$\forall\, x \in \mathbb{R}^d, \quad F_\lambda(x) = \frac{1}{|W|} \sum_{w \in W} G_\lambda(wx). \tag{2.10}$$

This function is the unique analytic function on $\mathbb{R}^d$, which satisfies the differential system

$$\begin{cases} p(T)F_\lambda(x) &= p(i\lambda)F_\lambda(x), \quad x \in \mathbb{R}^d, \\ F_\lambda(0) &= 1 \end{cases} \tag{2.11}$$

for all W-invariant polynomials p on $\mathbb{C}^d$ and $p(T) = p(T_1, T_2, ..., T_d)$.

The function $F_\lambda(x)$ called the Heckman-Opdam's hypergeometric function, is W-invariant both in λ and x.

The functions G_λ and F_λ possess the following properties

i) - For all $\lambda \in \mathbb{C}^d$, the functions $x \to G_\lambda(x)$ and $x \to F_\lambda(x)$ are of class C^∞ on $\mathbb{R}^d$.
- For all $x \in \mathbb{R}^d$, the functions $\lambda \to G_\lambda(x)$ and $\lambda \to F_\lambda(x)$ are entire on $\mathbb{C}^d$.

ii) - For all $x \in \mathbb{R}^d$ and $\lambda \in \mathbb{C}^d$, we have

$$\overline{G_\lambda(x)} = G_{-\bar{\lambda}}(x) \quad \text{and} \quad \overline{F_\lambda(x)} = F_{-\bar{\lambda}}(x). \tag{2.12}$$

iii) For all $x \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}^d$, we have

$$|G_\lambda(x)| \leq |W|^{1/2} \text{ and } |F_\lambda(x)| \leq |W|^{1/2}\, . \tag{2.13}$$

iv) As the function $f(x) = G_{i\rho}(x)$ is the unique solution of the system

$$\begin{cases} \dfrac{\partial}{\partial x_j} f(x) + \displaystyle\sum_{\alpha \in \mathcal{R}_+} \frac{k(\alpha)\alpha^j}{1 - e^{-\langle \alpha, x\rangle}} \{f(x) - f(r_\alpha x)\} = 0, j = 1, 2, ..., d \\ f(0) = 1 \end{cases}$$

and the constant function $f(x) = 1$, is also a solution of this system. Then, from the unicity of the solution of this system, we obtain

$$\forall\, x \in \mathbb{R}^d, \quad G_{i\rho}(x) = 1. \tag{2.14}$$

From this relation and (2.10), we get

$$\forall\, x \in \mathbb{R}^d, F_{i\rho}(x) = 1. \tag{2.15}$$

v) Let p and q be polynomials of degree m and n. Then there exists a positive constant M such that for all $\lambda \in \mathbb{C}^d$ and $x \in \mathbb{R}^d$, we have

$$|p(\frac{\partial}{\partial\lambda})q(\frac{\partial}{\partial x})G_\lambda(x)| \leq M(1+\|x\|)^m(1+\|\lambda\|)^n F_0(x)e^{-\max_{w\in W} Im\langle w\lambda,x\rangle}. \tag{2.16}$$

The same inequality is also true for the function $F_\lambda(x)$.

vi) The function $F_0(x)$ satisfies the estimate

$$\forall\, x \in \overline{\mathfrak{a}}_+, F_0(x) \asymp e^{-\langle\rho,x\rangle} \prod_{a\in\mathcal{R}_+^0} (1+\langle\alpha,x\rangle). \tag{2.17}$$

vii) The function $G_\lambda, \lambda \in \mathbb{C}^d$, admits the following Laplace type representation

$$\forall\, x \in \mathbb{R}^d, G_\lambda(x) = \langle K_x, e^{i\langle\lambda,.\rangle}\rangle, \tag{2.18}$$

where K_x is a distribution on $\mathbb{R}^d$ with support in $\Gamma = conv\{wx, w \in W\}$ (the convex hull for the orbit of x under W).

ix) From (2.10), (2.18) we deduce that the function $F_\lambda(x), \lambda \in \mathbb{C}^d$, possesses the Laplace type representation

$$\forall\, x \in \mathbb{R}^d, F_\lambda(x) = \langle K_x^W, e^{i\langle\lambda,.\rangle}\rangle, \tag{2.19}$$

where K_x^W is the distribution on $\mathbb{R}^d$ with support in Γ, given by

$$K_x^W = \frac{1}{|W|}\sum_{\alpha\in\mathcal{R}_+} K_{wx}. \tag{2.20}$$

3. The harmonic analysis associated to the Heckman-Opdam's theory on W-invariant C^∞-functions

Notations. We denote by

- $\mathcal{E}(\mathbb{R}^d)^W$ the space of C^∞-functions on $\mathbb{R}^d$, which are W-invariant.
- $\mathcal{D}(\mathbb{R}^d)^W$ the space of C^∞ functions on $\mathbb{R}^d$, with compact support and W-invariant.
- $\mathcal{S}(\mathbb{R}^d)^W$ the space of W-invariant functions of the classical Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

- $\mathcal{S}_2(\mathbb{R}^d)^W$ the space of C^∞-functions on $\mathbb{R}^d$, which are W-invariant, and such that for all $\ell, n \in \mathbb{N}$,

$$p_{\ell,n}(f) = \sup_{\substack{|\mu|\leq n \\ x\in\mathbb{R}^d}} (1+\|x\|)^\ell (F_0(x))^{-1}|D^\mu f(x)| < +\infty, \tag{3.1}$$

where

$$D^\mu = \frac{\partial^{|\mu|}}{\partial x_1^{\mu_1}...\partial x_d^{\mu_d}}\ , \quad \mu = (\mu_1, ..., \mu_d) \in \mathbb{N}^d, \quad |\mu| = \sum_{i=1}^d \mu_i.$$

Its topology is defined by the semi-norms $p_{\ell,n}, \ell, n \in \mathbb{N}$.

- $PW_a(\mathbb{C}^d)^W$, $a > 0$, the space of entire functions on $\mathbb{C}^d$, which are W-invariant and satisfy

$$\forall\ m \in \mathbb{N}, q_m(g) = \sup_{\lambda\in\mathbb{C}^d}(1+\|\lambda\|)^m e^{-a\|Im\lambda\|}|g(\lambda)| < +\infty. \tag{3.2}$$

The topology of $PW_a(\mathbb{C}^d)$ is defined by the semi-norms $q_m, m \in \mathbb{N}$.

We set

$$PW(\mathbb{C}^d)^W = \cup_{a>0} PW_a(\mathbb{C}^d)^W. \tag{3.3}$$

This space is called the Paley-Wiener's space. It is equipped with the inductive limit topology.

3.1. The hypergeometric Fourier transform.

Definition 3.1. The hypergeometric Fourier transform $\mathcal{H}^W$ is defined for f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) by

$$\forall\ \lambda \in \mathbb{C}^d, \mathcal{H}^W(f)(\lambda) = \int_{\mathbb{R}^d} f(x)F_{-\lambda}(x)\mathcal{A}_k(x)dx. \tag{3.4}$$

Remark 3.2. We have also the relation

$$\forall\ \lambda \in \mathbb{C}^d, \mathcal{H}^W(f)(\lambda) = \int_{\mathbb{R}^d} f(x)F_\lambda(-x)\mathcal{A}_k(x)dx. \tag{3.5}$$

(see [19]).

Proposition 3.3. *For all f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) we have the following relations*

$$\forall\ \lambda \in \mathbb{R}^d, \mathcal{H}^W(\bar{f})(\lambda) = \overline{\mathcal{H}^W(\check{f})(\lambda)}, \tag{3.6}$$

$$\forall\ \lambda \in \mathbb{R}^d, \mathcal{H}^W(f)(\lambda) = \mathcal{H}^W(\check{f})(-\lambda), \tag{3.7}$$

where $\breve{f}$ is the function defined by

$$\forall\, x \in \mathbb{R}^d, \quad \breve{f}(x) = f(-x).$$

Proof. We deduce these relations from (2.12), (3.4), (3.5). □

THEOREM 3.4.

i) *The hypergeometric Fourier transform $\mathcal{H}^W$ is a topological isomorphism from*
  - $\mathcal{D}(\mathbb{R}^d)^W$ *onto* $PW(\mathbb{C}^d)^W$.
  - $\mathcal{S}_2(\mathbb{R}^d)^W$ *onto* $\mathcal{S}(\mathbb{R}^d)^W$.

ii) *A function f belongs to $\mathcal{D}(\mathbb{R}^d)^W$ with supp $f \subset B(0,a)$ the closed ball of center 0 and radius $a > 0$, if and only if its hypergeometric Fourier transform $\mathcal{H}^W(f)$ belongs to $PW_a(\mathbb{C}^d)^W$.*

iii) *The inverse transform $(\mathcal{H}^W)^{-1}$ is given by*

$$\forall\, x \in \mathbb{R}^d, (\mathcal{H}^W)^{-1}(h)(x) = \int_{\mathbb{R}^d} h(\lambda) F_\lambda(x) \mathcal{C}_k^W(\lambda) d\lambda, \tag{3.8}$$

where

$$\mathcal{C}_k^W(\lambda) = c_o |C_k(\lambda)|^{-2}, \tag{3.9}$$

with c_o a positive constant chosen in such a way that $\mathcal{C}_k^W(-\rho) = 1$, and

$$C_k(\lambda) = \prod_{\alpha \in \mathcal{R}_+} \frac{\Gamma(\langle i\lambda, \breve{\alpha}\rangle + \frac{1}{2}k(\frac{\alpha}{2}))}{\Gamma(\langle i\lambda, \breve{\alpha}\rangle + k(\alpha) + \frac{1}{2}k(\frac{\alpha}{2}))}\,, \tag{3.10}$$

with $k(\frac{\alpha}{2}) = 0$ if $\frac{\alpha}{2} \notin \mathcal{R}$.

REMARK 3.5. The function $\mathcal{C}_k^W$ is continuous on $\mathbb{R}^d$ and satisfies the estimate

$$\forall\, \lambda \in \mathbb{R}^d, |\mathcal{C}_k^W(\lambda)| \le const.(1 + \|\lambda\|)^b, \tag{3.11}$$

for some $b > 0$.

3.2. The hypergeometric transmutation operators V_k^W and ${}^tV_k^W$ (see also [22]). By using the distribution K_x^W given by (2.19) we define the hypergeometric transmutation operator V_k^W on $\mathcal{E}(\mathbb{R}^d)^W$ by

$$\forall\, x \in \mathbb{R}^d, \quad V_k^W(g)(x) = \langle K_x^W, g\rangle. \tag{3.12}$$

This operator is called also the trigonometric Dunkl intertwining operator. It satisfies the relation

$$\forall\, x \in \mathbb{R}^d,\ \forall\, \lambda \in \mathbb{C}^d,\ V_k^W(e^{i\langle\lambda,.\rangle}) = F_\lambda(x). \tag{3.13}$$

The operator V_k^W is the unique linear topological isomorphism from $\mathcal{E}(\mathbb{R}^d)^W$ onto itself satisfying the transmutation relations

$$\forall\, x \in \mathbb{R}^d, p(T)V_k^W(g)(x) = V_k^W(p(D)g)(x),\ g \in \mathcal{E}(\mathbb{R}^d)^W, \qquad (3.14)$$

for all W-invariant polynomials p on $\mathbb{C}^d$, $p(T) = p(T_1, T_2, ..., T_d)$ and $p(D) = p(D_1, D_2, ..., D_d)$ with $D_j = \frac{\partial}{\partial x_j}$, $j = 1, 2, ..., d$, and the condition

$$V_k^W(g)(0) = g(0). \qquad (3.15)$$

The dual ${}^tV_k^W$ of the operator V_k^W is defined by the following duality relation

$$\int_{\mathbb{R}^d} {}^tV_k^W(f)(y)g(y)dy = \int_{\mathbb{R}^d} V_k^W(g)(x)f(x)\mathcal{A}_k(x)dx, \qquad (3.16)$$

with f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) and g in $\mathcal{E}(\mathbb{R}^d)^W$.

The operator ${}^tV_k^W$ is a linear topological isomorphism from
- $\mathcal{D}(\mathbb{R}^d)^W$ onto itself
- $\mathcal{S}_2(\mathbb{R}^d)^W$ onto $\mathcal{S}(\mathbb{R}^d)^W$, satisfying the transmutation relations

$$\forall\, y \in \mathbb{R}^d, {}^tV_k^W(p(T)f)(y) = p(D_\rho){}^tV_k^W(f)(y), f \in \mathcal{D}(\mathbb{R}^d)^W(resp.\mathcal{S}_2(\mathbb{R}^d)^W) \qquad (3.17)$$

for all W-invariant polynomials p on $\mathbb{C}^d, p(T) = p(T_1, T_2, .., T_d)$ and $p(D_\rho) = p(D_{1,\rho_1}, D_{2,\rho_2}, ..., D_{d,\rho_d})$ with $D_{j,\rho_j} = \dfrac{\partial}{\partial x_j} - 2\rho_j$ $j = 1, 2, ..., d$.

Remark 3.6. By applying the relation (3.16) with the function $g(y) = e^{-i\langle\lambda,y\rangle}, \lambda \in \mathbb{R}^d$, we deduce from the relations (3.13),(3.4) that the operator ${}^tV_k^W$ satisfies for f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$), the following relation

$$\forall\, \lambda \in \mathbb{R}^d,\ \mathcal{F}o\, {}^tV_k(f)(\lambda) = \mathcal{H}^W(f)(\lambda), \qquad (3.18)$$

where $\mathcal{F}$ is the classical Fourier transform on $\mathbb{R}^d$.

3.3. The hypergeometric translation operator $\mathcal{T}_x^W$ and its dual ${}^t\mathcal{T}_x^W$ (see also [23]). By using the hypergeometric transmutation operator V_k^W we define the hypergeometric translation operator $\mathcal{T}_x^W, x \in \mathbb{R}^d$, on $\mathcal{E}(\mathbb{R}^d)^W$ by

$$\forall\, y \in \mathbb{R}^d, \mathcal{T}_x^W(f)(y) = (V_k^W)_x(V_k^W)_y[(V_k^W)^{-1}](f)(x+y)]. \qquad (3.19)$$

The operator $\mathcal{T}_x^W, x \in \mathbb{R}^d$, satisfies the following properties:

1. For all $x \in \mathbb{R}^d$, the operator $\mathcal{T}_x^W$ is continuous from $\mathcal{E}(\mathbb{R}^d)^W$ into itself.

2. For all f in $\mathcal{E}(\mathbb{R}^d)^W$ and $x, y \in \mathbb{R}^d$, we have
$$\mathcal{T}_x^W(f)(0) = f(x) \text{ and } \mathcal{T}_x^W(f)(y) = \mathcal{T}_y^W(f)(x). \tag{3.20}$$
3. For all $x, y \in \mathbb{R}^d$ and $\lambda \in \mathbb{C}^d$, we have the product formula
$$\mathcal{T}_x^W(F_\lambda)(y) = F_\lambda(x).F_\lambda(y), \tag{3.21}$$
where F_λ is the Heckman-Opdam's hypergeometric function given by (2.10).
4. For all f in $\mathcal{E}(\mathbb{R}^d)^W$, and $x, y \in \mathbb{R}^d$, we have
$$p(T)_x \mathcal{T}_x^W(f)(y) = \mathcal{T}_x^W(p(T)f)(y), \tag{3.22}$$
$$p(T)_y \mathcal{T}_x^W(f)(y) = \mathcal{T}_x^W(p(T)f)(y), \tag{3.23}$$
for all W-invariant polynomials p on $\mathbb{C}^d$, and $p(T) = p(T_1, T_2, ..., T_d)$.
5. Let f be in $\mathcal{E}(\mathbb{R}^d)^W$. Then the function $u(x, y) = \mathcal{T}_x^W(f)(y)$ is the unique solution of class C^∞ on $\mathbb{R}^d$ with respect to each variable, of the system
$$\begin{cases} p(T)_x u(x, y) & = p(T)_y u(x, y), \\ u(0, y) & = f(y), \end{cases} \tag{3.24}$$

for all W-invariant polynomials p on $\mathbb{C}^d$, and $p(T) = p(T_1, T_2, ..., T_d)$.

By using the hypergeometric transmutation operators V_k^W and ${}^tV_k^W$ we define the hypergeometric translation operator dual ${}^t\mathcal{T}_x^W$, $x \in \mathbb{R}^d$, on $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) by
$$\forall\, y \in \mathbb{R}^d, {}^t\mathcal{T}_x^W(f)(y) = (V_k^W)_x({}^tV_k^W)_y^{-1}[{}^tV_k^W(f)(y - x)]. \tag{3.25}$$
The operator ${}^t\mathcal{T}_x^W, x \in \mathbb{R}^d$, possesses the following properties

1. For all $x \in \mathbb{R}^d$, the operator ${}^t\mathcal{T}_x^W$ is continuous from $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) into itself.
2. For all f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) and $x, y \in \mathbb{R}^d$, we have
$${}^t\mathcal{T}_x^W(f)(y) = {}^t\mathcal{T}_{-y}^W(f)(-x). \tag{3.26}$$
3. For all f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) and h in $\mathcal{E}(\mathbb{R}^d)^W$ we have
$$\int_{\mathbb{R}^d} {}^t\mathcal{T}_x^W(f)(y)h(y)\mathcal{A}_k(y)dy = \int_{\mathbb{R}^d} f(y)\mathcal{T}_x^W(h)(y)\mathcal{A}_k(y)dy. \tag{3.27}$$

Proposition 3.7

i) *For all f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) and $x \in \mathbb{R}^d$, we have*
$$\forall\, \lambda \in \mathbb{C}^d, \mathcal{H}^W({}^t\mathcal{T}_x^W(f))(\lambda) = F_{-\lambda}(x)\mathcal{H}^W(f)(\lambda). \tag{3.28}$$

ii) *For all f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) and $x, y \in \mathbb{R}^d$, we have*

$$ {}^t\mathcal{T}_x^W(f)(y) = \int_{\mathbb{R}^d} F_{-\lambda}(x) F_\lambda(y) \mathcal{H}^W(f)(\lambda) \mathcal{C}_k^W(\lambda) d\lambda. \tag{3.29}$$

Proof. i) From the relations (3.4), (3.27), (3.21), we have

$$\begin{aligned} \forall\, \lambda \in \mathbb{C}^d, \mathcal{H}^W({}^t\mathcal{T}_x^W(f))(\lambda) &= \int_{\mathbb{R}^d} {}^t\mathcal{T}_x^W(f)(y) F_{-\lambda}(y) \mathcal{A}_k(y) dy, \\ &= \int_{\mathbb{R}^d} f(y) \mathcal{T}_x^W(F_{-\lambda})(y) \mathcal{A}_k(y) dy, \\ &= F_{-\lambda}(x) \int_{\mathbb{R}^d} f(y) F_{-\lambda}(y) \mathcal{A}_k(y) dy, \end{aligned}$$

thus

$$\forall\, \lambda \in \mathbb{C}^d, \mathcal{H}^W({}^t\mathcal{T}_x^W(f))(\lambda) = F_{-\lambda}(x) \mathcal{H}^W(f)(\lambda).$$

ii) We deduce (3.29) from (3.28) and (3.8).

□

COROLLARY 3.8. *For all f in $\mathcal{D}(\mathbb{R}^d)^W$ with support in the closed ball $B(0, a)$ of center 0 and radius $a > 0$, and $x \in \mathbb{R}^d$, we have*

$$ supp\, {}^t\mathcal{T}_x^W(f) \subset B(0, a + \|x\|). \tag{3.30}$$

Proof. We obtain (3.30) from the relations (3.28), (2.16) and Theorem 3.4 ii). □

3.4. The hypergeometric convolution product.

DEFINITION 3.9. The hypergeometric convolution product $f *_{H^W} g$ of the functions f, g in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) is defined by

$$\forall\, x \in \mathbb{R}^d, f *_{\mathcal{H}^W} g(x) = \int_{\mathbb{R}^d} \mathcal{T}_x^W(f)(-y) g(y) \mathcal{A}_k(y) dy. \tag{3.31}$$

REMARK 3.10. We have

$$\forall\, x \in \mathbb{R}^d, f *_{\mathcal{H}^W} g(x) = \int_{\mathbb{R}^d} \mathcal{T}_x^W(f)(y) \breve{g}(y) \mathcal{A}_k(y) dy, \tag{3.32}$$

where $\breve{g}$ is the function defined by

$$\forall\, y \in \mathbb{R}^d, \breve{g}(y) = g(-y),$$

then by applying the relation (3.27), the relation (3.32) can also be written in the form

$$\forall\, x \in \mathbb{R}^d,\ f *_{\mathcal{H}^W} g(x) = \int_{\mathbb{R}^d} f(y)\,{}^t\mathcal{T}_x^W(\check{g})(y)\mathcal{A}_k(y)dy. \tag{3.33}$$

Proposition 3.11

i) *For all* f, g *in* $\mathcal{D}(\mathbb{R}^d)^W$ *(resp.* $\mathcal{S}_2(\mathbb{R}^d)^W$*) the function* $f *_{H^W} g$ *belongs to* $\mathcal{D}(\mathbb{R}^d)^W$ *(resp.* $\mathcal{S}_2(\mathbb{R}^d)^W$*).*

ii) *For all* f, g *in* $\mathcal{D}(\mathbb{R}^d)^W$ *(resp.* $\mathcal{S}_2(\mathbb{R}^d)^W$*) we have*

$$\forall\, \lambda \in \mathbb{R}^d, \mathcal{H}^W(f *_{\mathcal{H}^W} g)(\lambda) = \mathcal{H}^W(f)(\lambda).\mathcal{H}^W(g)(\lambda). \tag{3.34}$$

Proof. i) We deduce the result from the relation (3.33) and the properties of the function ${}^t\mathcal{T}_x^W(\check{g})(y)$.

ii) We have

$$\forall\, \lambda \in \mathbb{R}^d, \mathcal{H}^W(f *_{H^W} g)(\lambda) = \int_{\mathbb{R}^d} f *_{H^W} g(x) F_{-\lambda}(x)\mathcal{A}_k(x)dx.$$

By using the relations (3.33), (3.26) and Fubini's theorem we obtain $\forall\, \lambda \in \mathbb{R}^d, \mathcal{H}^W(f *_{H^W} g)(\lambda) =$

$$\int_{\mathbb{R}^d} f(y)\Big[\int_{\mathbb{R}^d} {}^t\mathcal{T}_{-y}^W(\check{g})(-x)F_{-\lambda}(x)\mathcal{A}_k(x)dx\Big]\mathcal{A}_k(y)dy. \tag{3.35}$$

But from (3.5), (3.4), (3.28), (3.7) we get

$$\begin{aligned}\int_{\mathbb{R}^d} {}^t\mathcal{T}_{-y}^W(\check{g})(-x)F_{-\lambda}(x)\mathcal{A}_k(x)dx &= \int_{\mathbb{R}^d} {}^t\mathcal{T}_{-y}^W(\check{g})(-x)F_{\lambda}(-x)\mathcal{A}_k(x)dx,\\ &= \int_{\mathbb{R}^d} {}^t\mathcal{T}_{-y}^W(\check{g})(x)F_{\lambda}(x)\mathcal{A}_k(x)dx,\\ &= \mathcal{H}^W({}^t\mathcal{T}_{-y}^W)(\check{g}))(-\lambda),\\ &= F_\lambda(-y)\mathcal{H}^W(\check{g})(-\lambda),\end{aligned}$$

thus

$$\int_{\mathbb{R}^d} {}^t\mathcal{T}_{-y}^W(\check{g})(-x)F_{-\lambda}(x)\mathcal{A}_k(x)dx = F_\lambda(-y)\mathcal{H}^W(g)(\lambda).$$

We put this relation in (3.35) and we obtain

$$\forall\, \lambda \in \mathbb{R}^d, \mathcal{H}^W(f *_{H^W} g)(\lambda) = \mathcal{H}^W(g)(\lambda)\int_{\mathbb{R}^d} f(y)F_\lambda(-y)\mathcal{A}_k(y)dy,$$

we deduce (3.34) by applying (3.5). □

Corollary 3.12

i) *The hypergeometric convolution product is commutative and associative on $\mathcal{D}(\mathbb{R}^d)^W$ and $\mathcal{S}_2(\mathbb{R}^d)^W$.*

ii) *For all f, g in $\mathcal{D}(\mathbb{R}^d)^W$ with supp $f \subset B(0,a), a > 0$, and supp $g \subset B(0,b), b > 0$, we have*

$$\text{supp} f *_{H^W} g \subset B(0, a+b), \tag{3.36}$$

where $B(0,c), c > 0$, is the closed ball of center 0 and radius c.

iii) *For all f, g in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) we have*

$$\forall\, y \in \mathbb{R}^d, p(T)(f *_{\mathcal{H}^W} g)(y) = (p(T)f) *_{\mathcal{H}^W} g(y), \tag{3.37}$$

and

$$\forall\, y \in \mathbb{R}^d, p(T)(f *_{H^W} g)(y) = (f *_{H^W} (p(T)g))(y), \tag{3.38}$$

for all W-invariant polynomials p on $\mathbb{C}^d$ and $p(T) = p(T_1, T_2, ..., T_d)$.

iv) *For all f, g in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$), we have*

$$\forall\, y \in \mathbb{R}^d, {}^tV_k^W(f *_{H^W} g)(y) = {}^tV_k^W(f) * {}^tV_k^W(g)(y), \tag{3.39}$$

where $*$ is the classical convolution product on $\mathbb{R}^d$.

Proof. i) We deduce the result from Proposition 3.11 ii) and Theorem 3.4.i).

ii) Proposition 3.11 ii), the relation (2.16) and Theorem 3.4 ii) imply the relation (3.36).

iii) We consider the hypergeometric Fourier transform of the first member of the relations (3.37), (3.38). Next we apply the relations (3.4), (2.11) and we deduce (3.37), (3.38), from Theorem 3.4.i).

iv) We obtain (3.39) from (3.34), (3.18).

□

Corollary 3.13. *For all f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$), we have*

$$\forall\, x, y \in \mathbb{R}^d, \quad \mathcal{T}_x^W(f)(y) = {}^t\mathcal{T}_x^W(\breve{f})(-y), \tag{3.40}$$

where $\breve{f}$ is the function defined by

$$\forall\, z \in \mathbb{R}^d,\ \breve{f}(z) = f(-z).$$

Proof. From Corollary 3.12 i) the hypergeometric convolution product is commutative, then we have

$$\forall\, x \in \mathbb{R}^d, \int_{\mathbb{R}^d} \mathcal{T}_x^W(g)(-y) f(y) \mathcal{A}_k(y) dy = \int_{\mathbb{R}^d} \mathcal{T}_x^W(f)(-y) g(y) \mathcal{A}_k(y) dy.$$

On the other hand from the relation (3.33) we have

$$\forall\, x \in \mathbb{R}^d, \int_{\mathbb{R}^d} \mathcal{T}_x^W(g)(-y) f(y) \mathcal{A}_k(y) dy = \int_{\mathbb{R}^d} {}^t\mathcal{T}_x^W(\breve{f})(y) g(y) \mathcal{A}_k(y) dy.$$

Thus for all $x \in \mathbb{R}^d$ and g in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) we have

$$\int_{\mathbb{R}^d} \left[\mathcal{T}_x^W(f)(-y) - {}^t\mathcal{T}_x^W(\breve{f})(y)\right] g(y) \mathcal{A}_k(y) dy = 0.$$

This relation implies (3.40). □

PROPOSITION 3.14

i) *For all f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) and $x \in \mathbb{R}^d$, we have*

$$\forall\, \lambda \in \mathbb{C}^d, \mathcal{H}^W(\mathcal{T}_x^W(f))(\lambda) = F_\lambda(x) \mathcal{H}^W(f)(\lambda). \tag{3.41}$$

ii) *For all f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$), we have*

$$\forall\, x, y \in \mathbb{R}^d, \mathcal{T}_x^W(f)(y) = \int_{\mathbb{R}^d} F_\lambda(x) F_\lambda(y) \mathcal{H}^W(f)(\lambda) \mathcal{C}_k^W(\lambda) d\lambda. \tag{3.42}$$

Proof. i) From the relations (3.5), (3.27) for all f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) and $x \in \mathbb{R}^d$, we have

$$\begin{aligned}\forall\, \lambda \in \mathbb{C}^d, \mathcal{H}^W(\mathcal{T}_x^W(f))(\lambda) &= \int_{\mathbb{R}^d} \mathcal{T}_x^W(f)(y) \breve{F}_\lambda(y) \mathcal{A}_k(y) dy,\\ &= \int_{\mathbb{R}^d} f(y) {}^t\mathcal{T}_x^W(\breve{F}_\lambda)(y) \mathcal{A}_k(y) dy,\\ &= \int_{\mathbb{R}^d} \breve{f}(y) {}^t\mathcal{T}_x^W(\breve{F}_\lambda)(-y) \mathcal{A}_k(y) dy.\end{aligned}$$

By using the relations (3.40), (3.21) we obtain

$$\begin{aligned}\forall\, \lambda \in \mathbb{C}^d, \mathcal{H}^W(\mathcal{T}_x^W(f))(\lambda) &= \int_{\mathbb{R}^d} \breve{f}(y) \mathcal{T}_x^W(F_\lambda)(y) \mathcal{A}_k(y) dy,\\ &= F_\lambda(x) \int_{\mathbb{R}^d} \breve{f}(y) F_\lambda(y) \mathcal{A}_k(y) dy,\\ &= F_\lambda(x) \int_{\mathbb{R}^d} f(y) F_\lambda(-y) \mathcal{A}_k(y) dy.\end{aligned}$$

The relation (3.5) implies (3.41).

ii) We deduce (3.42) from the relations (3.41), (3.8).

□

Corollary 3.15. *For all f in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) and $x \in \mathbb{R}^d$, we have*

$$\int_{\mathbb{R}^d} \mathcal{T}_x^W(f)(y)\mathcal{A}_k(y)dy = \int_{\mathbb{R}^d} f(y)\mathcal{A}_k(y)dy. \tag{3.43}$$

Proof. From the relations (3.5), (3.41) we have

$$\int_{\mathbb{R}^d} \mathcal{T}_x^W(f)(y)F_{i\rho}(-y)\mathcal{A}_k(y)dy = F_{i\rho}(x)\int_{\mathbb{R}^d} f(y)F_{i\rho}(-y)\mathcal{A}_k(y)dy.$$

We obtain (3.43) from the relation (2.15). □

Theorem 3.16 (Plancherel's formula). *For all f, g in $\mathcal{D}(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) we have*

$$\int_{\mathbb{R}^d} f(y)\overline{g(y)}\mathcal{A}_k(y)dy = \int_{\mathbb{R}^d} \mathcal{H}^W(f)(\lambda)\overline{\mathcal{H}^W(g)(\lambda)}\mathcal{C}_k^W(\lambda)d\lambda. \tag{3.44}$$

Proof. By applying the relation (3.8) to the relation (3.34) we obtain

$$\forall\, x \in \mathbb{R}^d, f *_{H^W} \overline{g(x)} = \int_{\mathbb{R}^d} F_\lambda(x)\mathcal{H}^W(f)(\lambda)\mathcal{H}^W(\bar{g})(\lambda)\mathcal{C}_k^W(\lambda)d\lambda.$$

The relations (3.31), (3.6) permit to write this relation in the following form

$$\forall\, x \in \mathbb{R}^d, \int_{\mathbb{R}^d} \mathcal{T}_x^W(f)(y)\overline{\breve{g}(y)}\mathcal{A}_k(y)dy = \int_{\mathbb{R}^d} F_\lambda(x)\mathcal{H}^W(f)(\lambda)\overline{\mathcal{H}^W(\breve{g}(\lambda)}\mathcal{C}_\lambda^W(\lambda)d\lambda.$$

We obtain (3.44) by changing $\breve{g}$ by g in the two members, by taking $x = 0$, and by using the relations

$$\forall\, y \in \mathbb{R}^d, \mathcal{T}_0^W(f)(y) = f(y) \quad \text{and} \quad \forall\, \lambda \in \mathbb{R}^d, F_\lambda(0) = 1.$$

□

4. The harmonic analysis associated to the Heckman-Opdam's theory on the $L^p_{\mathcal{A}_k}(\mathbb{R}^d)^W$ spaces

4.1. The hypergeometric Fourier transform.

NOTATIONS. We denote by
- $L^p_{\mathcal{A}_k}(\mathbb{R}^d)^W$, $1 \leq p \leq +\infty$, the space of measurable functions f on $\mathbb{R}^d$ which are W-invariant and satisfy

$$\begin{aligned} \|f\|_{\mathcal{A}_k,p} &= \Big(\int_{\mathbb{R}^d} |f(x)|^p \mathcal{A}_k(x)dx\Big)^{1/p} < +\infty, \quad 1 \leq p < +\infty, \\ \|f\|_{\mathcal{A}_k,\infty} &= ess \sup_{x\in\mathbb{R}^d} |f(x)| < +\infty. \end{aligned}$$

- $L^p_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$, $1 \leq p \leq +\infty$, the space of measurable functions f on $\mathbb{R}^d$, which are W-invariant and satisfy

$$\begin{aligned} \|f\|_{\mathcal{C}_k^W,p} &= \Big(\int_{\mathbb{R}^d} |f(\lambda)|^p \mathcal{C}_k^W(\lambda)d\lambda\Big)^{1/p} < +\infty, 1 \leq p < +\infty, \\ |f\|_{\mathcal{C}_k^W,\infty} &= ess \sup_{\lambda\in\mathbb{R}^d} |f(\lambda)|. \end{aligned}$$

PROPOSITION 4.1. *Let $p \in [1,2[$ and q its conjugate exponent. Then*

$$\forall\, \lambda \in \mathbb{R}^d, \|F_\lambda\|_{\mathcal{A}_k,q} \leq C(p), \tag{4.1}$$

where $C(p)$ is a positive constant which depends only on p.

Proof. - For $q \in]2,+\infty[$.
As the function $x \to F_\lambda(x)$ and $\mathcal{A}_k(x)$ are W-invariant, then to obtain the result it suffices to prove that for all $\lambda \in \mathbb{R}^d$, we have

$$I = \int_{\overline{\mathfrak{a}^+}} |F_\lambda(x)|^q \mathcal{A}_k(x)dx < +\infty.$$

But from the relations (2.4), (2.16) (2.17) we have

$$\forall\, x \in \overline{\mathfrak{a}^+}, \mathcal{A}_k(x) \leq 2^{2\gamma} e^{2\langle\rho,x\rangle},$$

and

$$\forall\, x \in \overline{\mathfrak{a}^+}, |F_0(x)| \leq const.(\prod_{\alpha\in\mathcal{R}_+^0} (1+\langle\alpha,x\rangle))e^{-\langle\rho,x\rangle}.$$

Thus

$$I \leq const. \int_{\overline{\mathfrak{a}^+}} \left(\prod_{\alpha \in \mathcal{R}_+^0} (1 + \langle \alpha, x \rangle) \right)^q e^{-(q-2)\langle \rho, x \rangle} dx.$$

The integral of the second member is convergent for $q \in]2, +\infty[$. From this result we deduce that there exists a positive constant $C_0(p)$ which depends only on $p \in]1, 2[$ such that

$$\forall \ \lambda \in \mathbb{R}^d, \ \ \|F_\lambda\|_{\mathcal{A}_k, q} \leq C_0(p). \tag{4.2}$$

- For $q = +\infty$

From (2.13) we have

$$\forall \ \lambda \in \mathbb{R}^d, \sup_{x \in \mathbb{R}^d} |F_\lambda(x)| \leq |W|^{1/2}. \tag{4.3}$$

We obtain (4.1) from (4.2), (4.3), with $C(p) = Max(C_0(p), |W|^{1/2})$. □

Theorem 4.2. (Plancherel's theorem). *The hypergeometric Fourier transform $\mathcal{H}^W$ defined by (3.4) extends uniquely to an isometric isomorphism from $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ onto $L^2_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$.*

Proof. We deduce the result from the relation (3.44) and the fact that the space $\mathcal{S}(\mathbb{R}^d)^W$ is dense in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$. □

Corollary 4.3. *For all f in $L^2_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$, such that $\mathcal{H}^W(f)$ belongs to $L^1_{\mathcal{C}_k^W}(\mathbb{R}^d)$, we have the inversion formula*

$$f(x) = \int_{\mathbb{R}^d} \mathcal{H}^W(f)(\lambda) F_\lambda(x) \mathcal{C}_k^W(\lambda) d\lambda, a.e. \quad x \in \mathbb{R}^d \,. \tag{4.4}$$

Corollary 4.4. *The hypergeometric Fourier transform $\mathcal{H}^W$ is injective on $L^p_{\mathcal{A}_k}(\mathbb{R}^d), 1 \leq p \leq 2$.*

Proof. - For $p = 2$, the result follows from Theorem 4.2.

- We assume that $p \in [1, 2[$. Let q be the conjugate exponent of p. For f in $L^p_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and g in $\mathcal{D}(\mathbb{R}^d)^W$, we obtain from Hölder's inequality and Proposition 4.1 the following relations

$$|\langle f, g \rangle_{\mathcal{A}_k}| = | \int_{\mathbb{R}^d} f(x) \overline{g(x)} \mathcal{A}_k(x) dx| \leq \|f\|_{\mathcal{A}_k, p} \|g\|_{\mathcal{A}_k, q},$$

and

$$\begin{aligned} \left|\langle \mathcal{H}^W(f), \mathcal{H}^W(g)\rangle_{\mathcal{C}_k^W}\right| &= \left|\int_{\mathbb{R}^d} \mathcal{H}^W(f)(\lambda)\overline{\mathcal{H}^W(g)(\lambda)}\mathcal{C}_k^W(\lambda)d\lambda\right|, \\ &\leq \|\mathcal{H}^W(f)\|_{\mathcal{C}_k^W,\infty}\|\mathcal{H}^W(g)\|_{\mathcal{C}_k^W,1}, \\ &\leq C(p)\|f\|_{\mathcal{A}_k,p}\|\mathcal{H}^W(g)\|_{\mathcal{C}_k^W,1} \end{aligned}$$

Then the mappings $f \to \langle f, g\rangle_{\mathcal{A}_k}$ and $f \to \langle \mathcal{H}^W(f), \mathcal{H}^W(g)\rangle_{\mathcal{C}_k^W}$ are continuous functionals on $L^p_{\mathcal{A}_k}(\mathbb{R}^d)^W$.

On the other hand from Theorem 4.2, we have for all f in $L^p_{\mathcal{A}_k}(\mathbb{R}^d)^W \cap L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$:

$$\langle f, g\rangle_{\mathcal{A}_k} = \langle \mathcal{H}^W(f), \mathcal{H}^W(g)\rangle_{\mathcal{C}_k^W}. \tag{4.5}$$

Then by continuity this equality is also true for f in $L^p_{\mathcal{A}_k}(\mathbb{R}^d)^W$. We consider f in $L^p_{\mathcal{A}_k}(\mathbb{R}^d)^W$ such that $\mathcal{H}^W(f) = 0$. Then from (4.5), for all g in $\mathcal{D}(\mathbb{R}^d)^W$ we have

$$\langle f, g\rangle_{\mathcal{A}_k} = \langle \mathcal{H}^W(f), \mathcal{H}^W(g)\rangle_{\mathcal{C}_k^W} = 0.$$

Thus

$$f = 0.$$

This completes the proof. □

4.2. The hypergeometric translation operator.

Definition 4.5. The hypergeometric translation operator $\mathcal{T}_x^W, x \in \mathbb{R}^d$, is defined on $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ by

$$\mathcal{H}^W(\mathcal{T}_x^W(f))(\lambda) = F_\lambda(x)\mathcal{H}^W(f)(\lambda), \quad \lambda \in \mathbb{R}^d. \tag{4.6}$$

Remark 4.6. Note that this definition makes sense because the hypergeometric Fourier transform is, from Theorem 4.2, an isomorphism from $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ onto $L^2_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$, and from (2.13), for all $\lambda \in \mathbb{R}^d$, and $x \in \mathbb{R}^d$, the function $F_\lambda(x)$ is bounded.

Proposition 4.7.

i) *For all f in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ we have*

$$\|\mathcal{T}_x^W(f)\|_{\mathcal{A}_k,2} \leq |W|^{1/2}\|f\|_{\mathcal{A}_k,2}. \tag{4.7}$$

ii) *For all f in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ such that $\mathcal{H}^W(f)$ belongs to $L^1_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$, and $x \in \mathbb{R}^d$, we have*

$$\mathcal{T}_x^W(f)(y) = \int_{\mathbb{R}^d} F_\lambda(x)F_\lambda(y)\mathcal{H}^W(f)(\lambda)\mathcal{C}_k^W(\lambda)d\lambda, \quad a.e. \quad y \in \mathbb{R}^d. \tag{4.8}$$

Proof. i) We deduce the relation (4.7) from (4.6), Theorem 4.2 and (2.13).

ii) We obtain (4.8) from (4.6) and the inversion formula (4.4).

□

PROPOSITION 4.8. *For all f in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ the mapping $x \to \mathcal{T}^W_x(f)$ is continuous from $\mathbb{R}^d$ into $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$.*

Proof. Let $x_0 \in \mathbb{R}^d$. By using Theorem 4.2 and the relation (4.6), we obtain

$$\begin{aligned}\|\mathcal{T}^W_x(f) - \mathcal{T}^W_{x_0}(f)\|^2_{\mathcal{A}_k,2} &= \|\mathcal{H}^W(\mathcal{T}^W_x(f)) - \mathcal{H}^W(\mathcal{T}^W_{x_0}(f))\|^2_{\mathcal{A}_k,2},\\ &= \int_{\mathbb{R}^d} |F_\lambda(x) - F_\lambda(x_0)|^2 |\mathcal{H}^W(f)(\lambda)|^2 \mathcal{C}^W_k(\lambda) d\lambda.\end{aligned}$$

From the relation (2.13) and the fact that for all $\lambda \in \mathbb{R}^d$, the function $x \to F_\lambda(x)$ is continuous on $\mathbb{R}^d$, the dominated convergence theorem implies

$$\lim_{x \to x_0} \|\mathcal{T}^W_x(f) - \mathcal{T}^W_{x_0}(f)\|_{\mathcal{A}_k,2} = 0.$$

□

4.3. The hypergeometric convolution product.

THEOREM 4.9. *Let f be in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and g in $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$. Then the function $f *_{\mathcal{H}^W} g$ defined almost everywhere on $\mathbb{R}^d$ by*

$$f *_{\mathcal{H}^W} g(x) = \int_{\mathbb{R}^d} \mathcal{T}^W_x(f)(-y) g(y) \mathcal{A}_k(y) dy, \tag{4.9}$$

belongs to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and we have

$$\|f *_{\mathcal{H}^W} g\|_{\mathcal{A}_k,2} \leq |W|^{1/2} \|f\|_{\mathcal{A}_k,2} \|g\|_{\mathcal{A}_k,1}. \tag{4.10}$$

Proof. Let f, g, φ in $\mathcal{D}(\mathbb{R}^d)^W$. From (3.20) and Fubini's theorem we have

$$\begin{aligned}\int_{\mathbb{R}^d} f *_{\mathcal{H}^W} g(x) \overline{\varphi(x)} \mathcal{A}_k(x) dx &= \int_{\mathbb{R}^d} g(y) \left(\int_{\mathbb{R}^d} \mathcal{T}^W_x(f)(-y) \overline{\varphi(x)} \mathcal{A}_k(x) dx \right) \mathcal{A}_k(y) dy.\\ &= \int_{\mathbb{R}^d} \check{g}(y) \Big(\int_{\mathbb{R}^d} \mathcal{T}^W_y(f)(x) \overline{\varphi(x)} \mathcal{A}_k(x) dx \Big) \mathcal{A}_k(y) dy.\end{aligned}$$

By using Hölder's inequality and (4.7), we obtain

$$\left|\int_{\mathbb{R}^d} f *_{\mathcal{H}^W} g(x)\overline{\varphi(x)}\mathcal{A}_k(x)dx\right| \leq |W|^{1/2}\|f\|_{\mathcal{A}_k,2}\|g\|_{\mathcal{A}_k,1}\|\varphi\|_{\mathcal{A}_k,2}. \quad (4.11)$$

As the relation (4.11) remain true for all functions g in $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$, and f, φ in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$. So we obtain (4.10). □

THEOREM 4.10. *Let φ be a positive function in $\mathcal{D}(\mathbb{R}^d)^W$ such that supp$\varphi \subset B(0,1)$ the closed ball of center 0 and radius 1, and $\|\varphi\|_{\mathcal{A}_k,1} = 1$. For $\varepsilon > 0$, we consider the function φ_ε given by*

$$\forall\, x \in \mathbb{R}^d, \varphi_\varepsilon(x) = \frac{\mathcal{A}_k(\frac{x}{\varepsilon})}{\varepsilon^d \mathcal{A}_k(x)}\; \varphi(\frac{x}{\varepsilon}). \quad (4.12)$$

Then for all f in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, we have

$$\lim_{\varepsilon \to 0} \|f *_{\mathcal{H}^W} \varphi_\varepsilon - f\|_{\mathcal{A}_k,2} = 0. \quad (4.13)$$

Proof. Using the fact that $\|\varphi_\varepsilon\|_{\mathcal{A}_k,1} = 1$, we deduce that for $x \in \mathbb{R}^d$, we have

$$|f *_{\mathcal{H}^W} \varphi_\varepsilon(x) - f(x)| \leq \int_{\mathbb{R}^d} (\varphi_\varepsilon(y))^{1/2}(\varphi_\varepsilon(y))^{1/2}|\mathcal{T}^W_x(f)(-y) - f(x)|\mathcal{A}_k(y)dy.$$

By applying Hölder's inequality to the second member, we obtain

$$|f *_{\mathcal{H}^W} \varphi_\varepsilon(x) - f(x)|^2 \leq \int_{\mathbb{R}^d} \varphi_\varepsilon(y)|\mathcal{T}^W_x(f)(-y) - f(x)|^2\mathcal{A}_k(y)dy.$$

Thus

$$\int_{\mathbb{R}^d} |f *_{\mathcal{H}^W} \varphi_\varepsilon(x) - f(x)|^2\mathcal{A}_k(x)dx \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi_\varepsilon(y)|\mathcal{T}^W_x(f)(-y) - f(x)|^2\mathcal{A}_k(y)\mathcal{A}_k(x)dydx.$$

By using the relation (3.20) and Fubini-Tonelli's theorem we deduce that

$$\|f *_{\mathcal{H}^W} \varphi_\varepsilon - f\|^2_{\mathcal{A}_k,2} \leq \int_{\mathbb{R}^d} \varphi_\varepsilon(y)\|\mathcal{T}^W_{-y}(f) - f\|^2_{\mathcal{A}_k,2}\mathcal{A}_k(y)dy,$$

the change of variables $y = \varepsilon t$ gives

$$\|f *_{\mathcal{H}^*} \varphi_\varepsilon - f\|^2_{\mathcal{A}_k,2} \leq \int_{\mathbb{R}^d} \varphi(t)\|\mathcal{T}^W_{-\varepsilon t}(f) - f\|^2_{\mathcal{A}_k,2}\mathcal{A}_k(t)dt.$$

From Proposition 4.8, the relation (4.7) and the dominated convergence theorem, we deduce (4.13). □

Notation. We denote by $\tilde{\mathcal{H}}^W$ the mapping on $L^1_{\mathcal{C}_k^W}(\mathbb{R}^d)$ defined by

$$\forall\, x \in \mathbb{R}^d, \tilde{\mathcal{H}}^W(f)(x) = \int_{\mathbb{R}^d} h(\lambda)F_\lambda(x)\mathcal{C}_k^W(\lambda)d\lambda. \tag{4.14}$$

We remark that for h in $\mathcal{S}(\mathbb{R}^d)^W$, we have

$$\tilde{\mathcal{H}}^W(h) = (\mathcal{H}^W)^{-1}(h).$$

Theorem 4.11. *Let f and g be in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)$. Then*

i) *The function $f *_{\mathcal{H}^W} g$ defined on $\mathbb{R}^d$ by*

$$f *_{\mathcal{H}^W} g(x) = \int_{\mathbb{R}^d} \mathcal{T}_x^W(f)(-y)g(y)\mathcal{A}_k(y)dy, \tag{4.15}$$

is continuous on $\mathbb{R}^d$, tends to zero at the infinity and we have

$$\sup_{x\in\mathbb{R}^d} |f *_{\mathcal{H}^W} g(x)| \leq |W|^{1/2}\|f\|_{\mathcal{A}_k,2}\|g\|_{\mathcal{A}_k,2}. \tag{4.16}$$

ii) *We have*

$$\forall\, x \in \mathbb{R}^d, f *_{\mathcal{H}^W} g(x) = \tilde{\mathcal{H}}^W(\mathcal{H}^W(f)\mathcal{H}^W(g))(x). \tag{4.17}$$

Proof. i) Let $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be two sequences in $\mathcal{D}(\mathbb{R}^d)^W$ which converge respectively to f and g in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$. By using the fact that the operator $\mathcal{T}_x^W, x \in \mathbb{R}^d$, is continuous from $\mathcal{D}(\mathbb{R}^d)^W$ into itself, we deduce that the sequence $\{f_n *_{\mathcal{H}^W} g_n\}_{n\in\mathbb{N}}$ which belongs to $\mathcal{D}(\mathbb{R}^d)^W$, converges to $f *_{\mathcal{H}^W} g$ uniformly on $\mathbb{R}^d$. Then the function $f *_{\mathcal{H}^W} g$ is continuous on $\mathbb{R}^d$ and tends to zero at the infinity.

ii) Let $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be two sequences in $\mathcal{S}_2(\mathbb{R}^d)^W$ which converge respectively to f and g in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$. From (3.34) and Theorem 3.4, for all $n \in \mathbb{N}$, we have

$$\forall\, x \in \mathbb{R}^d, f_n *_{\mathcal{H}^W} g_n(x) = \int_{\mathbb{R}^d} \mathcal{H}^W(f_n)(\lambda)\mathcal{H}^W(g_n)(\lambda)F_\lambda(x)\mathcal{C}_k^W(\lambda)d\lambda. \tag{4.18}$$

On the other hand as the sequence $\{f_n *_{\mathcal{H}^W} g_n\}_{n\in\mathbb{N}}$ converges uniformly to $f *_{\mathcal{H}^W} g$, and the sequence $\{\mathcal{H}^W(f_n)\mathcal{H}^W(g_n)\}_{n\in\mathbb{N}}$ converges to $\mathcal{H}^W(f).\mathcal{H}^W(g)$ in $L^1_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$. Then we obtain (4.17) when n goes to infinity in (4.18). □

Corollary 4.12 *We have*

$$L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W *_{\mathcal{H}^W} L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W = \tilde{\mathcal{H}}^W(L^1_{\mathcal{C}_k^W}(\mathbb{R}^d)^W). \tag{4.19}$$

Proof. From Theorem 4.11, we have

$$L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W *_{\mathcal{H}^W} L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W \subset \tilde{\mathcal{H}}^W(L^1_{\mathcal{C}^W_k}(\mathbb{R}^d)^W).$$

conversely let f be in $\tilde{\mathcal{H}}^W(L^1_{\mathcal{C}^W_k}(\mathbb{R}^d)^W)$. There exists a function h in $L^1_{\mathcal{C}^W_k}(\mathbb{R}^d)^W$ such that $f = \tilde{\mathcal{H}}^W(h)$.

We write h in the following form

$$h = h_1.h_2,$$

with

$$h_1(\lambda) = |h(\lambda)|^{1/2}, \lambda \in \mathbb{R}^d,$$

and

$$h_2(\lambda) = \begin{cases} \dfrac{h(\lambda)}{|h(\lambda)|^{1/2}}, & \text{if } \lambda \in \mathbb{R}^d \backslash \{0\}, \\ 0, & \text{if } \lambda = 0. \end{cases}$$

These functions are in $L^2_{\mathcal{C}^W_k}(\mathbb{R}^d)$. By applying Theorem 4.11 to the functions $(\mathcal{H}^W)^{-1}(h_1)$ and $(\mathcal{H}^W)^{-1}(h_2)$, we obtain

$$(\mathcal{H}^W)^{-1}(h_1) *_{\mathcal{H}^W} (\mathcal{H}^W)^{-1}(h_2) = \tilde{\mathcal{H}}^W(h_1, h_2) = \tilde{\mathcal{H}}^W(h) = f.$$

Thus

$$\tilde{\mathcal{H}}^W)(L^1_{\mathcal{C}^W_k}(\mathbb{R}^d)^W) \subset L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W *_{\mathcal{H}^W} L^2_{\mathcal{A}_l}(\mathbb{R}^d)^W.$$

This completes the proof. □

4.4. The Kunze and Stein's phenomenon for the Heckman-Opdam's theory. The Kunze and Stein's phenomenon has been proved first by R.A.Kunze and E.M.Stein [9] for the harmonic analysis of the 2×2 real unimodular groups.

In this section we shall prove this phenomenon for the Heckman-Opdam's theory.

Theorem 4.13. Let $p \in [1, 2[$. For all function f in $L^p_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and g in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, the function $f *_{\mathcal{H}^W} g$ belongs to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and we have

$$\|f *_{\mathcal{H}^W} g\|_{\mathcal{A}_k,2} \leq C(p)\|f\|_{\mathcal{A}_k,p}\|g\|_{\mathcal{A}_k,2}, \tag{4.20}$$

where $C(p)$ is the constant given in Proposition 4.1.

Proof. Let f, g, φ be in $\mathcal{D}(\mathbb{R}^d)^W$. From (3.31) and Fubini's theorem, we have

$$\int_{\mathbb{R}^d} f *_{\mathcal{H}^W} g(x)\overline{\varphi(x)}\mathcal{A}_k(x)dx$$
$$= \int_{\mathbb{R}^d} f(y)\left(\int_{\mathbb{R}^d} \mathcal{T}_x^W(g)(-y)\overline{\varphi(x)}\mathcal{A}_k(x)dx\right)\mathcal{A}_k(y)dy.$$

But from Theorem 3.16 and the relations (3.41), (3.6) we have

$$\int_{\mathbb{R}^d} \mathcal{T}_x^W(g)(-y)\overline{\varphi(x)}\mathcal{A}_k(x)dx = \int_{\mathbb{R}^d} F_\lambda(-y)\mathcal{H}^W(g)(\lambda)\overline{\mathcal{H}^W(\check{\varphi})(\lambda)}\mathcal{C}_k^W(\lambda)d\lambda.$$

Thus

$$\int_{\mathbb{R}^d} f *_{\mathcal{H}^W} g(x)\overline{\varphi(x)}\mathcal{A}_k(x)dx$$
$$= \int_{\mathbb{R}^d} f(y)F_\lambda(-y)\Big(\int_{\mathbb{R}^d} \mathcal{H}^W(g)(\lambda)\overline{\mathcal{H}^W(\check{\varphi}(\lambda)}\mathcal{C}_k^W(\lambda)d\lambda\Big)\mathcal{A}_k(y)dy.$$

By applying Theorem 3.16, Hölder's inequality and Proposition 4.1 to the second member, we obtain

$$\left|\int_{\mathbb{R}^d} f *_{\mathcal{H}^W} g(x)\overline{\varphi(x)}\mathcal{A}_k(x)dx\right| \leq C(p)\|g\|_{\mathcal{A}_k,2}\|\varphi\|_{\mathcal{A}_k,2}\|f\|_{\mathcal{A}_k,p}. \quad (4.21)$$

This inequality remains true for the functions g and φ in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)$ and for all function f in $L^p_{\mathcal{A}_k}(\mathbb{R}^d)^W$. Then we obtain (4.20) from (4.21). □

THEOREM 4.14. *Let $p \in [1, 2[$ and q the conjugate exponent of p. Then for all functions f and g in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$, the function $f *_{\mathcal{H}^W} g$ belongs to $L^q_{\mathcal{A}_k}(\mathbb{R}^d)$, and we have*

$$\|f *_{\mathcal{H}^W} g\|_{\mathcal{A}_k,q} \leq C(p)\|f\|_{\mathcal{A}_k,2}\|g\|_{\mathcal{A}_k,2} \quad (4.22)$$

where $C(p)$ is the constant given in Proposition 4.1.

Proof. Let f, g be in $\mathcal{D}(\mathbb{R}^d)^W$ and φ in $L^p_{\mathcal{A}_k}(\mathbb{R}^d)^W$. From (3.33), (3.40), 3.20) and Fubini's theorem we have

$$\int_{\mathbb{R}^d} f *_{\mathcal{H}^W} g(x)\overline{\varphi(x)}\mathcal{A}_k(x)dx = \int_{\mathbb{R}^d} \check{f}(x)(g *_{\mathcal{H}^W} \check{\bar{\varphi}})(x)\mathcal{A}_k(x)dx.$$

By applying to the second member the same method used to obtain (4.21) we get

$$\left|\int_{\mathbb{R}^d} f *_{\mathcal{H}^W} g(x)\overline{\varphi(x)}\mathcal{A}_k(x)dx\right| \leq C(p)\|f\|_{\mathcal{A}_k,2}\|g\|_{\mathcal{A}_k,2}\|\varphi\|_{\mathcal{A}_k,p}.$$

Thus this inequality implies (4.22). □

5. Paley-Winer's theorem for the Hypergeometric Fourier transform on the $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ space

In this section we describe a class of holomorphe functions which arise in this manner. This class consists of all functions f of the form

$$\forall\, \lambda \in \mathbb{C}^d, f(\lambda) = \int_{\mathbb{R}^d} h(x) F_\lambda(-x) \mathcal{A}_k(x) dx, \tag{5.1}$$

where h belongs to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ ant its support is contained in the closed ball $B(0, a)$ of center 0 and radius $a > 0$. We can write also (5.1) in the form

$$\forall\, \lambda \in \mathbb{C}^d, f(\lambda) = \mathcal{H}^W(h)(\lambda). \tag{5.2}$$

From the relation (5.1) and the derivation theorem under the integral sign, the function f is entire on $\mathbb{C}^d$ and satisfies the growth condition

$$\forall\, \lambda \in \mathbb{C}^d, |f(\lambda)| \leq const.e^{a\|Im\lambda\|}. \tag{5.3}$$

Then every function f of the form (5.1) is an entire function on $\mathbb{C}^d$ which satisfies (5.3) and by Theorem 4.2 its restriction to $\mathbb{R}^d$ lies in $L^2_{\mathcal{C}_k^W}(\mathbb{R}^d)^W$.

It is remarkable fact that the converse of the previous result is true. This is the content of the following theorem called Paley-Wiener's theorem for the Hypergeometric Fourier transform $\mathcal{H}^W$ on the $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ space.

THEOREM 5.1. *Let f be an entire function on $\mathbb{C}^d$ satisfying the conditions*

i) *We have*

$$\forall\, \lambda \in \mathbb{C}^d, |f(\lambda)| \leq const.e^{a\|Im\lambda\|}, a > 0. \tag{5.4}$$

ii) *The restriction $f_{|\mathbb{R}^d}$ of f to $\mathbb{R}^d$, belongs to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$. Then there exists a function h in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)$ with support in the closed ball $B(0, a)$ such that*

$$\forall\, \lambda \in \mathbb{C}^d, f(\lambda) = \int_{\mathbb{R}^d} h(x) F_\lambda(-x) \mathcal{A}_k(x) dx. \tag{5.5}$$

To prove this theorem we need the following lemma.

LEMMA 5.2. *Let f be in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and g in $\mathcal{D}(\mathbb{R}^d)^W$. Then*

i) *The function* $f *_{\mathcal{H}^W} g$ *is continuous on* $\mathbb{R}^d$, *belongs to* $L^\infty_{\mathcal{A}_k}(\mathbb{R}^d)^W \cap L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ *and we have*

$$\|f *_{\mathcal{H}^W} g\|_{\mathcal{A}_k,2} \leq |W|^{1/2}\|f\|_{\mathcal{A}_k,2}\|g\|_{\mathcal{A}_k,1}. \tag{5.6}$$

ii) *We have*

$$\mathcal{H}^W(f *_{\mathcal{H}^W} g)(\lambda) = \mathcal{H}^W(f)(\lambda).\mathcal{H}^W(g)(\lambda), a.e.\lambda \in \mathbb{R}^d. \tag{5.7}$$

Proof. i) From Theorem 4.11 the function $f *_{\mathcal{H}^W} g$ is continuous on $\mathbb{R}^d$ and belongs to $L^\infty_{\mathcal{A}_k}(\mathbb{R}^d)^W$, and from Theorem 4.9 it is in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and satisfies the relation (5.6).

ii) Let $\{f_n\|_{n\in\mathbb{N}}$ be the sequence in $\mathcal{D}(\mathbb{R}^d)^W$ which converges to f in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$. From (3.34) we have

$$\forall\, \lambda \in \mathbb{R}^d, \mathcal{H}^W(f_n *_{\mathcal{H}^W} g) = \mathcal{H}^W(f_n)(\lambda).\mathcal{H}^W(g)(\lambda). \tag{5.8}$$

By using Theorem 4.2 and the relation (5.6), the sequence $\{\mathcal{H}^W(f_n *_{\mathcal{H}^W} g)\}_{n\in\mathbb{N}}$ converges to $\mathcal{H}^W(f *_{\mathcal{H}^W} g)$ in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$.
Theorem 4.2 implies also that the sequence $\{\mathcal{H}^W(f_n).\mathcal{H}^W(g)\}_{n\in\mathbb{N}}$ converges to $\mathcal{H}^W(f).\mathcal{H}^W(g)$ in $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$.
Thus we deduce (5.7) from the relation (5.8). □

Proof of Theorem 5.1

We consider the function h given by

$$h(x) = (\mathcal{H}^W)^{-1}(f_{/\mathbb{R}^d})(x), \quad x \in \mathbb{R}^d, \tag{5.9}$$

From Theorem 4.2 it belongs to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$.
Let $\varphi_\varepsilon, \varepsilon > 0$, be the function defined by (5.9). From Lemma 5.2 the function $h *_{\mathcal{H}^W} \varphi_\varepsilon$ is continuous on $\mathbb{R}^d$, belongs to $L^2_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and we have

$$\mathcal{H}^W(h *_{\mathcal{H}^W} \varphi_\varepsilon)(\lambda) = \mathcal{H}^W(h)(\lambda)\mathcal{H}^W(\varphi_\varepsilon)(\lambda), \quad a.e.\ \lambda \in \mathbb{R}^d.$$

Thus by using (5.9) we obtain

$$\mathcal{H}^W(h *_{\mathcal{H}^W} \varphi_\varepsilon)(\lambda) = f_{/\mathbb{R}^d}(\lambda).\mathcal{H}^W(\varphi_\varepsilon)(\lambda), \quad a.e.\ \lambda \in \mathbb{R}^d. \tag{5.10}$$

As the function φ_ε is in $\mathcal{D}(\mathbb{R}^d)^W$, then from Theorem 3.4 ii) the function $\mathcal{H}^W(\varphi_\varepsilon)$ is entire on $\mathbb{C}^d$ and satisfies

$$\forall\, \ell \in \mathbb{N}, \exists\, C^0_\ell > 0, \forall\, \lambda \in \mathbb{C}^d, |\mathcal{H}^W(\varphi_\varepsilon)(\lambda)| \leq C^0_\ell(1 + \|\lambda\|)^{-\ell} e^{\varepsilon\|Im\lambda\|}. \tag{5.11}$$

By using (5.11) and the fact that the function f is entire on $\mathbb{C}^d$ and satisfies the condition (5.4), we deduce that the function $f.\mathcal{H}^W(\varphi_\varepsilon)$ is entire on $\mathbb{C}^d$ verifying

$$\forall\, \ell \in \mathbb{N}, \exists\, C_\ell > 0, \forall\, \lambda \in \mathbb{C}^d, |f(\lambda).\mathcal{H}^W(\varphi_\varepsilon)(\lambda)| \leq C_\ell(1+\|\lambda\|)^{-\ell} e^{(a+\varepsilon)\|Im\lambda\|}$$

thus from Theorem 3.4 ii) we deduce that the function $f.\mathcal{H}^W(\varphi_\varepsilon)$ is the hypergeometric Fourier transform of a function in $\mathcal{D}(\mathbb{R}^d)^W$ which has its support contained in the closed ball $B(0, a+\varepsilon)$ of center 0 and radius $(a+\varepsilon)$. Then the relation (5.10) implies

$$\text{supp} h *_{\mathcal{H}^W} \varphi_\varepsilon \subset B(0, a+\varepsilon). \tag{5.12}$$

On the other hand from Theorem 4.9, we have

$$\lim_{\varepsilon \to 0} \|h *_{\mathcal{H}^W} \varphi_\varepsilon - h\|_{\mathcal{A}_k,2} = 0, \tag{5.13}$$

this relation and (5.12) imply that

$$\text{supp} h \subset B(0, a). \tag{5.14}$$

On the other hand from the relation (5.13) and Theorem 4.2, we have

$$\lim_{\varepsilon \to 0} \|\mathcal{H}^W(h *_{\mathcal{H}^W} \varphi_\varepsilon) - \mathcal{H}^W(h)\|_{\mathcal{A}_k,2} = 0. \tag{5.15}$$

By using this relation and (5.9), we obtain

$$\lim_{\varepsilon \to 0} \|\mathcal{H}^W(h *_{\mathcal{H}^W} \varphi_\varepsilon) - f_{|\mathbb{R}^d}\|_{\mathcal{A}_k,2} = 0. \tag{5.16}$$

Thus from the relations (5.14), (5.15), (5.16), we deduce that

$$f_{/\mathbb{R}^d}(\lambda) = \int_{\mathbb{R}^d} h(x) F_\lambda(-x) \mathcal{A}_k(x) dx, a.e. \quad \lambda \in \mathbb{R}^d\ ,$$

as the two members are entire on $\mathbb{C}^d$, then we obtain (5.5).

6. The harmonic analysis associated to the Heckman-Opdam's theory attached to a root system of type BC_d

The root system on $\mathbb{R}^d$ of type BC_d can be identified with the set $\mathcal{R}$ given by

$$\mathcal{R} = \{\ \pm 2e_i,\ \pm 4e_i,\ 1 \leq i \leq d\} \cup \{\ 2(\pm e_i \pm e_j),\ 1 \leq i < j \leq d\}. \tag{6.1}$$

We denote by $\mathcal{R}_+$ the set of positive roots

$$\mathcal{R}_+ = \{\ 2e_i,\ 4e_i,\ 1 \leq i \leq d\} \cup \{\ 2(e_i \pm e_j),\ 1 \leq i < j \leq d\}. \tag{6.2}$$

The Weyl group associated with $\mathcal{R}$ is isomorphic to the hyperoctahedral group which is generated by permutations and sign changes of the $e_i,\ i = 1, 2, ..., d$.

The multiplicity function $k : \mathcal{R} \longrightarrow [0, +\infty[$ can be written in the form $k = (k_1, k_2, k_3)$ where k_1 and k_2 are the values on the roots $\pm 2e_i$ and $\pm 4e_i,\ 1 \le i \le d$, respectively, and k_3 is the value on the roots $2(\pm e_i \pm e_j),\ 1 \le i < j \le d$.

The positive Weyl chamber denoted by $\mathfrak{a}^+$ is given by

$$\mathfrak{a}^+ = \{x = (x_1, x_2, ..., x_d) \in \mathbb{R}^d,\ x_1 > x_2 > ... > x_d > 0\}. \tag{6.3}$$

The closed chamber $\overline{\mathfrak{a}^+}$ corresponds to the set

$$C = \{x = (x_1, x_2, ..., x_d) \in \mathbb{R}^d,\ x_1 \ge x_2 \ge ... \ge x_d \ge 0\}. \tag{6.4}$$

One of the indefinite orthogonal, unitary or symplectic groups $SO_o(p, d)$, $SU(p, d)$ or $S_p(p, d)$ with maximal compact subgroup $K = SO(p) \times SO(d)$, $S(U(p) \times U(d))$ or $S_p(p) \times S_p(d)$, respectively (see [17]).

Let $p \in \mathbb{N}$ such that $p > d \ge 1,\ d_0 = 1, 2, 4$ and $\mu \in \mathbb{R}$ such that $\mu > \gamma_0 - 1$ with $\gamma_0 = d_0(d - \frac{1}{2}) + 1$.

We consider in this section the root system of type BC_d corresponding to the multiplicity function $k_\mu = (k_1, k_2, k_3)$ with $k_1 = \mu - \frac{d-d_0}{2}$, $k_2 = \frac{d_0-1}{2}$, $k_3 = \frac{d_0}{2}$.

We denote by $F^\mu_{BC}(\lambda, x)$ the Heckman-Opdam's hypergeometric function associated to this root system.

Remark 6.1. For $\mu = p\frac{d_0}{2}$, the function $F^\mu_{BC}(\lambda, x)$ can be identified with the spherical function on the Grasmann manifolds $G|K$ where G is one of the indefinite orthogonal, unitary or symplectic groups $SO_0(p, d),\ SU(p, d)$ or $S_p(p, d)$ with maximal compact subgroup $K = SO(p) \times SO(d),\ S(U(p) \times U(d))$ or $S_p(p) \times S_p(d)$, respectively (see [17]).

6.1. The product formula for the function F^μ_{BC}.

Notation. We denote by $\Sigma = \mathbb{R}^d + iconv(W.\rho)$, where $conv(W.\nu)$ is the convex hull of the Weyl group orbit $W.\rho$, and by $\overset{o}{\Sigma}$ the interior of Σ.

In [17] p.2789-2792, the author has proved that the function $F^\mu_{BC}(\lambda, x)$, $\lambda \in \mathbb{C}^d,\ x \in \mathbb{R}^d$, admits the following product formula

$$\forall x, y \in \mathbb{R}^d,\quad F^\mu_{BC}(\lambda, x).F^\mu_{BC}(\lambda, y) = \int_{\mathbb{R}^d} F^\mu_{BC}(\lambda, z)\, dm^\mu_{x,y}(z), \tag{6.5}$$

where $m^\mu_{x,y}$ is a probability measure on $\mathbb{R}^d$.

THEOREM 6.2. *For all $\lambda \in \Sigma$ we have*

$$\sup_{x\in\mathbb{R}^d} |F^\mu_{BC}(\lambda, x)| = 1. \tag{6.6}$$

Proof. In [17] p.2798-2799, the author has proved that for all $\lambda \in \Sigma$ the function $x \to F^\mu_{BC}(\lambda, x)$ is bounded.
On the other hand as for all $x, y \in \mathbb{R}^d$, the measure $m^\mu_{x,y}$ is a probability measure, then from the relation (6.5), for all $\lambda \in \Sigma$, we obtain

$$\forall x, y \in \mathbb{R}^d, \quad |F^\mu_{BC}(\lambda, x)||F^\mu_{BC}(\lambda, y)| \leq \sup_{z\in\mathbb{R}^d} |F^\mu_{BC}(\lambda, z)|.$$

We deduce (6.6) from this inequality and the fact that $F^\mu_{BC}(\lambda, 0) = 1$. □

REMARKS 6.3.

1. The function $x \longrightarrow F^\mu_{BC}(\lambda, x)$ is unbounded for $\lambda \notin \Sigma$. (see [17, Corollary 5.6]).

2. All the results of the previous five sections remain true for the root system of type BC_d considered in this section.

The hypergeometric translation operator defined by the relation (3.19), will be denoted by $\mathcal{T}^{W,\mu}_x$, $x \in \mathbb{R}^d$.

By using the relations (6.5), (3.21), the operator $\mathcal{T}^{W,\mu}_x$, $x \in \mathbb{R}^d$, possesses the following integral representation

$$\forall y \in \mathbb{R}^d, \quad \mathcal{T}^{W,\mu}_x(f)(y) = \int_{\mathbb{R}^d} f(z) dm^\mu_{x,y}(z), \quad f \in \mathcal{E}(\mathbb{R}^d)^W, \tag{6.7}$$

and then the relation (6.5) can also be written in the form

$$\forall x, y \in \mathbb{R}^d, \quad \tau^{W,\mu}_x(F^\mu_{BC})(y) = F^\mu_{BC}(x) F^\mu_{BC}(y). \tag{6.8}$$

PROPOSITION 6.4.

i) We have

$$m^\mu_{x,0} = \delta_x \quad \text{and} \quad m^\mu_{0,y} = \delta_y, \tag{6.9}$$

where δ_z is the Dirac measure at $z \in \mathbb{R}^d$.

ii) For all $x, y \in \mathbb{R}^d$, we have

$$supp\ m^\mu_{x,y} \subset \{z \in \mathbb{R}^d,\ ||z|| \leq ||x|| + ||y||\}. \tag{6.10}$$

Proof. i) We deduce the results from the relation (6.7),(3.20).

ii) The relation (6.10) is given in [17] p.2794.

□

6.2. Absolute continuity of the measure $m^{\mu}_{x,y}$.

NOTATIONS. We denote by
- $B(c,a)$ the open ball of $\mathbb{R}^d$ of center $c \in \mathbb{R}^d$ and radius $a > 0$, and by $\overline{B}(c,a)$ its closure.
- Λ the Lebesgue measure on $\mathbb{R}^d$.

In this subsection we prove that for all $x, y \in \mathbb{R}^d_{reg}$, the measure $m^{\mu}_{x,y}$ is absolute continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.

PROPOSITION 6.5. Let $t_0 \in \mathbb{R}^d$ and $a > 0$, We consider the sequence $\{f_n\}_{n\in\mathbb{N}\backslash\{0\}}$ of functions in $\mathcal{D}(\mathbb{R}^d)^W$, positive increasing such that

$$\forall n \in \mathbb{N}\backslash\{0\},\ supp\ f_n \subset \overline{B}(t_0, a - \frac{1}{n}),\ f_n(z) = 1,$$

and

$$\forall z \in \mathbb{R}^d,\ \lim_{n\to+\infty} f_n(z) = 1_{B(t_0,a)}(z),$$

where $1_{B(t_0,a)}$ is the characteristic function of the ball $B(t_0,a)$. We have

$$\forall y \in \mathbb{R}^d, \quad \lim_{n\to+\infty} \mathcal{T}^{W,\mu}_x(f_n)(y) = \lim_{n\to+\infty} \int_{\mathbb{R}^d} f_n(z) dm^{\mu}_{x,y}(z),$$

$$= \int_{\mathbb{R}^d} 1_{B(t_0,a)}(z) dm^{\mu}_{x,y}(z).$$

The function $y \longrightarrow m^{\mu}_{x,y}(B(t_0,a)) = \int_{\mathbb{R}^d} 1_{B(t_0,a)}(z) dm^{\mu}_{x,y}(z)$ which can also be denoted by $\mathcal{T}^{W,\mu}_x(1_{B(t_0,a)})(y)$ is defined almost every where on $\mathbb{R}^d$, measurable and for all function h in $\mathcal{D}(\mathbb{R}^d)^W$, we have

$$\int_{\mathbb{R}^d} m^{\mu}_{x,y}(B(t_0,a)) h(y) \mathcal{A}_k(y) dy = \int_{B(t_0,a)} \mathcal{T}^{W,\mu}_x(\check{h})(-z) \mathcal{A}_k(z) dz, \quad (6.11)$$

where $\check{h}$ is the function given by

$$\forall u \in \mathbb{R}^d, \quad \check{h}(u) = h(-u).$$

Proof. For all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}\backslash\{0\}$, the function $\mathcal{T}^{W,\mu}_x(f_n)$ belongs to $\mathcal{D}(\mathbb{R}^d)^W$. Then we obtain the results of this proposition from the monotonic convergence theorem and the relations (3.27),(3.40). □

REMARK 6.6. There exists a $\sigma-algebra$ $\mathfrak{M}$ in $\mathbb{R}^d$ which contains all Borel sets in $\mathbb{R}^d$. Then for all $E \in \mathfrak{M}$, the function $y \longrightarrow m^{\mu}_{x,y}(E)$ is defined almost every where on $\mathbb{R}^d$, measurable and we have the following relation

$$\int_{\mathbb{R}^d} m^{\mu}_{x,y}(E)h(y)\mathcal{A}_k(y)dy = \int_E \mathcal{T}^{W,\mu}_x(\check{h})(-z)\mathcal{A}_k(z)dz, \tag{6.12}$$

PROPOSITION 6.7. *For $x, y \in \mathbb{R}^d_{reg}$, there exists a unique positive function $\Theta^W(x, y, .)$ integrable on $\mathbb{R}^d$ with respect to the Lebesgue measure Λ, and a positive measure $m^{\mu,s}_{x,y}$ on $\mathbb{R}^d$ such that for every Borel set E, we have*

$$m^{\mu}_{x,y}(E) = \int_E \Theta^W(x, y, z)dz + m^{\mu,s}_{x,y}(E). \tag{6.13}$$

Proof. We deduce (6.13) from (6.7) and [18, Theorems 6.9 and 8.6]. □

REMARKS 6.8.

i) The supports of the function $z \longrightarrow \Theta(x, y, z)$ and the measure $m^{\mu,s}_{x,y}$, are contained in the set $\{z \in \mathbb{R}^d,\ ||z|| \leq ||x|| + ||y||\}$.

ii) Suppose ν_1 and ν_2 are measures on $\mathfrak{M}$ and suppose that there exists a pair of disjoint sets X_1 and X_2 such that ν_1 is concentrated on X_1 and ν_2 concentrated on X_2. Then we say that ν_1 and ν_2 are mutually singular (see Definition 6.7 of [18] p.128). In our case the measures $m^{\mu,s}_{x,y}$ and the Lebesgue measure Λ are mutually singular.

iii) From [18, Theorem 8.6 and Definition 8.3], we have

$$\Theta^W(x, y, z) = \lim_{a\to 0} \frac{m^{\mu}_{x,y}(B(z, a))}{\Lambda(B(z, a))}. \tag{6.14}$$

PROPOSITION 6.9. *We consider $x \in \mathbb{R}^d_{reg}$ and a positive function h in $\mathcal{D}(\mathbb{R}^d)^W$ with support contained in the ball $\overline{B}(0, R),\ R > 0$.*

i) *For all Borel set E, we have*

$$\int_E \mathcal{N}^h_x(z)dz = \int_{\overline{B}(0,R)} h(y)m^{\mu,s}_{x,y}(E)\mathcal{A}_k(y)dy, \tag{6.15}$$

where

$$\mathcal{N}^h_x(z) = \mathcal{T}^{W,\mu}_x(\check{h})(-z)\mathcal{A}_k(z) - \int_{\overline{B}(0,R)} \Theta^W(x, y, z)h(y)\mathcal{A}_k(y)dy. \tag{6.16}$$

ii) *We have*

$$\forall z \in \mathbb{R}^d, \quad \mathcal{N}_x^h(z) \geq 0. \tag{6.17}$$

Proof. i) By using the relations (6.12),(6.13) we obtain

$$\begin{aligned}
&\int_E \mathcal{T}_x^{W,\mu}(\check{h})(-z)\mathcal{A}_k(z)dz \\
&= \int_{\overline{B}(0,R)} m_{x,y}^{\mu}(E)h(y)\mathcal{A}_k(y)dy \\
&= \int_{\overline{B}(0,R)} [\int_E \Theta^W(x,y,z)h(y)dz + m_{x,y}^{\mu,s}(E)]h(y)\mathcal{A}_k(y)dy.
\end{aligned}$$

We deduce (6.15) by applying Fubini-Tonelli's theorem to the last member.

ii) From the relation (6.15), the positivity of the measure $m_{x,y}^{\mu,s}$ implies that for all Borel sets E, we have

$$\int_E \mathcal{N}_x^h(z)dz \geq 0.$$

Thus

$$\forall z \in \mathbb{R}^d, \quad \mathcal{N}_x^h(z) \geq 0.$$

□

Proposition 6.10. *The measure η_x^h on $\mathbb{R}^d$ given for all Borel sets E by*

$$\eta_x^h(E) = \int_E \mathcal{N}_x^h(z)dz \tag{6.18}$$

is positive and bounded.

Proof. - The relation (6.17) gives the positivity of the measure η_x^h.
- From the relations (6.18),(6.15), for all Borel sets E we have

$$\eta_x^h(E) \leq \int_{\overline{B}(0,R)} ||m_{x,y}^{\mu,s}||h(y)\mathcal{A}_k(y)dy. \tag{6.19}$$

On the other hand by using (6.13), we obtain for all $y \in \mathbb{R}^d_{reg}$:

$$m_{x,y}^{\mu,s}(E) \leq m_{x,y}^{\mu}(E).$$

Thus

$$||m_{x,y}^{\mu,s}|| \leq ||m_{x,y}^{\mu}|| = 1.$$

By using this result, the relation (6.19) implies that for all Borel sets E, we have

$$\eta_x^h(E) \leq M_h,$$

where

$$M_h = \int_{\overline{B}(0,R)} h(y)\mathcal{A}_k(y)dy.$$

Thus the measure η_x^h is bounded. □

PROPOSITION 6.11.

i) For all Borel sets E we have

$$\eta_x^h(E) = 0. \tag{6.20}$$

ii) For $x, z \in \mathbb{R}^d_{reg}$, we have

$$\mathcal{T}_x^{W,\mu}(\check{h})(-z) = \frac{1}{\mathcal{A}_k(z)} \int_{\overline{B}(0,R)} \Theta^W(x,y,z)h(y)\mathcal{A}_k(y)dy. \tag{6.21}$$

Proof. i) From the relations (6.18), (6.15), for all Borel set E the measure η_x^h possesses also the following form

$$\eta_x^h(E) = \int_{\overline{B}(0,R)} m_{x,y}^{\mu,s}(E)h(y)\mathcal{A}_k(y)dy. \tag{6.22}$$

On the other hand from Proposition 6.10 the measure η_x^h is absolute continuous with respect to the Lebesgue measure Λ and from Remark 6.8 ii) the measure $m_{x,y}^{\mu,s}$ and the Lebesgue measure Λ are mutually singular.
Then from [18, Proposition 6.8 (f)], the measures η_x^h and $m_{x,y}^{\mu,s}$, are mutually singular. We deduce (6.20) from (6.22) and Remark 6.8 i).

ii) By using the i) and (6.18), (6.16), we get

$$\mathcal{T}_x^{W,\mu}(\check{h})(-z)\mathcal{A}_k(z)dz = \int_{\overline{B}(0,R)} \Theta^W(x,y,z)h(y)\mathcal{A}_k(y)dy.$$

As

$$z \in \mathbb{R}^d_{reg} \Longleftrightarrow \mathcal{A}_k(z) \neq 0,$$

then we deduce (6.21) from this relation.

□

Theorem 6.12. *For all f in $\mathcal{E}(\mathbb{R}^d)^W$ and $x, y \in \mathbb{R}^d_{reg}$ we have*

$$\mathcal{T}_x^{W,\mu}(f)(y) = \int_{\mathbb{R}^d} f(z)\mathcal{W}^W(x, y, z)\mathcal{A}_k(z)dz, \tag{6.23}$$

where

$$\mathcal{W}^W(x, y, z) = \frac{\Theta^W(x, -z, -y)}{\mathcal{A}_k(y)}. \tag{6.24}$$

Proof. We obtain (6.23),(6.24), by writing $f = f^+ - f^-$ and by using the relation (6.21) and the properties of the operator $\mathcal{T}_x^{W,\mu}$. □

Remark 6.13. Theorem 6.12 shows that for all $x, y \in \mathbb{R}^d_{reg}$, the measure $m^\mu_{x,y}$ is absolute continuous with respect to the measure $\mathcal{A}_k(z)dz$. More precisely for all $z \in \mathbb{R}^d$, we have

$$dm^\mu_{x,y}(z) = \mathcal{W}^W(x, y, z)\mathcal{A}_k(z)dz. \tag{6.25}$$

Corollary 6.14

i) *For all $\lambda \in \mathbb{C}^d$ and $x, y \in \mathbb{R}^d_{reg}$, we have*

$$F_\lambda(x)F_\lambda(y) = \int_{\mathbb{R}^d} F_\lambda(z)\mathcal{W}^W(x, y, z)\mathcal{A}_k(z)dz. \tag{6.26}$$

ii) *For all $x, y \in \mathbb{R}^d_{reg}$, we have*

$$\int_{\mathbb{R}^d} \mathcal{W}^W(x, y, z)\mathcal{A}_k(z)dz = 1. \tag{6.27}$$

iii) *For all $x, y \in \mathbb{R}^d_{reg}$, the support of the function $z \longrightarrow \mathcal{W}^W(x, y, z)$ is contained in the set $\{z \in \mathbb{R}^d, ||z|| \leq ||x|| + ||y||\}$.*

Proof. We deduce the results of this Corollary from (6.5),(6.25) and Theorem 6.12. □

Corollary 6.15.

i) *We have*

$$\forall x, y, z \in \mathbb{R}^d, \quad \mathcal{W}^W(x, y, z) = \mathcal{W}^W(y, x, z). \tag{6.28}$$

ii) *We have*

$$\forall x, y, z \in \mathbb{R}^d, \mathcal{W}^W(x, -y, z)\mathcal{A}_k(y)\mathcal{A}_k(z)dzdy$$
$$= \mathcal{W}^W(x, -z, y)\mathcal{A}_k(y)\mathcal{A}_k(z)dzdy. \tag{6.29}$$

Proof. i) We deduce the result from the relations (6.23),(3.20).

ii) Let f, g be in $\mathcal{D}(\mathbb{R}^d)^W$. From Corollary 3.12.i) we have

$$\forall x \in \mathbb{R}^d, \quad f *^{\mu}_{\mathcal{H}^W} g(x) = g *^{\mu}_{\mathcal{H}^W} f(x).$$

Then be using (3.31),(6.23) we obtain for all $x \in \mathbb{R}^d$:

$$\int_{\mathbb{R}^d} \mathcal{T}^{W,\mu}_x(f)(-y)g(y)\mathcal{A}_k(y)dy = \int_{\mathbb{R}^d} \mathcal{T}^{W,\mu}_x(g)(-z)f(z)\mathcal{A}_k(z)dz,$$

thus

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathcal{T}^{W,\mu}_x(f)(-y)g(y)\mathcal{A}_k(y)\mathcal{A}_k(z)dzdy =$$

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathcal{T}^{W,\mu}_x(g)(-z)f(z)\mathcal{A}_k(z)\mathcal{A}_k(y)dydz.$$

We deduce (6.29) from this relation and (6.23).

□

6.3. The hypergeometric convolution product on W-invariant $L^p_{\mathcal{A}_k}$-spaces.

PROPOSITION 6.16. *The operator $\mathcal{T}^{W,\mu}_x, x \in \mathbb{R}^d$, is bounded on $L^p_{\mathcal{A}_k}(\mathbb{R}^d)^W$, $1 \leq p \leq +\infty$, and we have*

$$||\mathcal{T}^{W,\mu}_x(f)||_{\mathcal{A}_k,p} \leq ||f||_{\mathcal{A}_k,p}. \tag{6.30}$$

Proof. From the relation (6.23), for all f in $\mathcal{D}(\mathbb{R}^d)^W$ we have

$$\forall y \in \mathbb{R}^d, \quad \mathcal{T}^{W,\mu}_x(f)(y) = \int_{\mathbb{R}^d} f(z)\mathcal{W}^W(x,y,z)\mathcal{A}_k(z)dz.$$

As the function equal to 1 belongs to the space $L^q(\mathbb{R}^d, \mathcal{W}^W(x,y,z)\mathcal{A}_k(z)dz)$, $1 \leq q \leq +\infty$, then Hölder's inequality and the relation (6.27) imply

$$\forall y \in \mathbb{R}^d, \quad |\mathcal{T}^{W,\mu}_x(f)(y)|^p \leq \int_{\mathbb{R}^d} |f(z)|^p\mathcal{W}^W(x,y,z)\mathcal{A}_k(z)dz.$$

Thus

$$\begin{aligned}\int_{\mathbb{R}^d} |\mathcal{T}^{W,\mu}_x(f)(y)|^p\mathcal{A}_k(y)dy &\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(z)|^p\mathcal{W}^W(x,y,z)\mathcal{A}_k(z)\mathcal{A}_k(y)dzdy \\ &\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(z)|^p\mathcal{W}^W(x,-y,z)\mathcal{A}_k(z)\mathcal{A}_k(y)dzdy.\end{aligned}$$

By applying the relation (6.29), Fubini-Tonelli's theorem and the relation (6.27) to the second member, we obtain

$$\int_{\mathbb{R}^d} |\mathcal{T}_x^{W,\mu}(f)(y)|^p \mathcal{A}_k(y)dy \leq \int_{\mathbb{R}^d} |f(z)|^p \mathcal{A}_k(z)dz.$$

This completes the proof of the relation (6.30). □

THEOREM 6.17. *Let f, g be in $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$. Then*

i) *The function $f *^{\mu}_{\mathcal{H}^W} g$ defined almost every where by*

$$f *^{\mu}_{\mathcal{H}^W} g(x) = \int_{\mathbb{R}^d} \mathcal{T}_x^{W,\mu}(f)(-y)g(y)\mathcal{A}_k(y)dy, \quad x \in \mathbb{R}^d, \tag{6.31}$$

belongs to $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and we have

$$||f *^{\mu}_{\mathcal{H}^W} g||_{\mathcal{A}_k,1} \leq ||f||_{\mathcal{A}_k,1}||g||_{\mathcal{A}_k,1}. \tag{6.32}$$

ii) *We have*

$$\forall \lambda \in \mathbb{R}^d, \quad \mathcal{H}^{W,\mu}(f *^{\mu}_{\mathcal{H}^W} g)(\lambda) = \mathcal{H}^{W,\mu}(f)(\lambda).\mathcal{H}^{W,\mu}(g)(\lambda). \tag{6.33}$$

Proof. i) From Fubini-Tonelli's theorem and (6.30) we have

$$\int_{\mathbb{R}^d}[\int_{\mathbb{R}^d} |\mathcal{T}_x^{W,\mu}(f)(-y)|\mathcal{A}_k(x)dx]|g(y)|\mathcal{A}_k(y)dy$$
$$\leq \int_{\mathbb{R}^d} ||f||_{\mathcal{A}_k,1}|g(y)|\mathcal{A}_k(y)dy \leq ||f||_{\mathcal{A}_k,1}||g||_{\mathcal{A}_k,1}.$$

Thus the function $(x, y) \longrightarrow \mathcal{T}_x^{W,\mu}(f)(-y)g(y)$ belongs to $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$. Then from Fubini's theorem for almost all $x \in \mathbb{R}^d$, the function $y \longrightarrow \mathcal{T}_x^{W,\mu}(f)(-y)g(y)$ is in $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and the function $f *^{\mu}_{\mathcal{H}^W} g$ defined by (6.31) belongs to $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$.
On the other hand we have

$$|f *^{\mu}_{\mathcal{H}^W} g|(x) \leq \int_{\mathbb{R}^d} |\mathcal{T}_x^{W,\mu}(f)(-y)||g(y)|\mathcal{A}_k(y)dy.$$

Thus

$$\int_{\mathbb{R}^d} |f *^{\mu}_{\mathcal{H}^W} g(x)|\mathcal{A}_k(x)dx \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\mathcal{T}_x^{W,\mu}(f)(-y)||g(y)|\mathcal{A}_k(y)\mathcal{A}_k(x)dydx.$$

By applying Fubini's theorem and the relation (6.30), to the second member, we obtain (6.32).

ii) For all $\lambda \in \mathbb{R}^d$ we have

$$\mathcal{H}^{W,\mu}(f *^{\mu}_{\mathcal{H}^W} g)(\lambda) = \int_{\mathbb{R}^d} f *^{\mu}_{\mathcal{H}^W} g(x)F^{\mu}_{BC}(-\lambda, x)\mathcal{A}_k(x)dx.$$

By using Fubini's theorem and (6.31) we obtain
$\forall\lambda \in \mathbb{R}^d, \quad \mathcal{H}^{W,\mu}(f *^{\mu}_{\mathcal{H}^W} g)(\lambda)$

$$= \int_{\mathbb{R}^d} [\int_{\mathbb{R}^d} \mathcal{T}^{W,\mu}_x(f)(-y) F^{\mu}_{BC}(-\lambda, x)\mathcal{A}_k(x)dx] g(y)\mathcal{A}_k(y)dy. \qquad (6.34)$$

But by using (3.20), (3.27),(3.40),(6.8) we obtain

$$\int_{\mathbb{R}^d} \mathcal{T}^{W,\mu}_x(f)(-y) F^{\mu}_{BC}(-\lambda, x)\mathcal{A}_k(x)dx$$

$$\begin{aligned} &= \int_{\mathbb{R}^d} \mathcal{T}^{W,\mu}_{-y}(f)(x) F^{\mu}_{BC}(-\lambda, x)\mathcal{A}_k(x)dx \\ &= \int_{\mathbb{R}^d} f(x)^t\mathcal{T}^{W,\mu}_{-y}(F^{\mu}_{BC}(-\lambda, .))(x)\mathcal{A}_k(x)dx \\ &= \int_{\mathbb{R}^d} f(x)\mathcal{T}^{W,\mu}_{-y}(\breve{F}^{\mu}_{BC}(-\lambda, .))(-x)\mathcal{A}_k(x)dx \\ &= \int_{\mathbb{R}^d} f(x)\breve{F}^{\mu}_{BC}(-\lambda, -x)\breve{F}^{\mu}_{BC}(-\lambda, -y)\mathcal{A}_k(x)dx \\ &= F^{\mu}_{BC}(-\lambda, y)\int_{\mathbb{R}^d} f(x) F^{\mu}_{BC}(-\lambda, x)\mathcal{A}_k(x)dx. \end{aligned}$$

Thus

$$\int_{\mathbb{R}^d} \mathcal{T}^{W,\mu}_x(f)(-y) F^{\mu}_{BC}(-\lambda, x)\mathcal{A}_k(x)dx = F^{\mu}_{BC}(-\lambda, y)\mathcal{H}^{W,\mu}(f)(\lambda). \quad (6.35)$$

The relations (6.34), (6.35) imply (6.33). □

REMARK 6.18 From the relation (6.33) we deduce that the hypergeometric convolution product given by (6.31) is commutative and associative

COROLLARY 6.19. *The space $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$ with the hypergeometric convolution product $*^{\mu}_{\mathcal{H}^W}$ is a commutative Banach algebra.*

THEOREM 6.20. *Let f be in $L^p_{\mathcal{A}_k}(\mathbb{R}^d)^W$, $1 < p \leq +\infty$, and g in $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$. Then the function $f *^{\mu}_{\mathcal{H}^W} g$ defined almost everywhere on $\mathbb{R}^d$ by*

$$f *^{\mu}_{\mathcal{H}^W} g(x) = \int_{\mathbb{R}^d} \mathcal{T}^{W,\mu}_x(f)(-y) g(y)\mathcal{A}_k(y)dy, \qquad (6.36)$$

belongs to $L^p_{\mathcal{A}_k}(\mathbb{R}^d)^W$ and we have

$$||f *^{\mu}_{\mathcal{H}^W} g||_{\mathcal{A}_k,p} \leq ||f||_{\mathcal{A}_k,p}||g||_{\mathcal{A}_k,1}. \qquad (6.37)$$

Proof. We obtain the results of this theorem by using the relation (6.30), and by making the same proof as for Theorem 4.9 □

THEOREM 6.21. *Let* φ_ϵ, $\epsilon > 0$, *the function given by the relation (4.12). Then for all* f *in* $L^p_{\mathcal{A}_k}(\mathbb{R}^d)^W$, $1 \leq p < +\infty$, *we have*

$$\lim_{\epsilon \to 0} ||f *^{\mu}_{\mathcal{H}^W} \varphi_\epsilon - f||_{\mathcal{A}_k,p} = 0. \tag{6.38}$$

Proof. The relation (6.30) and the same proof as for Theorem 4.10 imply the relation (6.38). □

THEOREM 6.22. *Let* f *be in* $L^p_{\mathcal{A}_k}(\mathbb{R}^d)^W$, $1 < p \leq +\infty$, *and* g *in* $L^q_{\mathcal{A}_k}(\mathbb{R}^d)^W$ *with* q *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then the function* $f *^{\mu}_{\mathcal{H}^W} g$ *defined on* $\mathbb{R}^d$ *by*

$$f *^{\mu}_{\mathcal{H}^W} g(x) = \int_{\mathbb{R}^d} \mathcal{T}^{W,\mu}_x(f)(-y)g(y)\mathcal{A}_k(y)dy, \tag{6.39}$$

is continuous on $\mathbb{R}^d$, *tends to zero at the infinity and we have*

$$\sup_{x \in \mathbb{R}^d} |f *^{\mu}_{\mathcal{H}^W} g| \leq ||f||_{\mathcal{A}_k,p}||g||_{\mathcal{A}_k,q}. \tag{6.40}$$

Proof. By using the relation (6.30) and by applying the same proof as for Theorem 4.11, we obtain the results of this theorem. □

6.4. The maximal ideal space of the algebra $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$. In this subsection we give the maximal ideal space S of the algebra $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$, and we prove that S is homeomorphic to the set Σ equipped with the usual topology.

THEOREM 6.23. *To each complex homomorphism* $\mathcal{X}$ *of* $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$ *corresponds to a unique element* $\lambda \in \Sigma$ *such that*

$$\forall f \in L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W, \quad \mathcal{X}(f) = \mathcal{H}^{W,\mu}(f)(\lambda). \tag{6.41}$$

To prove this theorem we need the following Lemma.

LEMMA 6.24. *Let* ψ *be a function in* $L^\infty_{\mathcal{A}_k}(\mathbb{R}^d)^W$, *satisfying the relation*

$$\mathcal{T}^{W,\mu}_x(\breve{\psi})(y) = \breve{\psi}(x)\breve{\psi}(y), \quad a.e.\ x, y \in \mathbb{R}^d, \tag{6.42}$$

where $\breve{\psi}$ *is the function given by*

$$\breve{\psi}(x) = \psi(-x), \quad x \in \mathbb{R}^d.$$

Then

i) *The function* $\breve{\psi}$ *is of class* C^∞ *on* $\mathbb{R}^d$.

ii) *We have*

$$\breve{\psi}(0) = 1. \tag{6.43}$$

iii) *There exists a unique* $\lambda \in \Sigma$ *such that*

$$\forall x \in R^d, \quad \breve{\psi}(x) = F^{\mu}_{BC}(\lambda, x). \tag{6.44}$$

Proof. i) We choose h in $\mathcal{D}(\mathbb{R}^d)^W$ satisfying

$$\int_{\mathbb{R}^d} \breve{\psi}(x)h(x)\mathcal{A}_k(x)dx = 1. \tag{6.45}$$

We have for $x \in \mathbb{R}^d$:

$$\breve{\psi} *^{\mu}_{\mathcal{H}^W} h(x) = \int_{\mathbb{R}^d} \mathcal{T}^{W,\mu}_x(\breve{\psi})(-y)h(y)\mathcal{A}_k(y)dy,$$

the relations (6.42),(6.45) imply for $x \in \mathbb{R}^d$:

$$\breve{\psi} *^{\mu}_{\mathcal{H}^W} h(x) = \breve{\psi}(x),$$

thus by using the relations (3.33),(3.40) we obtain

$$\breve{\psi}(x) = \int_{\mathbb{R}^d} \breve{\psi}(y)\mathcal{T}^{W,\mu}_x(h)(y)\mathcal{A}_k(y)dy, \quad x \in \mathbb{R}^d. \tag{6.46}$$

From this relation we deduce that the function $\breve{\psi}$ is of class C^∞ on $\mathbb{R}^d$, because from (3.40),(3.30) for all $x \in \mathbb{R}^d$ the function $\mathcal{T}^{W,\mu}_x(h)$ belongs to $\mathcal{D}(\mathbb{R}^d)^W$.

ii) We obtain (6.43) from (6.42) and the i).

iii) By using (6.46),(3.20), we obtain

$$\mathcal{T}^{W,\mu}_z(\breve{\psi})(x) = \int_{\mathbb{R}^d} \breve{\psi}(y)\mathcal{T}^{W,\mu}_z(\mathcal{T}^{W,\mu}_y(h))(x)\mathcal{A}_k(y)dy. \tag{6.47}$$

For all operator $p(T) = p(T_1, T_2, ..., T_d)$, where p is a W-invariant polynomial on $\mathbb{C}^d$, the relations (3.22), (3.23) imply

$$\begin{aligned} p(T)_z\mathcal{T}^{W,\mu}_z(\breve{\psi})(x) &= \int_{\mathbb{R}^d} \breve{\psi}(y)p(T)_z\mathcal{T}^{W,\mu}_z(\mathcal{T}^{W,\mu}_y(h))(y)\mathcal{A}_k(y)dy, \\ &= \int_{\mathbb{R}^d} \breve{\psi}(y)p(T)_x\mathcal{T}^{W,\mu}_z(\mathcal{T}^{W,\mu}_y(h))(y)\mathcal{A}_k(y)dy. \end{aligned}$$

Thus by using the relation (6.47) we get

$$p(T)_z\mathcal{T}_z^{W,\mu}(\breve{\psi})(x) = p(T)_x\mathcal{T}_z^{W,\mu}(\breve{\psi})(x), \tag{6.48}$$

and

$$\begin{aligned} p(T)\breve{\psi}(x) &= p(T)_x\mathcal{T}_z^{W,\mu}(\breve{\psi})(x)_{/z=0}, \\ &= p(T)_z\mathcal{T}_z^{W,\mu}(\breve{\psi})(x)_{/z=0}. \end{aligned}$$

Then

$$p(T)\breve{\psi}(x) = \sigma_{\breve{\psi}}(p)\breve{\psi}(x), \tag{6.49}$$

with

$$\sigma_{\breve{\psi}}(p) = p(T)\breve{\psi}(0).$$

Thus from ([17] p.2796), there exists a unique $\lambda \in \mathbb{C}^d$ such that

$$\sigma_{\breve{\psi}}(p) = p(i\lambda). \tag{6.50}$$

From the relations (6.49),(6.50),(6.43) the function $\breve{\psi}$ satisfies the differential system

$$\begin{cases} p(T)\breve{\psi}(x) = p(i\lambda)\breve{\psi}(x), \quad x \in \mathbb{R}^d, \\ \breve{\psi}(0) = 1, \end{cases}$$

corresponding to the root system of type BC_d attached to the multiplicity function k_μ.
As the solution of this system is unique, it follows from (2.11) that

$$\forall x \in \mathbb{R}^d, \quad \breve{\psi}(x) = F_{BC}^{\mu}(\lambda, x).$$

But the function $\breve{\psi}$ is bounded, then from (6.6) and Remarks 6.3 i), λ belongs to Σ. □

Proof of Theorem 6.23

Let $\mathcal{X}$ be the linear functional from $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$ into $\mathbb{C}^d$ defined for $\lambda \in \Sigma$ by

$$\mathcal{X}(f) = \mathcal{H}^{W,\mu}(f)(\lambda) = \int_{\mathbb{R}^d} f(x)F_{BC}^{\mu}(\lambda, -x)\mathcal{A}_k(x)dx.$$

From the relation (6.6) we have

$$|\mathcal{X}(f)| \leq ||f||_{\mathcal{A}_k,1},$$

and from (6.33), for all f, g in $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$ we get

$$\mathcal{X}(f *_{\mathcal{H}^W}^{\mu} g) = \mathcal{X}(f)\mathcal{X}(g).$$

Then $\mathcal{X}$ is a complex homomorphism of the algebra $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$.
Conversely, Let $\mathcal{X}$ be a complex homomorphism of the algebra $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$.

The mapping $f \longrightarrow \mathcal{X}(f)$ is a linear functional from $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$ into $\mathbb{C}^d$ of norm at most 1. Then there exists a function ψ in $L^\infty_{\mathcal{A}_k}(\mathbb{R}^d)^W$ such that

$$\mathcal{X}(f) = \int_{\mathbb{R}^d} f(x)\psi(x)\mathcal{A}_k(x)dx. \tag{6.51}$$

From the relation

$$\mathcal{X}(f *^\mu_{\mathcal{H}^W} g) = \mathcal{X}(f)\mathcal{X}(g),$$

with f, g in $\mathcal{D}(\mathbb{R}^d)^W$, we obtain from (6.30), (6.31) and Fubini's theorem

$$\begin{aligned}\mathcal{X}(f *^\mu_{\mathcal{H}^W} g) &= \int_{\mathbb{R}^d} f *^\mu_{\mathcal{H}^W} g(x)\psi(x)\mathcal{A}_k(x)dx \\ &= \int_{\mathbb{R}^d}[\int_{\mathbb{R}^d} \mathcal{T}^{W,\mu}_x(f)(-y)\psi(x)\mathcal{A}_k(x)dx]g(y)\mathcal{A}_k(y)dy.\end{aligned}$$

But from (3.20),(3.27) (3.40) we have

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{T}^{W,\mu}_x(f)(-y)\psi(x)\mathcal{A}_k(x)dx &= \int_{\mathbb{R}^d} \mathcal{T}^{W,\mu}_{-y}(f)(x)\psi(x)\mathcal{A}_k(x)dx \\ &= \int_{\mathbb{R}^d} f(x)\mathcal{T}^{W,\mu}_{-y}(\psi)(x)\mathcal{A}_k(x)dx \\ &= \int_{\mathbb{R}^d} f(x)\mathcal{T}^{W,\mu}_{-y}(\breve{\psi})(-x)\mathcal{A}_k(x)dx.\end{aligned}$$

Thus

$$\mathcal{X}(f *^\mu_{\mathcal{H}^W} g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathcal{T}^{W,\mu}_{-y}(\breve{\psi})(-x)f(x)g(y)\mathcal{A}_k(x)\mathcal{A}_k(y)dxdy. \tag{6.52}$$

On the other hand we have

$$\mathcal{X}(f)\mathcal{X}(g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi(x)\psi(y)f(x)g(y)\mathcal{A}_k(x)\mathcal{A}_k(y)dxdy. \tag{6.53}$$

From the relations (6.52), (6.53) we deduce

$$\mathcal{T}^{W,\mu}_{-y}(\breve{\psi})(-x) = \psi(x).\psi(y), \quad a.e, x, y \in \mathbb{R}^d.$$

This relation can also be written in the form of the relation (6.42).
Thus from Lemma 6.24 and the relation (6.51), there exists a unique $\lambda \in \Sigma$ such that for all f in $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$ we have

$$\mathcal{X}(f) = \int_{\mathbb{R}^d} f(x)F^\mu_{BC}(\lambda, -x)\mathcal{A}_k(x)dx,$$

Thus

$$\mathcal{X}(f) = \mathcal{H}^{W,\mu}(f)(\lambda).$$

This completes the proof.

REMARKS 6.25.

i) Theorem 6.23 proves that the hypergeometric Fourier transform $\mathcal{H}^{W,\mu}$ is the Gelfand transform defined on $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$ by

$$\mathcal{G}(f)(x) = \mathcal{X}(f), \quad \mathcal{X} \in S, \tag{6.54}$$

where S denotes the set of all complex homomorphisms $\mathcal{X}$ of $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$.

ii) Let $\mathcal{G}(L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W)$ be the space of all $\mathcal{G}(f)$ for f in $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$. The Gelfand topology of S is the weak topology induced by $\mathcal{G}(L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W)$ that is the weakest topology that makes every $\mathcal{G}(f)$ continuous. Then we have

$$\mathcal{G}(L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W) \subset C(S), \tag{6.55}$$

where $C(S)$ is the space of complex continuous functions on S.

iii) The set S equipped with the Gelfand topology is usually called the maximal ideal space of $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$.

THEOREM 6.26. *The maximal ideal space S of $L^1_{\mathcal{A}_k}(\mathbb{R}^d)^W$ is homeomorphic to Σ equipped with the usual topology.*

Proof. We deduce the result from Theorem 6.22, the relations (6.54),(6.55) and Theorem 5 G of [10] p.12. □

6.5. The hypergeometric Fourier transform on the W-invariant measures spaces.

NOTATIONS. We denote by

- $C_b(\mathbb{R}^d)^W$ the space of continuous and bounded functions on $\mathbb{R}^d$.
- $\mathcal{M}_b(\mathbb{R}^d)^W$ the space of bounded Borel measures on $\mathbb{R}^d$, which are W-invariant.
- $\mathcal{M}^1(\mathbb{R}^d)^W$ the subset of probability measures on $\mathbb{R}^d$ which are W-invariant.

DEFINITION 6.27. The hypergeometric Fourier transform of a measure η in $\mathcal{M}_b(\mathbb{R}^d)^W$ is the function $\mathcal{H}^{W,\mu}(\eta)$ defined on Σ by

$$\mathcal{H}^{W,\mu}(\eta)(\lambda) = \int_{\mathbb{R}^d} F^{\mu}_{BC}(-\lambda, x)d\eta(x). \tag{6.56}$$

PROPOSITION 6.28.

i) *For η in $\mathcal{M}_b(\mathbb{R}^d)^W$ the function $\mathcal{H}^{W,\mu}(\eta)$ is continuous in Σ and holomorphic in $\overset{o}{\Sigma}$.*

ii) *For* η *in* $\mathcal{M}_b(\mathbb{R}^d)^W$ *we have*

$$\forall \lambda \in \Sigma, \quad |\mathcal{H}^{W,\mu}(\eta)(\lambda)| \leq ||\eta||. \tag{6.57}$$

Proof. i) - For all $x \in \mathbb{R}^d$, the function $\lambda \longrightarrow F^{\mu}_{BC}(-\lambda, x)$ is continuous in Σ and from the relation (6.6) it satisfies

$$\forall \lambda \in \Sigma, \forall x \in \mathbb{R}^d, \quad |F^{\mu}_{BC}(-\lambda, x)| \leq 1.$$

Then the dominated convergence theorem implies the continuity of the function $\mathcal{H}^{W,\mu}(\eta)$ in Σ.

- As for all $x \in \mathbb{R}^d$, the function $\lambda \longrightarrow F^{\mu}_{BC}(-\lambda, x)$ is entire on $\mathbb{C}^d$, then from Fubini's theorem and Cauchy's formula we deduce that the function $\mathcal{H}^{W,\mu}(\eta)$ is holomorphic in $\overset{o}{\Sigma}$.

ii) We deduce (6.57) from (6.56) and (6.6).

□

PROPOSITION 6.29. *Let* η, ν *two measures in* $\mathcal{M}_b(\mathbb{R}^d)^W$ *such that*

$$\forall \lambda \in \mathbb{R}^d, \quad \mathcal{H}^{W,\mu}(\eta)(\lambda) = \mathcal{H}^{W,\mu}(\nu)(\lambda). \tag{6.58}$$

Then

$$\eta = \nu. \tag{6.59}$$

Proof. We denote by σ the measure of $\mathcal{M}_b(\mathbb{R}^d)^W$ given by

$$\sigma = \eta - \nu.$$

We have

$$\forall \lambda \in \mathbb{R}^d, \quad \mathcal{H}^{W,\mu}(\sigma)(\lambda) = 0.$$

On the other hand for all f in $\mathcal{D}(\mathbb{R}^d)^W$, we deduce from Theorem 3.4, iii) the relation (3.7) and Fubuni's theorem

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x) d\sigma(x) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathcal{H}^{W,\mu}(f)(\lambda) F^{\mu}_{BC}(\lambda, x) \mathcal{C}^W_k(\lambda) d\lambda d\sigma(x) \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathcal{H}^{W,\mu}(\check{f})(-\lambda) F^{\mu}_{BC}(\lambda, x) \mathcal{C}^W_k(\lambda) d\lambda d\sigma(x) \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathcal{H}^{W,\mu}(\check{f})(\lambda) F^{\mu}_{BC}(-\lambda, x) \mathcal{C}^W_k(\lambda) d\lambda d\sigma(x) \\
&= \int_{\mathbb{R}^d} \mathcal{H}^{W,\mu}(\check{f})(\lambda) \mathcal{H}^{W,\mu}(\sigma)(\lambda) \mathcal{C}^W_k(\lambda) d\lambda = 0.
\end{aligned}$$

Thus for all f in $\mathcal{D}(\mathbb{R}^d)^W$ we have

$$\int_{\mathbb{R}^d} f(x)d\sigma(x) = 0,$$

Then

$$\sigma = 0 \Longleftrightarrow \eta = \nu.$$

□

6.6. The hypergeometric convolution product on the W-invariant measures spaces.

DEFINITION 6.30. The hypergeometric convolution product $\eta *^{\mu}_{\mathcal{H}^W} \nu$ of the measures η, ν in $\mathcal{M}_b(\mathbb{R}^d)^W$ is defined by

$$\eta *^{\mu}_{\mathcal{H}^W} \nu(f) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathcal{T}^{W,\mu}_x(f)(y)d\eta(x)d\nu(y), \quad f \in C_b(\mathbb{R}^d)^W. \quad (6.60)$$

The following propositions give some properties of the convolution product $*^{\mu}_{\mathcal{H}^W}$.

PROPOSITION 6.31.

i) *We have*

$$\forall x, y \in \mathbb{R}^d, \quad \delta_x *^{\mu}_{\mathcal{H}^W} \delta_y(f) = \mathcal{T}^{W,\mu}_x(f). \quad (6.61)$$

ii) - *The measure* $\eta *^{\mu}_{\mathcal{H}^W} \nu$ *belongs to* $\mathcal{M}_b(\mathbb{R}^d)^W$ *and we have*

$$||\eta *^{\mu}_{\mathcal{H}^W} \nu|| \leq ||\eta||||\nu||. \quad (6.62)$$

- *For all* η, ν *in* $\mathcal{M}^1(\mathbb{R}^d)^W$, *the measure* $\eta *^{\mu}_{\mathcal{H}^W} \nu$ *belongs to* $\mathcal{M}^1(\mathbb{R}^d)^W$.

PROPOSITION 6.32. *Let* η, ν *in* $\mathcal{M}_b(\mathbb{R}^d)^W$. *Then we have*

$$\forall \lambda \in \mathbb{R}^d, \quad \mathcal{H}^{W,\mu}(\eta *^{\mu}_{\mathcal{H}^W} \nu)(\lambda) = \mathcal{H}^{W,\mu}(\eta)(\lambda).\mathcal{H}^{W,\mu}(\nu)(\lambda). \quad (6.63)$$

Proof. We deduce (6.63) from the relations (6.60), (6.56),(6.8). □

COROLLARY 6.33. *The hypergeometric convolution product* $*^{\mu}_{\mathcal{H}^W}$ *of measures in* $\mathcal{M}_b(\mathbb{R}^d)^W$ *is commutative and associative.*

Proof. We deduce these results from Proposition 6.32. □

6.7. The hypergroup associated to the root system BC_d relating to the multiplicity function k_μ. The notion of an abstract algebraic hypergroup has its origins in the studies of F.Marty and H.S.Wail in the 1930s, and harmonic analysis on hypergroups dates back to J.Delsart's and B.M.Levitan's work during the 1930s and 1940s, but the substantial development had to wait till the 1970s when C.F. Dunkl [5], R. Spector [20] and R.I. Jewett [8] put hypergroups in the right setting for harmonic analysis. There have been many fruitful development of the theory of hypergroups and their applications in analysis, probability theory and approximation theory (see [1,21]).

In the subsection 6.6 we have considered the hypergeometric convolution product $*^{\mu}_{\mathcal{H}W}$ of measure on $\mathbb{R}^d$, parameterized by μ. In this subsection we shall show that $(\mathbb{R}^d, *^{\mu}_{\mathcal{H}W})$ are commutative hypergroups, having the Heckman-Opdam's hypergeometric function $F^\mu(\lambda, x)$, $x \in \mathbb{R}^d$, $\lambda \in \Sigma$, as characters. In the group theory cases corresponding to $\mu = p\frac{d_0}{2}$ these hypergroups are given by the double coset convolution associated with the Gelfand pairs (G, K). In the rank one case, they coincide with the one variable Jacobi hypergroups (see [1] p.235) which are particular cases of the Chébli-Trimèche's hypergroups (see [1] p.202 and 209, [21]).

In [17] M.Rösler has proved that $(C, *^{\mu}_{\mathcal{H}W})$ where C is the closure of the Weyl chamber $\mathfrak{a}^+$ given by (6.4), are commutative hypergroups.

We consider the probability measure $m^\mu_{x,y}$ given for $x, y \in \mathbb{R}^d$ by (6.5). From the relations (6.7), (6.61) we obtain

$$\delta_x *^{\mu}_{\mathcal{H}W} \delta_y = m^\mu_{x,y}. \tag{6.64}$$

THEOREM 6.34. *The relation (6.64) define the commutative hypergroups $(\mathbb{R}^d, *^{\mu}_{\mathcal{H}W})$. The neutral element is zero and the involution is the identity mapping.*

Proof. We obtain the results of this theorem by applying the proof of ([17] p.2793-2799), used to prove that $(C, *^{\mu}_{\mathcal{H}W})$, are commutative hypergroups. □

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