

A HIGH-ORDER MODEL FOR SPIKE AND BUBBLE IN IMPULSIVELY ACCELERATED INTERFACE

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ABSTRACT. We present a high-order potential flow model for the motion of the impulsively accelerated unstable interface of infinite density jump. The Layzer model for the evolution of the interface is extended to high-order. The time-evolution solutions of the bubble and the spike in the interface are obtained from the high-order model. We show that the high-order model gives improvement on the prediction of the evolution of the bubble and the spike.

1. Introduction

An interface between two fluids of different densities accelerated by a shock wave is hydrodynamically unstable and is known as Richtmyer-Meshkov (RM) instability [1]. The RM instability plays important roles in many fields ranging from astrophysics to inertial confinement fusion, and has many common features with Rayleigh-Taylor (RT) instability [2, 3], which is driven by a gravitational acceleration.

The key characteristic of the RT and RM unstable interfaces is fingers, known as *bubble* and *spike*, of each phase extending into the region occupied by the opposite phase [4]. Thus a bubble (spike) is a portion of the light (heavy) fluid penetrating into the heavy (light) fluid. Eventually, a turbulent mixing caused by vortex structures around spikes breaks the ordered fluid motion.

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The main purpose of this paper is to develop a high-order model for the evolution of the bubble and the spike in the single-mode RM instability. Layzer [5] proposed a potential flow model for comprehensive description of the motion of hydrodynamically unstable interfaces. The Layzer model approximates the shape of the interface near the bubble (or spike) tip as a parabola and give a set of ordinary differential equations to determine the position, velocity and curvature of the bubble (or spike). In the last decades there have been significant progresses in the Layzer-type model. For the RM instability of infinite density jump (fluid/vacuum), Hecht *et al.* [6] and Zhang [7] obtained the asymptotic solutions of the bubble and the spike, respectively. Goncharov [8] and Sohn [9, 10] then extended the Layzer-type model to the interfaces of finite density jumps. Sohn [11, 12] recently applied the model to the multiple bubble interaction, and succeeded in the extension of the model to the unstable interfaces with surface tension and viscosity.

The Layzer model provides good predictions for the growth of the bubble in the RM instability, but there were differences in the spike velocity and the bubble curvature between the solution of the model and numerical results [7, 13]. In this paper, we present a high-order Layzer model to give improvement on the solutions of the RM bubble and spike of infinite density jump. Note that there have been few studies on the motion of the spike in the RM instability, while various models have been proposed for the bubble motion.

In Section 2, we describe the Layzer model, of low-order, for the evolution of the unstable interface. In Section 3, we present the high-order Layzer model for the motion of the interface. Section 4 gives the time-evolution solutions of the RM bubble and spike from the high-order model, in comparisons with the low-order model and numerical results. Section 5 gives conclusions.

2. Layzer model

In this section, we briefly describe the Layzer model for the evolution of hydrodynamically unstable interfaces. We consider an initial single-mode interface of infinite density jump in two dimensions

$$(1) \quad y = \eta(x, t = 0) = h_0 \cos(kx),$$

where $k = 2\pi/L$ represents the wave-number of the interface and L is the wave-length of the interface. The fluid is assumed as incompressible and inviscid. In the Layzer model [5], the interface near the tip of the bubble or the spike is approximated as

$$(2) \quad y = \eta(x, t) = \zeta_0(t) + \zeta_1(t) x^2.$$

Here, ζ_0 represents the longitudinal position of the bubble (or spike) tip, and $d\zeta_0/dt$ is the velocity of the bubble (or spike) tip. The velocity potential is given by

$$(3) \quad \phi(x, y, t) = a_1(t) \cos(kx) e^{-ky}.$$

The evolution of the interface is determined by the kinematic condition and the Bernoulli equation

$$(4) \quad \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} = v \quad \text{at } y = \eta,$$

$$(5) \quad \frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + g\eta = \text{const} \quad \text{at } y = \eta,$$

where u and v are x and y components of the interface velocity, and g is an external acceleration. The RM instability is modeled by setting $g = 0$ and giving a nonzero initial velocity, via the impulsive acceleration [1]. The kinematic condition implies the continuity of the normal component of the fluid velocity across the interface.

Substituting the expressions (2) and (3) into the kinematic condition and satisfying up to the second-order in x , one can obtain the following equations

$$(6) \quad \frac{d\zeta_0}{dt} = -a_1 k e^{-k\zeta_0},$$

$$(7) \quad \frac{d\zeta_1}{dt} = a_1 k^2 \left(3\zeta_1 + \frac{1}{2}k \right) e^{-k\zeta_0}.$$

The second order equation from the Bernoulli equation is given by

$$(8) \quad k \left(\zeta_1 + \frac{1}{2}k \right) e^{-k\zeta_0} \frac{da_1}{dt} = -a_1^2 k^3 \zeta_1 e^{-2k\zeta_0} + g\zeta_1.$$

Equations (6)-(8) determine the evolution of the bubble and the spike.

The initial condition for the bubble is $\zeta_0 > 0$, $U_0 > 0$ and $\zeta_1 < 0$, while the spike has the initial condition $\zeta_0 < 0$, $U_0 < 0$ and $\zeta_1 > 0$, where U_0 represents the initial velocity. The asymptotic solution of the

bubble and the spike can be obtained by taking the large time limit for Eqs. (6)-(8). The asymptotic velocity and curvature of the bubble [6] is

$$(9) \quad U^{bb} \sim \frac{2}{3kt}, \quad \xi^{bb} \rightarrow -\frac{k}{3},$$

and the asymptotic velocity and curvature of the spike [7] is

$$(10) \quad U^{sp} \rightarrow U_0 \sqrt{\frac{6\zeta_2 + 3k}{6\zeta_2 + k}}, \quad \xi^{sp} \rightarrow \infty,$$

where $\xi = 2\zeta_1$ represents the curvature. This solution implies that the growth rate of the bubble decays to zero, and the spike velocity converges to a constant limit which is dependent on the initial velocity and the curvature. The curvature of the spike diverges to infinity, which indicates a shape of a long filament.

3. High-order model

We present the high-order extension of the Layzer model. The interface near the tip of the bubble or the spike can be written as

$$(11) \quad y = \eta(x, t) = \sum_{j=0}^{\infty} \zeta_j(t) x^{2j},$$

and the velocity potential is generalized to

$$(12) \quad \phi(x, y, t) = \sum_{\substack{j=1 \\ j: \text{ odd}}}^{\infty} a_j(t) \cos(jkx) e^{-jk y}.$$

The evolution of the interface is again governed by the kinematic condition and the Bernoulli equation. One may apply the similar procedure as Section 2, to derive high-order equations. The velocities at the interface are given approximately by

$$\begin{aligned} u = \frac{\partial \phi}{\partial x} &= - \sum_j a_j(t) jk \sin(jkx) e^{-jk y} \\ &\approx -k^2 \sum_j a_j j^2 \left[x - jk \left(\zeta_1 + \frac{1}{6} jk \right) x^3 \right] e^{-jk \zeta_0}, \end{aligned}$$

$$\begin{aligned}
v = \frac{\partial \phi}{\partial y} &= - \sum_j a_j(t) j k \cos(j k x) e^{-j k y} \\
&\approx -k \sum_j a_j j \left[1 - \left(\frac{1}{2} j^2 k^2 + j k \zeta_1 \right) x^2 \right. \\
&\quad \left. + \left(\frac{1}{2} j^2 k^2 \zeta_1^2 - j k \zeta_2 + \frac{1}{2} j^3 k^3 \zeta_1 + \frac{1}{24} j^4 k^4 \right) x^4 \right] e^{-j k \zeta_0}.
\end{aligned}$$

Using this expression and satisfying the kinematic condition up to the fourth order in x , we obtain the equations

$$(13) \quad \frac{d\zeta_0}{dt} = -k \sum_j j a_j e^{-j k \zeta_0},$$

$$(14) \quad \frac{d\zeta_1}{dt} = k^2 \sum_j j^2 \left(3\zeta_1 + \frac{1}{2} j k \right) a_j e^{-j k \zeta_0},$$

$$(15) \quad \frac{d\zeta_2}{dt} = k^2 \sum_j j^2 \left(5\zeta_2 - \frac{5}{2} j k \zeta_1^2 - \frac{5}{6} j^2 k^2 \zeta_1 - \frac{1}{24} j^3 k^3 \right) a_j e^{-j k \zeta_0},$$

where all the summations are taken for $j = 1$ and 3 . The second and fourth order equations from the Bernoulli equation are given by

$$\begin{aligned}
(16) \quad k \sum_j j \left(\zeta_1 + \frac{1}{2} j k \right) e^{-j k \zeta_0} \frac{da_j}{dt} &= \frac{1}{2} k^4 \left(\sum_j j^2 a_j e^{-j k \zeta_0} \right)^2 \\
&\quad - k^3 \left(\sum_j j a_j e^{-j k \zeta_0} \right) \left[\sum_j j^2 a_j \left(\zeta_1 + \frac{1}{2} j k \right) e^{-j k \zeta_0} \right] + g \zeta_1,
\end{aligned}$$

$$\begin{aligned}
(17) \quad k \sum_j j \left(\zeta_2 - \frac{1}{2} j k \zeta_1^2 - \frac{1}{2} j^2 k^2 \zeta_1 - \frac{1}{24} j^3 k^3 \right) e^{-j k \zeta_0} \frac{da_j}{dt} \\
= \frac{1}{2} k^4 \left[\sum_j j^2 \left(\zeta_1 + \frac{1}{2} j k \right) a_j e^{-j k \zeta_0} \right]^2 + k^3 \left(\sum_j j a_j e^{-j k \zeta_0} \right) \\
\times \left[\sum_j j^2 \left(-\zeta_2 + \frac{1}{2} j k \zeta_1^2 + \frac{1}{2} j^2 k^2 \zeta_1 + \frac{1}{24} j^3 k^3 \right) a_j e^{-j k \zeta_0} \right] \\
- k^5 \left(\sum_j j^2 a_j e^{-j k \zeta_0} \right) \left[\sum_j j^3 \left(\zeta_1 + \frac{1}{6} j k \right) a_j e^{-j k \zeta_0} \right] + g \zeta_2.
\end{aligned}$$

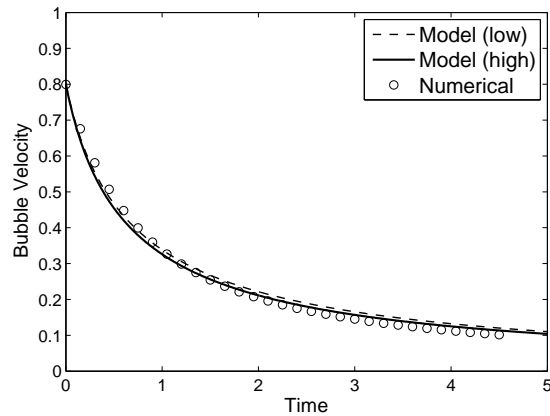


FIGURE 1. Bubble velocity from the low- and high-order models and the numerical result.

Equations (13)-(17) are the main equations we study in this paper. The asymptotic solution is not obtained in the high-order model, due to the complexity of the equations. We calculate the time-evolution solutions of the bubble and the spike from the model in the next section.

4. Results of the model

We now examine the agreement of the model by comparing the finite-time solutions of the model with numerical results. The time-evolution solution of the high-order model can be obtained by solving Eqs. (13)-(17) numerically. We employ the standard fourth-order Runge-Kutta method for numerical integrations.

In Figure 1, we compare the solutions of the bubble velocity from the low- and high-order Layzer models with the numerical result taken from Sohn [13]. The numerical simulations in [13] are performed by the point vortex method based on the vortex sheet model. The wave number is set to $k = 1$. The initial amplitude of the interface is given by 0.5 and the initial velocity of the interface is 0.8. In Fig. 1, we see that the bubble velocity decays to zero at a late time. The low-order and high-order solutions have little difference, and both solutions agree well with the numerical result.

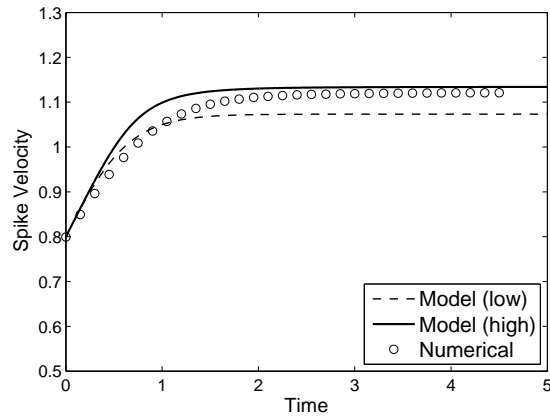


FIGURE 2. Spike velocity from the low- and high-order models and the numerical result.

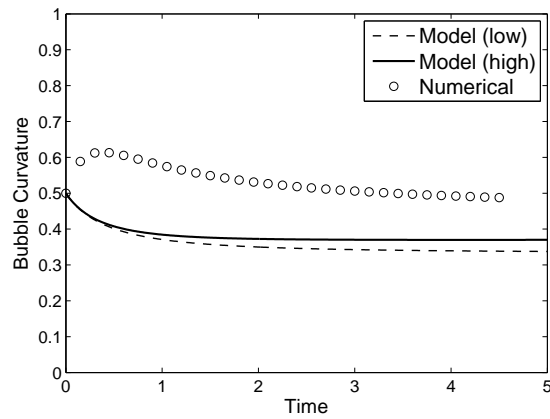


FIGURE 3. Bubble curvature from the low- and high-order models and the numerical result.

Figure 2 is the plot of the spike velocity of the low- and high-order models and the numerical result. In Fig. 2, the sign of the velocity is reversed. The solution of the high-order model converges to a constant limit and is in excellent agreement with the numerical solution. This indicates that the present model indeed improves the solution of the Layzer model.

Figure 3 shows the comparison of the bubble curvature from the low- and high-order models and the numerical result. In Fig. 3, the sign of the

curvature is reversed. The high-order solution of the bubble curvature also converges asymptotically to a constant limit. The difference of the solution of the model and the numerical result is reduced by the high-order model, but is still fairly large. The terminal values of the bubble curvature are 0.488 for the numerical result, 0.377 for the high-order model, and 0.333 for the low-order model.

5. Conclusions

We have presented the high-order solution for the bubble and spike evolution in the RM instability from the Layzer-type potential flow model. The high-order model gives better predictions for both the bubble and spike velocity than the low-order model. The results presented in the paper validates that the high-order model provides an appropriate description for the evolution of the unstable interface.

We have also found a limitation of the present model. The difference of the high-order solution for the bubble curvature with the numerical result is fairly large. In order to give a quantitatively accurate solution for the bubble curvature, the model would require even higher-order expansions, which is a formidable work.

The present high-order model is only for the interface of the infinite density jump. In fact, it is possible to develop a high-order Layzer model for the cases of finite density jump, but we have found that the equations in that model are quite coupled and it is difficult to solve them. Even if one succeeds in solving the high-order model for the cases of finite density jump, it may not be directly applicable to the spike evolution. It is because for finite density jump, the mushroom-like vortex structure is pronounced around the spike, which results increase of the drag, and therefore this effect should be considered in the modelling. So far, no quantitative model for the spike evolution of finite density jump has been established. Development of the model for the motion of the spike of finite density jump would be a challenging subject.

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