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ELLIPTIC BOUNDARY VALUE PROBLEM WITH TWO SINGULARITIES

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Abstract. We investigate existence and multiplicity of the solutions for elliptic boundary value problem with two singularities. We obtain one theorem which shows that there exists at least one nontrivial weak solution under some conditions on which the corresponding functional of the problem satisfies the Palais-Smale condition. We obtain this result by variational method and critical point theory.

1. Introduction

Let Ω be a bounded domain of R^n with smooth boundary $\partial\Omega$, $n \geq 3$. In this paper we investigate existence and multiplicity of the solutions for the perturbation problem of a singular elliptic equation with Dirichlet boundary condition

$$-\Delta u = au + \frac{1}{|u|^{p+1}} + \frac{1}{|u-\alpha|^{q+1}} + |u|^{r-1} \qquad \text{in } \Omega, \tag{1.1}$$

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$$u = 0 \qquad \text{on} \quad \partial\Omega,$$

where a, p, q and α are real constants, $2 < r < p < q$ and $r < \frac{2n}{n-2}$.

Our problems are characterized as singular eliptic problems with singularities at $\{u = 0\}$ and $\{u = \alpha\}$. We recommend the book [5] for the singular elliptic problems. When $p+1 > 0$, since the pioneering work on the subject in [2], these problem have been investigated in many ways. For a survey on the scalar case we recommend the paper [3] and the references therein. In the last decades, some works on the matter were published focusing some other obstacles added to this kind of nonlinearities problems having critical growth and the case involving systems. Ambrosetti-Prodi type problems for the critical growth case were studied in [4]. For systems, we recommend the papers [1] and [3]. Essentially, we work with variational techniques: We first prove that the associated functional of (1.1) satisfies Palais-Smale condition, and then we use critical point theory.

Let $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ be eigenvalues of the eigenvalue problem $-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$, and ϕ_k be eigenfunctions belonging to the eigenvalues λ_k, $k \geq 1$. The eigenvalue problem $(-\Delta - a)u = \mu u$ in Ω, $u = 0$ on $\partial\Omega$ has infinitely many eigenvalues $\mu_{\lambda_i} = \lambda_i - a$ and corresponding eigenfunctions ϕ_k, $k \geq 1$. If $a < \lambda_1$, then

$$\mu_{\lambda_i} > 0 \qquad \forall i \geq 1$$

and

$$\lim_{i\to\infty} \frac{\mu_{\lambda_i}}{\lambda_i} = 1.$$

Let $c_{\mu_{\lambda_i}}$ be eigenvectors corresponding to eigenvalues $\mu_{\lambda_i} = \lambda_i - a$ respectively. Let us define the space

$$E = W_0^{1,r}(\Omega, R) = \{u|\ \nabla u \in L^r(\Omega, R) \text{ with compact support in } \Omega\}$$

with the norm

$$\|u\|_{W_0^{1,r}(\Omega,R)} = \Big(\int_\Omega |\nabla u|^r dx\Big)^{\frac{1}{r}} \ \text{ for all } \ r \geq 1, \text{ for all } \ u \in W_0^{1,r}(\Omega).$$

Let us set

$$\begin{aligned} W_{\lambda_i} &= \text{span}\{\phi_i|\ -\Delta\phi_i = \lambda_i\phi_i\}, \\ E_{\mu_{\lambda_i}} &= \{c_{\mu_{\lambda_i}}\phi \in E|\ c \in R, \phi \in W_{\lambda_i}\}. \end{aligned}$$

Then

$$E = \oplus_{i\geq 1} E_{\mu_{\lambda_i}}.$$

Let us set

$$U = \{u(x) \in E |\ u(x) \neq 0,\ u(x) \neq \alpha \text{ for all } x \in \Omega\},$$

$$\partial U = \{u(x) \in U |\ u(x_0) = 0 \text{ for some } x_0 \text{ and } \ u(x_1) = \alpha \text{ for some } x_1\}.$$

In this paper we are trying to find weak solutions of (1.1) in $W_0^{1,r}(\Omega, R)$ with singularity at $u = 0$ and $u = \alpha$. The weak solutions of (1.1) in U satisfy

$$\int_\Omega [-\Delta u \cdot v - auv - \frac{1}{|u|^{p+1}} v - \frac{1}{|u-\alpha|^{q+1}} v - |u|^{r-1} v] dx = 0 \qquad \forall v \in U. \tag{1.2}$$

We note that there exists one to one corresponding between weak solutions of (1.1) and critical points of the continuous and *Frechét* differentiable functional

$$F(u) \in C^1(U),$$

$$F(u) = \Psi_a(u) - \int_\Omega [-\frac{1}{p}\frac{1}{|u|^p} - \frac{1}{q}\frac{1}{|u-\alpha|^q} + \frac{1}{r}|u|^r] dx, \tag{1.3}$$

where

$$\Psi_a(u) = \frac{1}{2}\int_\Omega [|\nabla u|^2 - au^2] dx,$$

which will be proved in Section 2.

Our main result is as follows:

Theorem 1.1. *Assume that $2 < r < p < q$, $r < \frac{2n}{n-2}$, $a < \lambda_1$ and α is a real constant. Then (1.1) has at least one nontrivial weak solution $u(x)$ such that*

$$u(x) \neq 0 \qquad u(x) \neq \alpha.$$

For the proof of Theorem 1.1 we approach the variational technique. When $2 < r < p < q$, $r < \frac{2n}{n-2}$ and $a < \lambda_1$, the functional $F(u)$ satisfies Palais-Smale condition, so we can use the variational linking method in the critical point theory. The Outline of the proof of Theorem 1.1 is as follows: In Section 2, we introduce eigenvalues and eigenfunctions of the eigenvalue problem $(-\Delta - a) = \mu u$ in Ω, $u = 0$ on $\partial\Omega$, introduce eigenspaces spanned by the eigenfunctions corresponding to $\lambda_i - a$, investigate the properties of eigenspaces and prove that when $2 < r < p < q$, $r < \frac{2n}{n-2}$ and $a < \lambda_1$, the functional $F(u)$ satisfies Palais-Smale condition. In Section 3, we divide the whole space E into two subspaces, investigate the geometry of the sublevel sets of corresponding functional

F of (1.1), find some inequalities of $F(u)$ on two linked sublevel sets, and prove Theorem 1.1.

2. Variational Properties

LEMMA 2.1. *Assume that $2 < r < p < q$, $r < \frac{2n}{n-2}$, $a < \lambda_1$ and α be a real constant. Let $u \in U$ and $au + \frac{1}{|u|^{p+1}} + \frac{1}{|u-\alpha|^{q+1}} + |u|^{r-1} \in L^{1,r}(\Omega)\backslash\{0\}$. Then all the solutions of*

$$-\Delta u = au + \frac{1}{|u|^{p+1}} + \frac{1}{|u-\alpha|^{q+1}} + |u|^{r-1}$$

belong to U.

Proof. Equation (1.1) can be rewritten by

$$u = (-\Delta)^{-1}(au + \frac{1}{|u|^{p+1}} + \frac{1}{|u-\alpha|^{q+1}} + |u|^{r-1}) \qquad \text{in } \Omega, \tag{2.1}$$

$$u = 0 \qquad \text{on } \partial\Omega.$$

Then there exist constants $D_1 > 0$ such that

$$\begin{aligned} \|u\|_E^2 &= \|(-\Delta)^{-1}(au + \frac{1}{|u|^{p+1}} + \frac{1}{|u-\alpha|^{q+1}} + |u|^{r-1})\|_E^2 \\ &= \|\nabla(-\Delta)^{-1}(au + \frac{1}{|u|^{p+1}} + \frac{1}{|u-\alpha|^{q+1}} + |u|^{r-1})\|_{L^r(\Omega)}^2 \\ &\le D_1\|au + \frac{1}{|u|^{p+1}} + \frac{1}{|u-\alpha|^{q+1}} + |u|^{r-1}\|_{L^r(\Omega)}^2. \end{aligned}$$

Thus

$$\|u\|_E < \infty.$$

Thus the lemma is proved. □

LEMMA 2.2. *Assume that $2 < r < p < q$, $r < \frac{2n}{n-2}$, $a < \lambda_1$ and α be a real constant. Then the functional $F(u)$ is continuous, Fréchet differentiable with Fréchet derivative in U,*

$$DF(u)\cdot v = \int_\Omega [-\Delta u \cdot v - au \cdot v - \frac{v}{|u|^{p+1}} - \frac{v}{|u-\alpha|^{q+1}} - |u|^{r-1} \cdot v]dx \qquad \forall v \in E. \tag{2.2}$$

Moreover $DF \in C$. That is $F \in C^1$.

Proof. Let us set $H(x,u)=\frac{1}{2}au^2-\frac{1}{p}\frac{1}{|u|^p}-\frac{1}{q}\frac{1}{|u-\alpha|^q}+\frac{1}{r}|u|^r$, $H_u(x,u)=au+\frac{1}{|u|^{p+1}}+\frac{1}{|u-\alpha|^{q+1}}+|u|^{r-1}$. First we shall prove that $F(u)$ is continuous. For $u,v\in U$,

$$\begin{aligned}|F(u+v)-F(u))| &= |\frac{1}{2}\int_\Omega(-\Delta u-\Delta v)\cdot(u+v)dx-\int_\Omega H(x,u+v)dx\\ &\quad -\frac{1}{2}\int_\Omega(-\Delta u)\cdot udx+\int_\Omega H(x,u)dx|\\ &= |\frac{1}{2}\int_\Omega[(-\Delta u\cdot v-\Delta v\cdot u-\Delta v\cdot v)dx\\ &\quad -\int_\Omega(H(x,u+v)-H(x,u))dx|.\end{aligned}$$

We have

$$|\int_\Omega[H(x,u+v)-H(x,u)]dx|\le|\int_\Omega[H_u(x,u)\cdot v+O(\|v\|_E)]dx|=O(\|v\|_E). \tag{2.3}$$

Thus we have

$$|F(u+v)-F(u)|=O(\|v\|_E).$$
$$|F(u+v)-F(u)-DF(u)\cdot v|=O(\|v\|_E^2).$$

Next we shall prove that $F(u)$ is *Fréchet* differentiable. For $u,v\in U$,

$$\begin{aligned}&|F(u+v)-F(u)-DF(u)\cdot v|\\ &=|\frac{1}{2}\int_\Omega(-\Delta u-\Delta v)\cdot(u+v)dx-\int_\Omega H(x,u+v)dx\\ &-\frac{1}{2}\int_\Omega(-\Delta u)\cdot udx+\int_\Omega H(x,u)dx-\int_\Omega(-\Delta u-H_u(x,u))\cdot vdx|\\ =\ &|\frac{1}{2}\int_\Omega[-\Delta u\cdot v-\Delta v\cdot u-\Delta v\cdot v]dx\\ &-\int_\Omega[H(x,u+v)-H(x,u)]dx-\int_\Omega[(-\Delta u-H_u(x,u))\cdot v]dx|.\end{aligned}$$

By (2.3),

$$||F(u+v)-F(u)-DF(u)\cdot v||=O(\|v\|_E^2).$$

Thus $F\in C^1$. □

(1.1) can be rewritten by

$$u=(-\Delta-a)^{-1}(\frac{1}{|u|^{p+1}}+\frac{1}{|u-\alpha|^{q+1}}+|u|^{r-1})\qquad\text{in}\ \ \Omega, \tag{2.4}$$

$$u = 0 \qquad \text{on} \ \ \partial\Omega.$$

If $a < \lambda_1$, then $(-\Delta - a)^{-1}$ is positive operator. Since $\frac{1}{|u|^{p+1}} + \frac{1}{|u-\alpha|^{q+1}} + |u|^{r-1}$ is positive, so if the weak solution of (1.1) exists, the weak solution of (1.1) is positive. We shall show that if we choose a sequence $(u_n)_n \in U$ such that $F(u_n) \to c > 0$ and $DF(u_n) \to 0$, then the sequence $(u_n)_n$ is bounded as follows:

Lemma 2.3. *(A priori estimate)*
Assume that $2 < r < p < q$, $r < \frac{2n}{n-2}$, $a < \lambda_1$ *and* α *be a real constant. Let* $(u_n)_n$ *be any sequence in* U *and* $c \in R$ *be any positive real number. Then there exists a constant* $C = C(c)$ *such that if* $(u_n)_n \in U$ *satisfies that* $F(u_n) \to c$ *and* $DF(u_n) \to 0$, *then*

$$\lim_{n\to\infty} \|u_n\|_{L^r(\Omega)} \leq C,$$

$$\lim_{n\to\infty} \int_\Omega \frac{1}{|u_n|^p} dx \leq C, \qquad \lim_{n\to\infty} \int_\Omega \frac{1}{|u_n-\alpha|^q} dx \leq C.$$

Proof. Let $c \in R$ be any positive real number. Let $(u_n)_n$ be any sequence in U such that $F(u_n) \to c$ and $DF(u_n) \to 0$. By $a < \lambda_1$, there exists a constant $D > 0$ such that

$$\frac{1}{2}\int_\Omega [-\Delta u_n \cdot u_n - a u_n^2] dx \geq D\|u_n\|^2_{L^\beta(\Omega)} > 0.$$

Thus we have

$$\begin{aligned} F(u_n) &= \frac{1}{2}\int_\Omega [-\Delta u_n \cdot u_n - a u_n^2] dx - \int_\Omega [-\frac{1}{p}\frac{1}{|u_n|^p} - \frac{1}{q}\frac{1}{|u_n-\alpha|^q} + \frac{1}{r}|u_n|^r] dx \\ &> -\int_\Omega [-\frac{1}{p}\frac{1}{|u_n|^p} - \frac{1}{q}\frac{1}{|u_n-\alpha|^q} + \frac{1}{r}|u_n|^r] dx. \end{aligned}$$

By $F(u_n) \to c$ and $DF(u_n) \to 0$, there exists a small number $\epsilon > 0$ such that

$$\begin{aligned}
c+\epsilon &\geq \lim_{n\to\infty} F(u_n) - \lim_{n\to\infty} \frac{1}{2} DF(u_n)\cdot u_n \\
&= \lim_{n\to\infty} \frac{1}{2}\int_\Omega [-\Delta u_n \cdot u_n - a u_n^2]dx - \lim_{n\to\infty}\int_\Omega [-\frac{1}{p}\frac{1}{|u_n|^p} - \frac{1}{q}\frac{1}{|u_n-\alpha|^q} + \frac{1}{r}|u_n|^r]dx \\
&\quad - \lim_{n\to\infty} \frac{1}{2}\int_\Omega [-\Delta u_n \cdot u_n - a u_n^2]dx \\
&\quad + \lim_{n\to\infty} \frac{1}{2}\int_\Omega [\frac{1}{|u_n|^{p+1}} u_n + + \frac{1}{|u_n-\alpha|^{q+1}} u_n n|^{r-1} u_n]dx \\
&= \lim_{n\to\infty}\int_\Omega [(\frac{1}{2}\frac{u_n}{|u_n|} + \frac{1}{p})\frac{1}{|u_n|^p} + (\frac{1}{2}\frac{u_n}{|u_n-\alpha|} + \frac{1}{q})\frac{1}{|u_n-\alpha|^q} + (\frac{1}{2}\frac{u_n}{|u_n|} - \frac{1}{r})|u_n|^r]dx.
\end{aligned}$$

By $\lim_{n\to\infty} DF(u_n) = 0$, we have

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (-\Delta - a)^{-1} (\frac{1}{|u_n|^{p+1}} + \frac{1}{|u_n-\alpha|^{q+1}} + |u_n|^{r-1}) \qquad \text{in} \quad \Omega,$$

$$u = 0 \qquad \text{on} \quad \partial\Omega.$$

Since $(-\Delta - a)^{-1}$ is a positive operator and $\frac{1}{|u_n|^{p+1}} + \frac{1}{|u_n-\alpha|^{q+1}} + |u_n|^{r-1} > 0$, $\lim_{n\to\infty} u_n > 0$, $\|\frac{u_n}{|u_n|}\|_{L^r(\Omega)} = 1$ and $1 < \|\frac{u_n}{|u_n-\alpha|}\|_{L^r(\Omega)} < 1+d$ for some constant $d > 0$. Thus we have

$$\begin{aligned}
c+\epsilon &\geq \lim_{n\to\infty} F(u_n) - \lim_{n\to\infty} \frac{1}{2} DF(u_n)\cdot u_n \\
&= \lim_{n\to\infty}\int_\Omega [(\frac{1}{2} + \frac{1}{p})\frac{1}{|u_n|^p} + (\frac{1}{2} + \frac{1}{q})\frac{1}{|u_n-\alpha|^q} + (\frac{1}{2} - \frac{1}{r})|u_n|^r]dx.
\end{aligned}$$

By $\frac{1}{q} < \frac{1}{p} < \frac{1}{r} < \frac{1}{2}$, $\frac{1}{2} + \frac{1}{p} > 0$, $\frac{1}{2} + \frac{1}{q} > 0$ and $\frac{1}{2} - \frac{1}{r} > 0$. Since $2 < r < p < q$, $r < \frac{2n}{n-2}$, it follows that there exists a constant $C > 0$ such that

$$\lim_{n\to\infty} \|u_n\|^r_{L^r(\Omega)} < C,$$

$$\lim_{n\to\infty}\int_\Omega \frac{1}{|u_n|^p}dx < C, \qquad \lim_{n\to\infty}\int_\Omega \frac{1}{|u_n-\alpha|^q}dx < C.$$

□

LEMMA 2.4. *If any sequence $(u_n)_n$ in U satisfies*

$$u_n \to u_0 \in \partial U.$$

Then

$$F(u_n) \to \infty.$$

Proof. The proof can be checked easily. □

Now, we shall prove that $F(u)$ satisfies $(P.S.)_c$ with $c > 0$ as follows:

LEMMA 2.5. *(Palais-Smale condition)*
Assume that $2 < r < p < q$, $r < \frac{2n}{n-2}$, $a < \lambda_1$ *and* α *be a real constant. Let* c *be any positive real number. Then* $F(u)$ *satisfies the Palais-Smale condition: if* $(u_n)_n \in U$ *is any sequence such that* $F(u_n) \to c$ *and* $DF(u_n) \to 0$, *then* (u_n) *has a convergent subsequence* (u_{n_i}) *such that*

$$u_{n_i} \to u_0 \in U.$$

Proof. Let $(u_n)_n$ be any sequence in U such that $F(u_n) \to c$, $c > 0$ and $DF(u_n) \to 0$. By Lemma 2.3, $\lim_{n\to\infty} \|u_n\|_{L^r(\Omega)}$ is finite. Thus $(u_n)_n$ is bounded in $L^r(\Omega)$. Then up to subsequence, $(u_n)_n$ converges weakly to some u_0. From $DF(u_n) \to 0$ we have

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (-\Delta - a)^{-1}\left(\frac{1}{|u_n|^{p+1}} + \frac{1}{|u_n - \alpha|^{q+1}} + |u_n|^{r-1}\right) \qquad \text{in } \Omega.$$

By Lemma 2.3, $(u_n)_n$ and $(\frac{1}{|u_n|^{p+1}} + \frac{1}{|u_n-\alpha|^{q+1}} + |u_n|^{r-1})_n$ is bounded in $L^r(\Omega)$. Since the embedding E into $L^{r-1}(\Omega)$, $2 < r < p < q$, $r-1 < \frac{n+2}{n-2}$, is compact and $(-\Delta - a)^{-1}$ is a compact operator, it follows that $(u_n)_n$ has a convergent subsequence (u_{n_i}) converging strongly to some u_0 such that

$$DF(u_0) = \lim_{n\to\infty} DF(u_{n_i}) = 0.$$

We claim that $u_0 \neq 0$ and $u_0 \neq \alpha$. By contradiction, we suppose that $u_0 = 0$ and $u_0 \neq \alpha$. Then $u_0 \in \partial U$. Then by Lemma 2.4, $F(u_0) = \infty$, which is absurd. Thus

$$u_0 \neq 0, \qquad u_0 \neq \alpha.$$

□

3. Proof of Theorem 1.1

Let $E = W_0^{1,r}(\Omega, R)$ and let

$$\begin{aligned} W_{\lambda_i} &= \text{span}\{\phi_i | -\Delta\phi_i = \lambda_i\phi_i\}, \\ E_{\mu_{\lambda_i}} &= \{c_{\mu_{\lambda_i}}\phi \in E | \ c \in R, \phi \in W_{\lambda_i}\}. \end{aligned}$$

Then we have $E = \oplus_{i\geq 1} E_{\lambda_i}$. Let us set

$$\begin{aligned} E^+ &= (\oplus_{\mu_{\lambda_i}>0} E_{\mu_{\lambda_i}}), \\ E^- &= (\oplus_{\mu_{\lambda_i}<0} E_{\mu_{\lambda_i}}), \\ E^0 &= (\oplus_{\mu_{\lambda_i}=0} E_{\mu_{\lambda_i}}). \end{aligned}$$

Then

$$E = E^+ \oplus E^- \oplus E^0.$$

Because $\mu_{\lambda_i} > 0 \ \forall i \geq 1$,

$$E^0 = \emptyset \qquad E^- = \emptyset$$

and

$$E = E^+.$$

We note that E can be split by two subspaces Y_1 and Y_2 such that

$$Y_1 = \text{span}\{\text{eigenfunctions corresponding to eigenvalues } \mu_{\lambda_i} \\ \text{with } 1 \leq i \leq m,\ m \geq 1\}.$$

$$Y_2 = \text{span}\{\text{eigenfunctions corresponding to eigenvalues } \mu_{\lambda_i}, \\ \text{with } i \geq m+1,\ m \geq 1\},$$

$\dim Y_1 < \infty$ and

$$E = Y_1 \oplus Y_2.$$

Let us set

$$X_1 = Y_1 \cap U,$$
$$X_2 = Y_2 \cap U.$$

Then

$$U = X_1 \oplus X_2.$$

Let us set

$$B_\rho = \{(u \in U|\ \|u\|_E \leq \rho\},$$
$$\partial B_\rho = \{u \in U|\ \|u\|_E = \rho\},$$
$$Q = \bar{B_R} \cap X_1 \oplus \{\rho e|\ e \in \partial B_1 \cap E_{\mu_{\lambda_{m+1}}} \subset \partial B_1 \cap X_2,\ 0 < \rho < R\}.$$

Let us define

$$\Gamma = \{\gamma \in C(\bar{Q}, U)|\ \gamma = id \text{ on } \ \partial Q\}.$$

LEMMA 3.1. *Assume that $2 < r < p < q$, $r < \frac{2n}{n-2}$, $a < \lambda_1$ and α be a real constant. Let $e \in \partial B_1 \cap E_{\mu_{\lambda_i}} \subset \partial B_1 \cap X_2$. Then there* *exist a small number* $\rho > 0$ *and a large number* $R > 0$ *such that if* $u \in \partial Q = \partial(\bar{B_R} \cap X_1 \oplus \{\rho e|\ 0 < \rho < R\})$*, then*

$$\sup_{u \in \partial Q} F(u) < 0$$

and

$$\sup_{u \in Q} F(u) < \infty.$$

Proof. Let us choose an element $e \in \partial B_1 \cap X_2$ and $u \in X_1 \oplus \{\rho e|\ \rho > 0\}$. Then we have

$$\begin{aligned} F(u) &= \frac{1}{2}\int_\Omega [-\Delta u \cdot u - au^2]dx - \int_\Omega \frac{1}{r}|u|^r dx + \int_\Omega [\frac{1}{p}\frac{1}{|u|^p} + \frac{1}{q}\frac{1}{|u-\alpha|^q}]dx \\ &\leq \frac{1}{2}\mu_{\lambda_m}\|u\|^2_{L^r(\Omega)} + \frac{1}{2}\rho^2\mu_{\lambda_{m+1}}\|u\|^2_{L^r(\Omega)} - \frac{1}{r}\|u\|^r_{L^r(\Omega)} \\ &\quad + \int_\Omega [\frac{1}{p}\frac{1}{|u|^p} + \frac{1}{q}\frac{1}{|u-\alpha|^q}]dx. \end{aligned}$$

By Lemma 2.3, $\int_\Omega \frac{1}{p}\frac{1}{|u|^p}dx < \bar{C}$ and $\int_\Omega \frac{1}{q}\frac{1}{|u-\alpha|^q}dx < \bar{C}$ for some $\tilde{C}$. Thus

$$F(u) \leq \frac{1}{2}\mu_{\lambda_m}\|u\|^2_{L^r(\Omega)} + \frac{1}{2}\rho^2\mu_{\lambda_{m+1}}\|u\|^2_{L^r(\Omega)} - \frac{1}{r}\|u\|^r_{L^r(\Omega)} + \bar{C}.$$

Since $2 < r$, there exists a large number $R > 0$ such that if $u \in \partial Q$, then $F(u) < 0$. Thus we have $\sup_{u \in \partial Q} F(u) < 0$. Moreover if $u \in Q$, then $F(u) \leq \frac{1}{2}\mu_{\lambda_m}\|u\|^2_{L^r(\Omega)} + \frac{1}{2}\rho^2\mu_{\lambda_{m+1}}\|u\|^2_{L^r(\Omega)} + \bar{C} < \infty$. □

Lemma 3.2. *Assume that* $2 < r < p < q$, $r < \frac{2n}{n-2}$, $a < \lambda_1$ *and* α *be a real constant. Then there exist a small number* $\rho > 0$ *such that*

$$\inf_{u \in \partial B_\rho \cap X_2} F(u) > 0$$

and

$$\inf_{u \in B_\rho \cap X_2} F(u) > -\infty.$$

Proof. Let $u \in \partial B_\rho \cap X_2$. Then we have

$$\begin{aligned} F(u) &= \frac{1}{2}\int_\Omega [-\Delta u \cdot u - au^2]dx - \int_\Omega \frac{1}{r}|u|^r dx + \int_\Omega [\frac{1}{p}\frac{1}{|u|^p} + \frac{1}{q}\frac{1}{|u-\alpha|^q}]dx \\ &\geq \frac{1}{2}\int_\Omega [-\Delta u \cdot u - au^2]dx - \int_\Omega \frac{1}{r}|u|^r dx \\ &\geq \frac{1}{2}\mu_{\lambda_{m+1}}\|u\|^2_{L^r(\Omega)} - \frac{1}{r}\|u\|^r_{L^r(\Omega)}. \end{aligned}$$

Since $2 < r$, there exists a small number $\rho > 0$ such that if $u \in \partial B_\rho \cap X_2$, then $F(u) > 0$. Thus $\inf_{u\in\partial B_\rho\cap X_2} F(u) > 0$. Moreover if $(u, v) \in B_\rho \cap X_2$, then $F(u) \geq -\frac{1}{r}\|u\|^r_{L^r(\Omega)} > -\infty$. Thus $\inf_{u\in B_\rho\cap X_2} F(u) > -\infty$. So the lemma is proved. □

Let us define

$$c = \inf_{h\in\Gamma}\sup_{u\in Q} F(h(u)).$$

Lemma 3.3. *Assume that $2 < r < p < q$, $r < \frac{2n}{n-2}$, $a < \lambda_1$ and α be a real constant. Then*

$$0 < \inf_{u\in\partial B_\rho\cap X_2} F(u) \leq c = \inf_{h\in\Gamma}\sup_{u\in Q} F(h(u)) \leq \sup_{u\in Q} F(u) < \infty.$$

Proof. By Lemma 3.1, we have

$$\inf_{h\in\Gamma}\sup_{u\in Q} F(h(u)) \leq \sup_{u\in Q} F(u) < \infty.$$

By Lemma 3.2, we have

$$\inf_{h\in\Gamma}\sup_{u\in Q} F(h(u)) \geq \inf_{u\in\partial B_\rho\cap X_2} F(u) > 0.$$

Thus the lemma is proved. □

Proof of Theorem 1.1

Assume that $2 < r < p < q$, $r < \frac{2n}{n-2}$, $a < \lambda_1$ and α be a real constant. We note that $F(u)$ is continuous and *Fréchet* differentiable in U and $DF \in C$. By Lemma 2.5, $F(u)$ satisfies Palais-Smale condition. We claim that $c > 0$ is a critical value of $F(u)$, that is, $F(u)$ has a critical point u_0 such that

$$F(u_0) = c,$$

$$DF(u_0) = 0.$$

In fact, by contradiction, we suppose that $c > 0$ is not a critical value of $F(u)$. Then by Theorem A.4 in [6], for any $\bar{\epsilon} \in (0, c) > 0$, there exists a constant $\epsilon \in (0, \bar{\epsilon})$ and a deformation $\eta \in C([0,1] \times U, U)$ such that
(i) $\eta(0,u) = u$ for all $u \in U$,
(ii) $\eta(s,u) = u$ for all $s \in [0,1]$ if $F(u) \notin [c - \bar{\epsilon}, c + \bar{\epsilon}]$,
(iii) $F(\eta(1,u)) \leq c - \epsilon$ if $F(u) \leq c + \epsilon$.
We can choose $h \in \Gamma$ such that

$$\sup_{u \in Q} F(h(u)) \leq c + \epsilon$$

and

$$F(h(u)) < c - \bar{\epsilon} \qquad \text{on} \quad \partial Q.$$

This lead to $F(h(u)) \notin [c - \bar{\epsilon}, c + \bar{\epsilon}]$. Thus by (ii),

$$\eta(1, h(u)) = h(u) \qquad \text{on} \quad \partial Q.$$

Hence $\eta(1, h(u,v)) \in \Gamma$. By (iii) and the definition of c,

$$c \leq \sup_{u \in Q} F(\eta(1, h(u))) = \sup_{u \in Q} F(h(u)) \leq c - \epsilon,$$

which is a contradiction. Thus c is a critical value of $F(u)$. Thus $F(u)$ has a critical point u_0 with a critical value

$$c = F(u_0)$$

such that

$$0 < \inf_{u \in \partial B_\rho \cap X_2} F(u) \leq c \leq \sup_{u \in Q} F(u) < \infty.$$

By Lemma 2.4,

$$u_0 \neq 0 \qquad u_0 \neq \alpha.$$

Thus (1.1) has at least one nontrivial solution u_0 such that $u_0 \neq 0$ and $u_0 \neq \alpha$. Thus Theorem 1.1 is proved.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

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