

## AN ITERATIVE ALGORITHM FOR EXTENDED GENERALIZED NONLINEAR VARIATIONAL INCLUSIONS FOR RANDOM FUZZY MAPPINGS

A.H. DAR, MOHD. SARFARAZ, AND M.K. AHMAD

**ABSTRACT.** In this slush pile, we introduce a new kind of variational inclusions problem stated as random extended generalized nonlinear variational inclusions for random fuzzy mappings. We construct an iterative scheme for the this variational inclusion problem and also discuss the existence of random solutions for the problem. Further, we show that the approximate solutions achieved by the generated scheme converge to the required solution of the problem.

### 1. Historical Perspective

The introductory literature in the fuzzy set theory was studied by Zadeh [18] in 1965. The fuzzy set theory became popular because it enables us to show pragmatically powerful concepts in which one can judge the transition from nonmembership to membership. The literature of variational inequalities is very vast and mainly devoted to the study of real world problems which arise in calculus of variations, nonlinear programming, network and transportation equilibrium, mechanics, mathematical physics and basic and applied sciences. The literature of variational inequalities have been translated in many formats and one of the important translation is enunciated as variational inclusions which was mainly due to the Hassouni and Moudafi [7] in 1994. The study

---

Received November 28, 2017. Revised March 12, 2018. Accepted March 19, 2018.  
2010 Mathematics Subject Classification: 49J40, 49J52, 90C30.

Key words and phrases: Variational inclusion, Random fuzzy mapping, Algorithm, Existence, Convergence.

© The Kangwon-Kyungki Mathematical Society, 2018.

This is an Open Access article distributed under the terms of the Creative Commons Attribution Non-Commercial License (<http://creativecommons.org/licenses/by-nc/3.0/>) which permits unrestricted non-commercial use, distribution and reproduction in any medium, provided the original work is properly cited.

of variational inclusions covers wide range of problems encountered by Noor [13–15], Isac [10], Siddiqui and Ansari [17], as a particular ones. Moreover, fuzzy mappings for variational inequalities was introduced by Chang and Zhu [6] in 1989. Afterwards, many researchers [3–5, 11, 12, 16] around the subject handle these techniques to enlarge the core of this concept. In fact, Huang [9] was the first person to introduce the concept of random fuzzy mappings.

Motivated by the work of Ahmad et. al [1] and Huang [9], we designed a new problem known as extended generalized nonlinear variational inclusions for random fuzzy mappings. Additionally, we prove the existence as well as convergence result for random solutions of the particular variational inclusion problem.

## 2. Prelude

In this section, we give some basic characterization about the fuzzy mappings and related problems. Throughout the slush pile, we can symbolize the notions by the underneath characterizations.

- $\mathfrak{H}$  stands for real Hilbert space;
- $\langle \cdot, \cdot \rangle$  and  $\| \cdot \|$  stand for inner product and norm, respectively ;
- $(\mathbb{S}, \Sigma)$  stands for measurable space;
- $B(\mathfrak{H})$  stands for class of Borel  $\sigma$ -fields in  $\mathfrak{H}$ ;
- $CB(\mathfrak{H})$  stands for family of closed, bounded subsets of  $\mathfrak{H}$ ;
- $H(\cdot, \cdot)$  stands for the Hausdorff metric on  $CB(\mathfrak{H})$ ;
- $\mathbf{F}(\mathfrak{H})$  stands for family of all fuzzy sets over  $\mathfrak{H}$ .

Let  $P$  be a map from  $\mathfrak{H}$  in to  $\mathbf{F}(\mathfrak{H})$ , i.e.,  $P : \mathfrak{H} \rightarrow \mathbf{F}(\mathfrak{H})$ , enunciated as fuzzy map. For any  $t \in \mathfrak{H}$  and a fuzzy map  $P$  on  $\mathfrak{H}$ ,  $P(t)$  (denoted as  $P_t$  in further places) characterizes as a fuzzy set on  $\mathfrak{H}$  and  $P_t y$  is pronounced as membership function of  $y$  in  $P_t$ . For  $\beta \in [0, 1]$  and a fuzzy set  $G$  lies in  $\mathbf{F}(\mathfrak{H})$ , define  $(G)_\beta = \{t \in \mathfrak{H} : G(t) \geq \beta\}$ , known as a  $\beta$ -cut of the set  $G$ .

**DEFINITION 2.1.** [9] For any  $B$  in  $B(\mathfrak{H})$ , a mapping  $\mathfrak{t} : \mathbb{S} \rightarrow \mathfrak{H}$  is enunciated as measurable if  $\{r \in \mathbb{S} : \mathfrak{t}(r) \in B\} \in \Sigma$ .

**DEFINITION 2.2.** [9] For any  $B$  in  $B(\mathfrak{H})$ , a set-valued map  $M : \mathbb{S} \rightarrow 2^\mathfrak{H}$  is enunciated as measurable if  $M^{-1}(B) = \{r \in \mathbb{S} : M(r) \cap B \neq \emptyset\}$  lies in  $\Sigma$ .

DEFINITION 2.3. [9] For any  $r \in \mathbb{S}$  and a set-valued measurable map  $M : \mathbb{S} \rightarrow 2^{\mathfrak{H}}$ , a map  $\mathfrak{z} : \mathbb{S} \rightarrow \mathfrak{H}$  is enunciated as measurable selection of  $M$  if  $\mathfrak{z}(r)$  lies in the set  $M(r)$ .

DEFINITION 2.4. [9] For any  $r \in \mathbb{S}$ , a map  $W : \mathbb{S} \times \mathfrak{H} \rightarrow 2^{\mathfrak{H}}$  enunciated as random multi-valued map if  $W(\cdot, t)$  is measurable. If  $W(r, \cdot)$  is continuous in  $H(\cdot, \cdot)$  for  $r \in \mathbb{S}$ , then map  $W : \mathbb{S} \times \mathfrak{H} \rightarrow CB(\mathfrak{H})$  is pronounced as  $H$ -continuous.

DEFINITION 2.5. [9] For any  $\beta \in (0, 1]$ , a fuzzy map  $G : \mathbb{S} \rightarrow \mathbf{F}(\mathfrak{H})$  is pronounced as measurable, if  $(G(\cdot))_{\beta} : \mathbb{S} \rightarrow 2^{\mathfrak{H}}$  be a measurable multi-valued map.

DEFINITION 2.6. [9] For any  $t \in \mathfrak{H}$ , a fuzzy map  $N : \mathbb{S} \times \mathfrak{H} \rightarrow \mathbf{F}(\mathfrak{H})$  is termed as random fuzzy map, if a map  $N(\cdot, t) : \mathbb{S} \rightarrow \mathbf{F}(\mathfrak{H})$  is fuzzy measurable.

DEFINITION 2.7. [9] For any  $t \in \mathfrak{H}$ , an operator  $P : \mathbb{S} \times \mathfrak{H} \rightarrow \mathfrak{H}$  is named as random operator, if  $P(r, t) = t(r)$  is measurable. If for arbitrary  $r \in \mathbb{S}$ , the operator  $P(r, \cdot) : \mathfrak{H} \rightarrow \mathfrak{H}$  is continuous, then  $P$  is termed as continuous map.

REMARK 2.8. The multi-valued mappings, random multi-valued mappings and fuzzy mappings are particular cases of the random fuzzy mappings.

### 3. Construction Of The Problem

Presume that the given random maps  $P, Q, M, N : \mathbb{S} \times \mathfrak{H} \rightarrow \mathbf{F}(\mathfrak{H})$  meet the underneath criteria:

**C1:** There does exist membership functions  $\mathfrak{c}, \mathfrak{d}, \mathfrak{e}, \mathfrak{h} : \mathfrak{H} \rightarrow (0, 1]$  with the suitable requirements for all  $(r, y) \in \mathbb{S} \times \mathfrak{H}$ :

$$\begin{aligned} (P_{r, \mathfrak{y}(r)})_{\mathfrak{c}(y)} &\in CB(\mathfrak{H}), \\ (Q_{r, \mathfrak{y}(r)})_{\mathfrak{d}(y)} &\in CB(\mathfrak{H}), \\ (M_{r, \mathfrak{y}(r)})_{\mathfrak{e}(y)} &\in CB(\mathfrak{H}) \text{ and} \\ (N_{r, \mathfrak{y}(r)})_{\mathfrak{h}(y)} &\in CB(\mathfrak{H}) . \end{aligned}$$

With the assistance of these random fuzzy maps  $P, Q, M, N : \mathbb{S} \times \mathfrak{H} \rightarrow \mathbf{F}(\mathfrak{H})$ , we can characterize random set-valued maps:

$$\begin{aligned} \hat{P} : \mathbb{S} \times \mathfrak{H} &\rightarrow CB(\mathfrak{H}), \quad y \rightarrow (P_{r, y})_{\mathfrak{c}(y)} \quad (r, y) \in \mathbb{S} \times \mathfrak{H}, \\ \hat{Q} : \mathbb{S} \times \mathfrak{H} &\rightarrow CB(\mathfrak{H}), \quad y \rightarrow (Q_{r, y})_{\mathfrak{d}(y)} \quad (r, y) \in \mathbb{S} \times \mathfrak{H}, \end{aligned}$$

$\hat{M} : \mathbb{S} \times \mathfrak{H} \rightarrow CB(\mathfrak{H})$ ,  $y \rightarrow (M_{r,y})_{\mathfrak{e}(y)}$  ( $r, y$ )  $\in \mathbb{S} \times \mathfrak{H}$ , and  
 $\hat{N} : \mathbb{S} \times \mathfrak{H} \rightarrow CB(\mathfrak{H})$ ,  $y \rightarrow (N_{r,y})_{\mathfrak{h}(y)}$  ( $r, y$ )  $\in \mathbb{S} \times \mathfrak{H}$ .

Maps  $\hat{P}, \hat{Q}, \hat{M}$  and  $\hat{N}$  are named as the random set-valued maps persuade by random fuzzy maps  $P, Q, M$  and  $N$ , respectively.

For the membership functions  $\mathfrak{c}, \mathfrak{d}, \mathfrak{e}, \mathfrak{h} : \mathfrak{H} \rightarrow (0, 1]$ , random fuzzy maps  $P, Q, M, N : \mathbb{S} \times \mathfrak{H} \rightarrow \mathbf{F}(\mathfrak{H})$  and random operators  $p, q, f, g : \mathbb{S} \times \mathfrak{H} \rightarrow \mathfrak{H}$  with  $Im(g) \cap dom(\partial\phi) \neq \emptyset$ . We discuss the underneath problem:

Find measurable mappings  $\eta, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathfrak{z} : \mathbb{S} \rightarrow \mathfrak{H}$  such that for  $g(r, \eta(r)) \cap dom(\partial\phi) \neq \emptyset$  and for all  $r \in \mathbb{S}$ ,  $\mathfrak{x}(r) \in \mathfrak{H}$  and  $\eta(r) \in \mathfrak{H}$ ,

- $Q_{r,\eta(r)}(\mathbf{v}(r)) \geq \mathfrak{d}(\eta(r))$
- $M_{r,\eta(r)}(\mathbf{w}(r)) \geq \mathfrak{e}(\eta(r))$
- $N_{r,\eta(r)}(\mathfrak{z}(r)) \geq \mathfrak{h}(\eta(r))$

Find measurable mappings  $\eta, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathfrak{z} : \mathbb{S} \rightarrow \mathfrak{H}$  such that for all  $r \in \mathbb{S}$ ,  $\mathfrak{x}(r) \in \mathfrak{H}$  and  $\eta(r) \in \mathfrak{H}$ ,  $P_{r,\eta(r)}(\mathbf{u}(r)) \geq \mathfrak{c}(\eta(r))$ ,  $Q_{r,\eta(r)}(\mathbf{v}(r)) \geq \mathfrak{d}(\eta(r))$ ,  $M_{r,\eta(r)}(\mathbf{w}(r)) \geq \mathfrak{e}(\eta(r))$ ,  $N_{r,\eta(r)}(\mathfrak{z}(r)) \geq \mathfrak{h}(\eta(r))$ ,  $g(r, \eta(r)) \cap dom(\partial\phi) \neq \emptyset$ , and

$$(3.1) \quad \langle p(r, \mathbf{u}(r)) - \{q(r, \mathbf{v}(r)) - f(r, \mathbf{w}(r))\}, \mathfrak{x}(r) - g(r, \mathfrak{z}(r)) \rangle + \phi(\mathfrak{x}(r)) - \phi(g(r, \mathfrak{z}(r))) \geq 0,$$

where  $\phi : \mathfrak{H} \rightarrow R \cup \{0, +\infty\}$  is a proper, convex and lower semi continuous function and  $\partial\phi$  is its sub-differential. The problem (3.1) is enunciated as random extended generalized non linear variational inclusions for random fuzzy mappings.

### 3.1. Special Cases.

Case 1. If we take  $f = g$ , then problem (3.1) reduces to random generalized non linear mixed variational inclusions for random fuzzy mappings, which is find measurable mappings  $\eta, \mathbf{u}, \mathbf{v}, \mathbf{w} : \mathbb{S} \rightarrow \mathfrak{H}$  such that for all  $r \in \mathbb{S}$ ,  $\mathfrak{x}(r) \in \mathfrak{H}$  and  $\eta(r) \in \mathfrak{H}$ ,  $P_{r,\eta(r)}(\mathbf{u}(r)) \geq \mathfrak{c}(\eta(r))$ ,  $Q_{r,\eta(r)}(\mathbf{v}(r)) \geq \mathfrak{d}(\eta(r))$ ,  $M_{r,\eta(r)}(\mathbf{w}(r)) \geq \mathfrak{e}(\eta(r))$ ,  $g(r, \eta(r)) \cap dom(\partial\phi) \neq \emptyset$ , and

$$(3.2) \quad \langle p(r, \mathbf{u}(r)) - \{q(r, \mathbf{v}(r)) - f(r, \mathbf{w}(r))\}, \mathfrak{x}(r) - f(r, \mathbf{w}(r)) \rangle + \phi(\mathfrak{x}(r)) - \phi(f(r, \mathbf{w}(r))) \geq 0.$$

This was investigated by Ahmad et. al [1]. They developed an iterative algorithm for this problem.

Case 2. If  $M$  and  $N$  are taken as zero mappings and also  $f : \mathbb{S} \times \mathfrak{H} \rightarrow \mathfrak{H}$  is taken as zero map, then problem (3.1) reduces to random generalized non linear variational inclusion for random fuzzy mappings, which is finding measurable mappings  $\eta, u, v : \mathbb{S} \rightarrow \mathfrak{H}$  such that for all  $r \in \mathbb{S}$ ,  $\mathfrak{x}(r) \in \mathfrak{H}, P_{r,\eta(r)}(u(r) \geq c(\eta(r)), Q_{r,\eta(r)}(v(r) \geq \mathfrak{d}(\eta(r)), g(r, \eta(r)) \cap \text{dom}(\partial\phi) \neq \emptyset$ ,

$$(3.3) \quad \langle p(r, u(r)) - q(r, v(r)), \mathfrak{x}(r) - g(r, \eta(r)) \rangle + \phi(\mathfrak{x}(r)) - \phi(g(r, \eta(r))) \geq 0.$$

This was studied by Huang [9]. He constructed an iterative algorithm for this variational inequality.

#### 4. Random iterative algorithm

First we list some Lemmas which are building blocks of our algorithm and main result.

LEMMA 4.1. [2] Let  $M : \mathbb{S} \times \mathfrak{H} \rightarrow CB(\mathfrak{H})$  be a  $\mathfrak{H}$  - continuous multi-valued mapping. Then for any measurable mapping  $u : \mathbb{S} \rightarrow \mathfrak{H}$ , the multi-valued mapping  $M(\cdot, u(\cdot)) : \mathbb{S} \rightarrow CB(\mathfrak{H})$  is measurable.

LEMMA 4.2. [2] Let  $M, N : \mathbb{S} \rightarrow CB(\mathfrak{H})$  be two multi-valued measurable mappings,  $\epsilon \geq 0$  be constant  $u_1 : \mathbb{S} \rightarrow \mathfrak{H}$  be a measurable selection of  $M$ . Then there exists a measurable selection  $u_2 : \mathbb{S} \rightarrow \mathfrak{H}$  of  $N$  such that for all  $r \in \mathbb{S}$ ,

$$\|u_1(r) - u_2(r)\| \leq (1 + \epsilon)\mathfrak{H}(M(r), N(r)).$$

LEMMA 4.3. The measurable mappings  $\eta, u, v, w, z : \mathbb{S} \rightarrow \mathfrak{H}$  are solutions of problem (3.1) if and only if for all  $u(r) \in \hat{P}(r, \eta(r)), v(r) \in \hat{Q}(r, \eta(r)), w(r) \in \hat{M}(r, \eta(r))$ ,  $z(r) \in \hat{N}(r, \eta(r))$  and

$$(4.1) \quad g(r, \eta(r)) = J_{\alpha(r)}^\phi [g(r, \eta(r)) - \alpha(r)\{p(r, u(r)) - (q(r, v(r)) - f(r, w(r)))\}],$$

where  $\alpha : \mathbb{S} \rightarrow (0, +\infty)$  is measurable function and  $J_{\alpha(r)}^\phi = (I + \alpha(r)\partial\phi)^{-1}$  is the proximal mapping on  $H$ .

*Proof.* By the definition of  $J_{\alpha(r)}^\phi$ , we have

$$\begin{aligned} &g(r, \eta(r)) - \alpha(r)\{p(r, u(r)) - (q(r, v(r)) - f(r, w(r)))\} \\ &\in g(r, \eta(r)) + \alpha(r)\partial\phi(g(r, \eta(r))), \forall r \in \mathbb{S}, \end{aligned}$$

so,

$$(q(r, \mathbf{v}(r)) - f(r, \mathbf{w}(r))) - p(r, \mathbf{u}(r)) \in \partial\phi(g(r, \boldsymbol{\eta}(r))), \forall r \in \mathbb{S}.$$

By using the definition of sub-differential, we can write

$$\phi(\mathbf{x}(r)) \geq \phi(g(r, \boldsymbol{\eta}(r))) + \langle p(r, \mathbf{u}(r)) - \{q(r, \mathbf{v}(r)) - f(r, \mathbf{w}(r))\} \rangle, \forall \mathbf{x}(r) \in \mathbb{S}.$$

Therefore,  $\boldsymbol{\eta}$ ,  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w}$  and  $\mathbf{z}$  are solutions of problem (3.1).  $\square$

The approximate solution of the problem (3.1) can be obtained by applying a successive approximation method to the problem of solving  $\boldsymbol{\eta}(r) \in G(r, \boldsymbol{\eta}(r))$ ,  $\forall r \in \mathbb{S}$ , where,

$$\begin{aligned} G(r, \boldsymbol{\eta}(r)) &= \boldsymbol{\eta}(r) - g(r, \boldsymbol{\eta}(r)) + J_{\alpha(r)}^{\phi}[g(r, \boldsymbol{\eta}(r)) \\ (4.2) \quad &- \alpha(r)\{p(s, \hat{P}(r, \boldsymbol{\eta}(r))) - (q(s, \hat{Q}(r, \boldsymbol{\eta}(r))) - f(s, \hat{P}(r, \boldsymbol{\eta}(r))))\}]. \end{aligned}$$

With the help of equations (4.1) and (4.2), we develop the following iterative algorithm to compute the approximate solutions of problem (3.1).

**Algorithm 4.1** Let  $P, Q, M, N : \mathbb{S} \times \mathfrak{H} \rightarrow \mathbf{F}(\mathfrak{H})$  be the random fuzzy mappings satisfying the condition  $(\mathbf{c}_1)$ . Let  $\hat{P}, \hat{Q}, \hat{M}, \hat{N} : \mathbb{S} \times \mathfrak{H} \rightarrow CB(\mathfrak{H})$  be four  $\mathfrak{H}$ -continuous random multi-valued mappings induced by  $P, Q, M$  and  $N$ , respectively. Suppose  $p, q, f, g : \mathbb{S} \times \mathfrak{H} \rightarrow \mathfrak{H}$  be continuous random operators. By Lemma 4.1, for a measurable mapping  $\boldsymbol{\eta}_0 : \mathbb{S} \rightarrow \mathfrak{H}$ , the multi-valued mappings  $\hat{P}(\cdot, \boldsymbol{\eta}_0(\cdot)), \hat{Q}(\cdot, \boldsymbol{\eta}_0(\cdot)), \hat{M}(\cdot, \boldsymbol{\eta}_0(\cdot)), \hat{N}(\cdot, \boldsymbol{\eta}_0(\cdot)) : \mathbb{S} \rightarrow CB(\mathfrak{H})$  are measurable. So, by Himmelberg [8] there exist measurable selection  $\mathbf{u}_0 : \mathbb{S} \rightarrow \mathfrak{H}$  of  $\hat{P}(\cdot, \boldsymbol{\eta}_0(r))$ ,  $\mathbf{v}_0 : \mathbb{S} \rightarrow \mathfrak{H}$  of  $\hat{Q}(\cdot, \boldsymbol{\eta}_0(r))$ ,  $\mathbf{w}_0 : \mathbb{S} \rightarrow \mathfrak{H}$  of  $\hat{M}(\cdot, \boldsymbol{\eta}_0(r))$  and  $\mathbf{z}_0 : \mathbb{S} \rightarrow \mathfrak{H}$  of  $\hat{N}(\cdot, \boldsymbol{\eta}_0(r))$ . Let

$$\begin{aligned} \boldsymbol{\eta}_1(r) &= \boldsymbol{\eta}_0(r) - g(r, \boldsymbol{\eta}_0(r)) + J_{\alpha(r)}^{\phi}[g(r, \boldsymbol{\eta}_0(r)) - \\ &\alpha(r)\{p(r, \mathbf{u}_0(r)) - (q(r, \mathbf{v}_0(r)) - f(r, \mathbf{w}_0(r)))\}]. \end{aligned}$$

It can be seen that  $\boldsymbol{\eta}_1 : \mathbb{S} \rightarrow \mathfrak{H}$  is measurable. By Lemma 4.2, there exist measurable selections  $\mathbf{u}_1 : \mathbb{S} \rightarrow \mathfrak{H}$  of  $\hat{P}(\cdot, \boldsymbol{\eta}_1(r))$ ,  $\mathbf{v}_1 : \mathbb{S} \rightarrow \mathfrak{H}$  of  $\hat{Q}(\cdot, \boldsymbol{\eta}_1(r))$ ,

$\mathfrak{w}_1 : \mathbb{S} \rightarrow \mathfrak{H}$  of  $\hat{M}(\cdot, \eta_1(r))$  and  $\mathfrak{z}_1 : \mathbb{S} \rightarrow \mathfrak{H}$  of  $\hat{N}(\cdot, \eta_1(r))$  such that

$$\begin{aligned} \|u_1(r) - u_o(r)\| &\leq (1 + 1)\hat{\mathfrak{H}}(\hat{P}(r, \eta_1(r)), \hat{P}(r, \eta_o(r))), \forall r \in \mathbb{S}, \\ \|v_1(r) - v_o(r)\| &\leq (1 + 1)\hat{\mathfrak{H}}(\hat{Q}(r, \eta_1(r)), \hat{Q}(r, \eta_o(r))), \forall r \in \mathbb{S}, \\ \|w_1(r) - w_o(r)\| &\leq (1 + 1)\hat{\mathfrak{H}}(\hat{M}(r, \eta_1(r)), \hat{M}(r, \eta_o(r))), \forall r \in \mathbb{S}, \\ \|z_1(r) - z_o(r)\| &\leq (1 + 1)\hat{\mathfrak{H}}(\hat{N}(r, \eta_1(r)), \hat{N}(r, \eta_o(r))), \forall r \in \mathbb{S}. \end{aligned}$$

Let

$$\begin{aligned} \eta_2(r) &= \eta_1(r) - g(r, \eta_1(r)) + J_{\alpha(r)}^\phi [g(r, \eta_1(r)) - \\ &\quad \alpha(r)\{p(r, u_1(r)) - (q(r, v_1(r)) - f(r, w_1(r)))\}]. \end{aligned}$$

then,  $\eta_2 : \mathbb{S} \rightarrow \mathfrak{H}$  is measurable. By using the method of induction we can define our algorithm as follows:

$$(4.3) \quad \eta_{n+1}(r) = \eta_n(r) - g(r, \eta_n(r)) + J_{\alpha(r)}^\phi [g(r, \eta_n(r)) - \alpha(r)\{p(r, u_n(r)) - (q(r, v_n(r)) - f(r, w_n(r)))\}],$$

where

$$\begin{aligned} u_n(r) &\in \hat{P}(r, \eta_n(r)), \quad v_n(r) \in \hat{Q}(r, \eta_n(r)), \quad w_n(r) \in \hat{M}(r, \eta_n(r)) \\ z_n(r) &\in \hat{N}(r, \eta_n(r)), \end{aligned}$$

$$(4.4) \quad \begin{aligned} \|u_{n+1}(r) - u_n(r)\| &\leq (1 + (1 + n)^{-1})\hat{\mathfrak{H}}(\hat{P}(r, \eta_{n+1}(r)), \hat{P}(r, \eta_n(r))), \\ \|v_{n+1}(r) - v_n(r)\| &\leq (1 + (1 + n)^{-1})\hat{\mathfrak{H}}(\hat{Q}(r, \eta_{n+1}(r)), \hat{Q}(r, \eta_n(r))), \\ \|w_{n+1}(r) - w_n(r)\| &\leq (1 + (1 + n)^{-1})\hat{\mathfrak{H}}(\hat{M}(r, \eta_{n+1}(r)), \hat{M}(r, \eta_n(r))), \\ \|z_{n+1}(r) - z_n(r)\| &\leq (1 + (1 + n)^{-1})\hat{\mathfrak{H}}(\hat{N}(r, \eta_{n+1}(r)), \hat{N}(r, \eta_n(r))), \end{aligned}$$

for any  $r \in \mathbb{S}$  and  $n = 0, 1, 2, 3, \dots$

REMARK 4.4. Algorithms developed by R. Ahmad [1] and Huang [9], are special cases of our algorithm.

## 5. Existence and convergence

Before giving the main result, we mention some definitions.

DEFINITION 5.1. [9] A random mapping  $g : \mathbb{S} \times \mathfrak{H} \rightarrow \mathfrak{H}$  is termed as strongly monotone if there exists some measurable function  $\beta : \mathbb{S} \rightarrow (0, +\infty)$  such that

$$\langle g(r, \eta_1) - g(r, \eta_2), \eta_1 - \eta_2 \rangle \geq \beta(r) \|\eta_1 - \eta_2\|^2 \quad \forall \eta_1, \eta_2 \in \mathfrak{H}, r \in \mathbb{S}.$$

DEFINITION 5.2. [9] A random mapping  $g : \mathbb{S} \times \mathfrak{H} \rightarrow \mathfrak{H}$  is enuciated as Lipschitz continuous, if there exists some measurable function  $\gamma : \mathbb{S} \rightarrow (0, +\infty)$  such that

$$\|g(r, \eta_1) - g(r, \eta_2)\| \leq \gamma(r) \|\eta_1 - \eta_2\| \quad \forall \eta_1, \eta_2 \in \mathfrak{H}, r \in \mathbb{S}.$$

DEFINITION 5.3. [9] A random multi-valued mapping  $P : \mathbb{S} \times \mathfrak{H} \rightarrow CB(\mathfrak{H})$  is enuciated as a strongly monotone with respect to measurable function  $g : \mathbb{S} \times \mathfrak{H} \rightarrow \mathfrak{H}$ , if there exists some measurable function  $\delta : \mathbb{S} \rightarrow (0, +\infty)$  such that

$$\begin{aligned} \langle g(r, u_1) - g(r, u_2), \eta_1 - \eta_2 \rangle &\geq \delta(r) \|\eta_1 - \eta_2\|^2 \\ \forall r \in \mathbb{S}, \forall \eta_1, \eta_2 \in \mathfrak{H}, \forall u_i \in P(r, \eta_i), i = 1, 2. \end{aligned}$$

DEFINITION 5.4. [9] A random set-valued mapping  $P : \mathbb{S} \times \mathfrak{H} \rightarrow CB(\mathfrak{H})$  is termed to be  $\mathfrak{H}$  Lipschitz continuous, if there exists a measurable function  $\sigma_1 : \mathbb{S} \rightarrow (0, +\infty)$  such that

$$\mathfrak{H}(P(r, \eta_1(r)), P(r, \eta_2(r))) \leq \sigma_1(r) \|\eta_1 - \eta_2\|, \quad \forall \eta_1, \eta_2 \in \mathfrak{H}.$$

Now we state our main result.

THEOREM 5.5. *Let  $g : \mathbb{S} \times \mathfrak{H} \rightarrow \mathfrak{H}$  be strongly monotone and Lipschitz continuous random operators. Let  $p, q, f : \mathbb{S} \times \mathfrak{H} \rightarrow \mathfrak{H}$  be Lipschitz continuous random operators. Let  $P, Q, M, N : \mathbb{S} \times \mathfrak{H} \rightarrow \mathbf{F}(\mathfrak{H})$  be random fuzzy mappings satisfying condition  $(\mathfrak{c}_1)$ ,  $\hat{P}, \hat{Q}, \hat{M}, \hat{N} : \mathbb{S} \times \mathfrak{H} \rightarrow CB(\mathfrak{H})$  be set valued mappings induced by  $P, Q, M$  and  $N$ , respectively. Assume that  $\hat{P}, \hat{Q}, \hat{M}$  and  $\hat{N}$  be  $\mathfrak{H}$  – Lipschitz continuous and  $\hat{P}$  be strongly*



monotone with respect to  $g$ . If the following conditions hold:

$$(5.1) \quad \left| \alpha(r) - \frac{\delta(r) + (\gamma_3(r)\sigma_2(r) - \gamma_4(r)\sigma_3(r))(k(r) - 1)}{\gamma_2^2(r)\sigma_1^2(r) - (\gamma_3(r)\sigma_2(r) - \gamma_4(r)\sigma_3(r))^2} \right| \leq \frac{\sqrt{(\delta(r) + (k(r) - 1)(\gamma_3(r)\sigma_2(r) - \gamma_4(r)\sigma_3(r)))^2 - l(r)}}{\gamma_2^2(r)\sigma_1^2(r) - (\gamma_3(r)\sigma_2(r) - \gamma_4(r)\sigma_3(r))^2},$$

$$(5.2) \quad \delta(r) \geq (1 - k(r))(\gamma_3(r)\sigma_2(r) - \gamma_4(r)\sigma_3(r)) + \sqrt{l(r)}$$

$$\gamma_2(r)\sigma_1(r) \geq \gamma_3(r)\sigma_2(r) - \gamma_4(r)\sigma_3(r),$$

$$(5.3) \quad l(r) = (\gamma_2^2(r)\sigma_1^2(r) - (\gamma_3(r)\sigma_2(r) - \gamma_4(r)\sigma_3(r))^2),$$

$$\alpha(r)(\gamma_3(r)\sigma_2(r) - \gamma_4(r)\sigma_3(r)) < 1 - k(r),$$

$$(5.4) \quad k(r) = 2\sqrt{1 - 2\beta(r) + \sigma_1^2(r)}, k(r) < 1.$$

for all  $r \in \mathbb{S}$ , where  $\delta(r)$  and  $\beta(r)$  are strongly monotone coefficients of  $\hat{P}$  and  $\mathfrak{f}$ , respectively.  $\sigma_1(r)$ ,  $\sigma_2(r)$ ,  $\sigma_3(r)$  and  $\sigma_4(r)$ , are  $\mathfrak{H}$  - Lipschitz coefficients of  $\hat{P}$ ,  $\hat{Q}$ ,  $\hat{M}$  and  $\hat{N}$ , respectively,  $\gamma_1(r)$ ,  $\gamma_2(r)$ ,  $\gamma_3(r)$  and  $\gamma_4(r)$  are Lipschitz coefficients of  $\mathfrak{p}$ ,  $\mathfrak{q}$ ,  $\mathfrak{g}$  and  $\mathfrak{f}$ , respectively. Then there exist measurable mappings  $\eta, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathfrak{z} : \mathbb{S} \rightarrow \mathfrak{H}$  such that (3.1) holds. Moreover  $\eta_n(r) \rightarrow \eta(r), \mathbf{u}_n(r) \rightarrow \mathbf{u}(r), \mathbf{v}_n(r) \rightarrow \mathbf{v}(r), \mathbf{w}_n(r) \rightarrow \mathbf{w}(r)$  and  $\mathfrak{z}_n(r) \rightarrow \mathfrak{z}(r), n \rightarrow \infty$ , where  $\{\eta_n(r)\}, \{\mathbf{u}_n(r)\}, \{\mathbf{v}_n(r)\}, \{\mathbf{w}_n(r)\}$  and  $\{\mathfrak{z}_n(r)\}$  are sequences as defined in equation (4.3).

*Proof.* It follows from (4.3) that

$$\|\eta_{n+1}(r) - \eta_n(r)\| = \|\eta_n(r) - \eta_{n-1}(r) - (g(r, \eta_n(r)) - g(r, \eta_{n-1}(r))) + J_{\alpha(r)}^\phi(K(r, \eta_n(r))) - J_{\alpha(r)}^\phi(K(r, \eta_{n-1}(r)))\|,$$

where

$$K(r, \eta_n(r)) = g(r, \eta_n(r)) - \alpha(r)\{p(r, \mathbf{u}_n(r)) - (q(r, \mathbf{v}_n(r)) - f(r, \mathbf{w}_n(r)))\}.$$

Also, we have

$$\begin{aligned} & \|J_{\alpha(r)}^\phi(K(r, \eta_n(r))) - J_{\alpha(r)}^\phi(K(r, \eta_{n-1}(r)))\| \\ & \leq \|K(r, \eta_n(r)) - K(r, \eta_{n-1}(r))\| \\ & \leq \|\eta_n(r) - \eta_{n-1}(r) - \alpha(r)(p(r, \mathbf{u}_n(r)) - p(r, \mathbf{u}_{n-1}(r)))\| \\ & \quad + \|\eta_n(r) - \eta_{n-1}(r) - (g(r, \eta_n(r)) - g(r, \eta_{n-1}(r)))\| \\ & \quad + \alpha(r)\|q(r, \mathbf{v}_n(r)) - q(r, \mathbf{v}_{n-1}(r))\| \\ & \quad + \alpha(r)\|f(r, \mathbf{w}_{n-1}(r)) - f(r, \mathbf{w}_n(r))\|, \end{aligned}$$

that is

$$\begin{aligned}
& \|\eta_{n+1}(r) - \eta_n(r)\| \\
(5.5) \quad & \leq 2\|\eta_n(r) - \eta_{n-1}(r) - (g(r, \eta_n(r)) - g(r, \eta_{n-1}(r)))\| \\
& \quad + \|\eta_n(r) - \eta_{n-1}(r) - \alpha(r)(p(r, \mathbf{u}_n(r)) - p(r, \mathbf{u}_{n-1}(r)))\| \\
& \quad + \alpha(r)\|q(r, \mathbf{v}_n(r)) - q(r, \mathbf{v}_{n-1}(r))\| \\
& \quad + \alpha(r)\|f(r, \mathbf{w}_{n-1}(r)) - f(r, \mathbf{w}_n(r))\|
\end{aligned}$$

As  $g$  is strongly monotone and Lipschitz continuous, one can write

$$\begin{aligned}
& \|\eta_n(r) - \eta_{n-1}(r) - (g(r, \eta_n(r)) - g(r, \eta_{n-1}(r)))\|^2 \\
(5.6) \quad & \leq (1 - 2\beta(r) + \gamma_1^2(r))\|\eta_n(r) - \eta_{n-1}(r)\|^2 \quad \forall r \in \mathbb{S}.
\end{aligned}$$

Now using  $\mathfrak{H}$ -Lipschitz continuity and strong monotonicity of  $\hat{P}$  and Lipschitz continuity of  $p$ , we see that

$$\begin{aligned}
(5.7) \quad & \|\eta_n(r) - \eta_{n-1}(r) - \alpha(r)(p(r, \mathbf{u}_n(r)) - p(r, \mathbf{u}_{n-1}(r)))\| \\
& \leq (1 - 2\delta(r)\alpha(r) + \alpha^2(r)\gamma_2^2(r)(1 + \mathbf{n}^{-1})^2\sigma_1^2(r))\|\eta_n(r) - \eta_{n-1}(r)\| \quad \forall r \in \mathbb{S}.
\end{aligned}$$

By  $\mathfrak{H}$ -Lipschitz continuity of  $\hat{Q}$ , Lipschitz continuity of  $q$  and (4.4), we see that

$$\begin{aligned}
& \alpha(r)\|q(r, \mathbf{v}_n(r)) - q(r, \mathbf{v}_{n-1}(r))\| \\
(5.8) \quad & \leq \alpha(r)\gamma_3(r)(1 + \mathbf{n}^{-1})\sigma_2(r)\|\eta_n(r) - \eta_{n-1}(r)\| \quad \forall r \in \mathbb{S}.
\end{aligned}$$

Again by  $\mathfrak{H}$ -Lipschitz continuity of  $\hat{M}$ , Lipschitz continuity of  $f$  and (4.4), we see that

$$\begin{aligned}
& \alpha(r)\|f(r, \mathbf{w}_{n-1}(r)) - f(r, \mathbf{w}_n(r))\| \\
(5.9) \quad & \leq \alpha(r)\gamma_4(r)(1 + \mathbf{n}^{-1})\sigma_3(r)\|\eta_n(r) - \eta_{n-1}(r)\| \quad \forall r \in \mathbb{S}.
\end{aligned}$$

Combining equations (5.5)-(5.9), we get

$$\|\eta_{n+1}(r) - \eta_n(r)\| \leq \lambda_n(r)\|\eta_n(r) - \eta_{n-1}(r)\| \quad \forall r \in \mathbb{S}.$$

where

$$\begin{aligned}
\lambda_n(r) : & = 2\sqrt{1 - \beta(r) + \gamma_1^2(r)} \\
& \quad + \sqrt{1 - 2\delta(r)\alpha(r) + \alpha^2(r)\gamma_2^2(r)(1 + \mathbf{n}^{-1})^2\sigma_1^2(r)} \\
& \quad + \alpha(r)\gamma_3(r)(1 + \mathbf{n}^{-1})\sigma_2(r) + \alpha(r)\gamma_4(r)(1 + \mathbf{n}^{-1})\sigma_3(r).
\end{aligned}$$

Letting

$$\lambda(r) : = 2\sqrt{1 - \beta(r) + \gamma_1^2(r)} + \sqrt{1 - 2\delta(r)\alpha(r) + \alpha^2(r)\gamma_2^2(r)\sigma_1^2(r)} \\ \alpha(r)(\gamma_3(r)\sigma_2(r) - \gamma_4(r)\sigma_3(r)).$$

We know that  $\lambda_n(r) \rightarrow \lambda(r)$ , for all  $r \in \mathbb{S}$ . From equations (5.1)-(5.4), we see that  $\lambda(r) < 1$  for all  $r \in \mathbb{S}$ . Hence for any  $r \in \mathbb{S}$ ,  $\lambda_n(r) \leq 1$ , for very large  $n$ . So,  $\{\eta_n(r)\}$  is a cauchy sequence and we can assume that  $\eta_n(r) \rightarrow \eta(r)$ , for all  $r \in \mathbb{S}$ . From (4.4), we have

$$\begin{aligned} \|\mathbf{u}_n(r) - \mathbf{u}_{n-1}(r)\| &\leq (1 + (1 + \mathbf{n})^{-1}\sigma_1(r))\|\eta_n(r) - \eta_{n-1}(r)\|, \forall r \in \mathbb{S}, \\ \|\mathbf{v}_n(r) - \mathbf{v}_{n-1}(r)\| &\leq (1 + (1 + \mathbf{n})^{-1})\sigma_2(r)\|\eta_n(r) - \eta_{n-1}(r)\|, \forall r \in \mathbb{S}, \\ \|\mathbf{w}_n(r) - \mathbf{w}_{n-1}(r)\| &\leq (1 + (1 + \mathbf{n})^{-1})\sigma_3(r)\|\eta_n(r) - \eta_{n-1}(r)\|, \forall r \in \mathbb{S}, \\ \|\mathbf{z}_n(r) - \mathbf{z}_{n-1}(r)\| &\leq (1 + (1 + \mathbf{n})^{-1})\sigma_4(r)\|\eta_n(r) - \eta_{n-1}(r)\|, \forall r \in \mathbb{S}. \end{aligned}$$

Hence all the four sequences  $\{\mathbf{u}_n(r)\}$ ,  $\{\mathbf{v}_n(r)\}$ ,  $\{\mathbf{w}_n(r)\}$  and  $\{\mathbf{z}_n(r)\}$  are cauchy sequences. Let  $\mathbf{u}_n(r) \rightarrow \mathbf{u}(r)$ ,  $\mathbf{v}_n(r) \rightarrow \mathbf{v}(r)$ ,  $\mathbf{w}_n(r) \rightarrow \mathbf{w}(r)$  and  $\mathbf{z}_n(r) \rightarrow \mathbf{z}(r)$ . Since  $\{\mathbf{u}_n(r)\}$ ,  $\{\mathbf{v}_n(r)\}$ ,  $\{\mathbf{w}_n(r)\}$  and  $\{\mathbf{z}_n(r)\}$  are sequences of measurable mappings, we know that  $\eta, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z} : \mathbb{S} \rightarrow \mathfrak{H}$  are measurable. Further for any  $r \in \mathbb{S}$ , we have

$$\begin{aligned} d(\mathbf{u}(r), \hat{P}(r, \eta(r))) &= \inf\{\|\mathbf{u}(r) - m\| : m \in \hat{P}(r, \eta(r))\} \\ &\leq \|\mathbf{u}(r) - \mathbf{u}_n(r)\| + d(\mathbf{u}(r), \hat{P}(r, \eta(r))) \\ &\leq \|\mathbf{u}(r) - \mathbf{u}_n(r)\| + \hat{\mathfrak{H}}(\hat{P}(r, \eta_n(r)), \hat{P}(r, \eta(r))) \\ &\leq \|\mathbf{u}(r) - \mathbf{u}_n(r)\| + \sigma_1(r)\|\eta_n(r) - \eta(r)\| \rightarrow 0. \end{aligned}$$

Hence,  $\mathbf{u}(r) \in \hat{P}(r, \eta(r))$ , for all  $r \in \mathbb{S}$ . Similarly,  $\mathbf{v}(r) \in \hat{Q}(r, \eta(r))$ ,  $\mathbf{w}(r) \in \hat{M}(r, \eta(r))$  and  $\mathbf{z}(r) \in \hat{N}(r, \eta(r))$ ,  $\forall r \in \mathbb{S}$ .  $\square$

**COROLLARY 5.6.** *If we take  $f = g$ , then we get corresponding result for (3.2), which was proved by Ahmad et.al [1].*

**COROLLARY 5.7.** *If  $M, N$  and  $f$  are taken as zero mappings, then we get the corresponding result for equation (3.3), which was proved by Huang [9].*

## References

- [1] R. Ahmad and F. F. Bazan, *An iterative algorithm for random generalized non-linear mixed variational inclusions for random fuzzy mappings*, Appl. Math. Comput. **167** (1996) 150–155.
- [2] S. S. Chang, *Fixed Point Theory with Applications*, Chongqing Publishing House, Chongqing, 1984.
- [3] S. S. Chang, *Variational Inequality and Complementarity Problem Theory with Applications*, Shanghai Scientific and Tech. Literature Publishing House, Shanghai, 1991.
- [4] S. S. Chang, *Coincidence theorems and variational inequalities for fuzzy mappings*, Fuzzy Sets and Systems **61** (1994) 359–368.
- [5] S. S. Chang and N. J. Huang, *Generalized complementarity problems for fuzzy mappings*, Fuzzy Sets and Systems **55** (1993) 227–234.
- [6] S. S. Chang and Y. G. Zhu, *On variational inequalities for fuzzy mappings*, Fuzzy Sets and Systems **32** (1989) 359–367.
- [7] A. Hassouni and A. Moudafi, *A perturbed algorithm for variational inclusions*, J. Math. Anal. Appl. **185** (1994) 706–712.
- [8] C. J. Himmelberg, *Measurable relations*, Fund. Math. **87** (1975) 53–72.
- [9] N. J. Huang, *Random generalized non linear variational inclusions for random fuzzy mappings*, Fuzzy Sets and Systems **105** (1999) 437–444.
- [10] G. Isac, *A special variational inequality and the implicit complementarity problem*, J. Fac. Sci. Univ. Tokyo **37** (1990) 109–127.
- [11] G. M. Lee, D. S. Kim, B. S. Lee and S.J. Cho, *Generalized vector variational inequality and fuzzy extension*, Appl. Math. Lett. **6** (1993) 47–51.
- [12] B. S. Lee, G. M. Lee, S. J. Cho and D. S. Kim, *A variational inequality for fuzzy mappings*, Proc. 5th Internat. Fuzzy Systems Association World Congress, Seoul, (1993) 326–329.
- [13] M. A. Noor, *Strongly nonlinear variational inequalities*, C.R. Math. Rep. Acad. Sci. Canada **4** (1982) 213–218.
- [14] M. A. Noor, *On the nonlinear complementarity problem*, J. Math. Anal. Appl. **123** (1987) 455–460.
- [15] M. A. Noor, *Quasivariational inequalities*, Appl. Math. Lett. **1** (1988) 367–370.
- [16] M. A. Noor, *Variational inequality for fuzzy mapping (I)*, Fuzzy Sets and Systems **55** (1993) 309–312.
- [17] A. H. Siddiqi and Q. H. Ansari, *Strongly nonlinear quasivariational inequalities*, J. Math. Anal. Appl. **149** (1990) 444–450.
- [18] L. A. Zadeh, *Fuzzy Sets*, Inform. Contr. **8** (1965) 338–353.

**Aadil Hussain Dar**

Department of Mathematics

Aligarh Muslim University

Aligarh 202002, India

*E-mail:* daraadil555@gmail.com

**Mohd. Sarfaraz**

Department of Mathematics

Aligarh Muslim University

Aligarh 202002, India

*E-mail:* sarfarazm820@gmail.com

**Md. Kalimuddin Ahmad**

Department of Mathematics

Aligarh Muslim University

Aligarh 202002, India

*E-mail:* ahmad.kalimuddin@yahoo.co.in